Sampling methods for time domain inverse scattering problems

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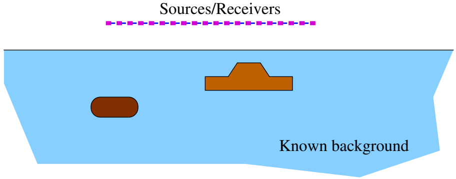
Joint work with

First part: Q. Chen, P. Monk, A. Lechleiter (Inverse Problems, 2010)
Second part: A. Lechleiter (in preparation)

The inverse problem / Motivation

Radar, Sonar, Mine detection, Infrastructure imaging, Non destructive testing, $\cdots$



Inverse problem: Determine the geometry of inclusions from the knowledge of diffracted fields associated with several incident waves.

- Physical properties of the inclusions are not known (a priori)
- Spectrum of the incident waves in the resonant region

$\Rightarrow$ Sampling methods are good candidates

Examples of applications

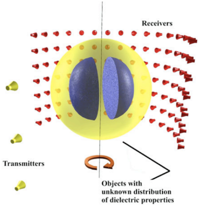
Radar, Sonar, Mine detection, Infrastructure imaging, Non destructive testing, · · ·

Imaging of urbain infrastructures with GPR:

Visualise complex structures (pipelines, deposits, mines, . . .) buried in the ground (few meters), from electromagnetic measurements

GPR device of WTI-inc

Microwave biomedical imaging:
Use microwaves (moderate frequencies) for diagnosis of malignancies or functional monitoring.
Advantages: cheap cost, absense of side effects.

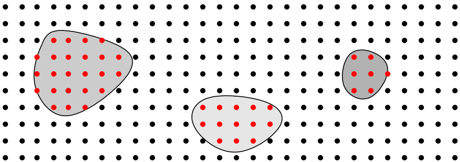


Prototype of experimental measurements

Sampling methods

Examples of sampling methods: *Linear Sampling Method* (Colton-Kirsch, 1996), *Factorization method* (Kirsch, 1998), *Probe Method* (Potthast, 2001), *Reciprocity Gap Sampling Method* (Colton-Haddar, 2005), ...)

Principle: Associate with a *sampling point* $\mathbf{z}$ of the probed domain a criterion $\mathcal{G}(\mathbf{z})$ that indicates whether $\mathbf{z}$ is in the interior or the exterior of the scatterer.



(+) Non-iterative, the computation of $\mathcal{G}$ does not require a forward solver.
(−) Require a large amount of multistatic-data (many transmitters-receivers).

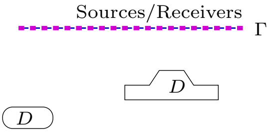
Goal: Reduce the required number of sources/receivers by using multiple frequencies, or even better: a *time dependent data*

Relevance of time dependent data

- Use of realistic measurments: causal sources and short pulses (GPR applications)
- Provide naturel "multi-frequency" reconstruction criteria
- Incorporate arrival time information in the reconstruction procedure
- Naturel dependance of regularization parameters on the frequency

A model problem

Inverse scattering from a perfect conductor



Total field $u_{\text{tot}}(\cdot, x_0)$

$$
\begin{aligned}
(\partial_{tt} - \Delta) u_{\text{tot}}(\cdot, x_0) &= \chi(t)\delta_{|x-x_0|} && (\mathbb{R}^3 \setminus \overline{D}) \times \mathbb{R}, \\
u_{\text{tot}}(\cdot, x_0) &= 0 && \partial D \times \mathbb{R}, \\
u_{\text{tot}}(\cdot, x_0) &= 0 && (\mathbb{R}^3 \setminus \overline{D}) \times \mathbb{R}_-
\end{aligned}
$$

χ: causal function with compact support.

$x_0 \in \Gamma$: location of the sources/receivers

Inverse problem: Determine D from the knowledge of $u_{\text{tot}}(x, t, x_0)$ for all $t \in \mathbb{R}$ and all x and x_0 in Γ

A model problem

Inverse scattering from a perfect conductor

Incident field:

$$u_{\text{inc}}(x,t,x_0) := \frac{\chi(t-|x-x_0|)}{4\pi|x-x_0|}$$

Scattered field:

$$u_{\text{sca}} := u_{\text{tot}} - u_{\text{inc}}$$

Near-Field operator:

$$(\mathcal{N}\phi)(x,t) := \int_{\mathbb{R}}\int_{\Gamma} u_{\text{sca}}(x,t-t_0,x_0)\phi(x_0,t_0)ds(x_0)dt_0 \quad x\in\Gamma,\ t\in\mathbb{R}$$

Principle of the Linear Sampling Method (LSM): characterize the inclusion D using the range of this operator.

Factorization of the operator $\mathcal{N}$

Linearity of the map: $u_{\text{inc}} \mapsto u_{\text{sca}}$

$\Rightarrow \mathcal{N}\phi$ is the scattered field associated with the incident field

$$\mathrm{SL}_{\Gamma}^{\chi}\,\phi(x,t) := \int_{\mathbb{R}} \int_{\Gamma} u_{\text{inc}}(x, t-t_0, x_0)\phi(x_0,t_0)ds(x_0)dt_0$$

Therefore:

$$\mathcal{N}\phi = \mathcal{G}(\mathrm{SL}_{\Gamma}^{\chi}\,\phi)$$

Where

$$\mathcal{G}f := u(x,t)|_{\Gamma\times\mathbb{R}}$$

$$\begin{cases} (\partial_{tt} - \Delta)u = 0 & (\mathbb{R}^3 \setminus \overline{D}) \times \mathbb{R}, \\ u = -f & \partial D \times \mathbb{R}, \\ u = 0 & (\mathbb{R}^3 \setminus \overline{D}) \times \mathbb{R}_- \end{cases}$$

Factorization of the operator $\mathcal{N}$

The solution u can be represented as a retarded potential

$$u(x,t) = (\mathrm{SL}_{\partial D}\phi)(x,t) := \int_{\partial D} \frac{\phi(x_0, t - |x - x_0|)}{4\pi|x - x_0|} ds(x_0)$$

$\Rightarrow$ Boundary integral equation: $S_{\partial D}\phi = -f$ on $\partial D \times \mathbb{R}$.

Analysis of retarded potentials via Laplace transform in $H^s_\sigma(\mathbb{R}, H^{-1/2}(\Gamma))$, $\sigma > 0$ (Bamberger & Ha-Duong, 1986)

$S_{\partial D} : H^p_\sigma(\mathbb{R}, H^{-1/2}(\partial D)) \to H^{p-1}_\sigma(\mathbb{R}, H^{1/2}(\partial D))$
$S^{-1}_{\partial D} : H^p_\sigma(\mathbb{R}, H^{1/2}(\partial D)) \to H^{p-2}_\sigma(\mathbb{R}, H^{-1/2}(\partial D))$

Hence: $\mathcal{G}\, f = -\mathrm{SL}_{\partial D}\, S^{-1}_{\partial D}\, f$, and

$$\boxed{\mathcal{N} = -\mathrm{SL}_{\partial D}\, S^{-1}_{\partial D}\, \mathrm{SL}^{\chi}_{\Gamma}}$$

Thm: $\mathcal{N} : H^2_\sigma(\mathbb{R}, H^{-1/2}(\Gamma)) \to H^{-2}_\sigma(\mathbb{R}, H^{1/2}(\Gamma))$ is bounded and injective with dense range.

Time Domain Sampling

- **Idea**: Test range of $\mathcal{N}$ with "point sources"

$$\phi_{z,\tau}(x,t) = \frac{\chi(t-\tau-|x-z|)}{4\pi|x-z|}, \quad (x,t) \in \mathbb{R}^3 \setminus \{z\} \times \mathbb{R}$$

- **Thm 1**: $\phi_{z,\tau}|_{\Gamma\times\mathbb{R}}$ is in the range of $\mathcal{G}$ if an only if $z \in D$
  - For $z \in D$: $\mathcal{G}(-\phi_{z,\tau}|_{\partial D\times\mathbb{R}}) = \phi_{z,\tau}|_{\Gamma\times\mathbb{R}}$
  - For $z \notin D$ the function $\phi_{z,\tau}|_{\Gamma\times\mathbb{R}}$ cannot belong to the range of $\mathcal{G}$: point source is singular at z but solutions to the wave equation are not (unique continuation argument is needed here)
- **Thm 2**: $\mathrm{SL}_\Gamma^\chi : H^2_\sigma(\mathbb{R}, H^{-1/2}(\Gamma)) \to H^1_\sigma(\mathbb{R}, H^{1/2}(\partial D))$ is injective with dense range.

Recall that:

$$\boxed{\mathcal{N} = \mathcal{G} \circ \mathrm{SL}_\Gamma^\chi}$$

Theoretical Justification of LSM

Main Theorem: Let $\tau \in \mathbb{R}$.

(1) If $z \in D$ then for all $\epsilon > 0$ there exists $g^{\epsilon}_{z,\tau} \in H^2_\sigma(\mathbb{R}, H^{-1/2}(\Gamma))$ such that

$$\|\mathcal{N} g^{\epsilon}_{z,\tau} - \phi_{z,\tau}\|_{H^{-2}_\sigma(\mathbb{R}, H^{1/2}(\Gamma))} \le \epsilon,$$

$$\lim_{\epsilon \to 0} \|\mathrm{SL}^{\chi}_{\Gamma}\, g^{\epsilon}_{z,\tau}\|_{H^1_\sigma(\mathbb{R}, H^1(D))} < \infty.$$

Moreover, for fixed ϵ:

$$\lim_{z \to \partial D} \|g^{\epsilon}_{z,\tau}\|_{H^2_\sigma(\mathbb{R}, H^{-1/2}(\Gamma))} = \infty, \qquad \text{and}$$

$$\lim_{z \to \partial D} \|\mathrm{SL}^{\chi}_{\Gamma}\, g^{\epsilon}_{z,\tau}\|_{H^1_\sigma(\mathbb{R}, H^1(D))} = \infty.$$

(2) If $z \notin (D \cup \Gamma)$ then for any $g^{\epsilon}_{z,\tau} \in H^2_\sigma(\mathbb{R}, H^{-1/2}(\Gamma))$ such that

$$\lim_{\epsilon \to 0} \|\mathcal{N} g^{\epsilon}_{z,\tau} - \phi_{z,\tau}\|_{H^{-2}_\sigma(\mathbb{R}, H^{1/2}(\Gamma))} = 0$$

it holds that

$$\lim_{\epsilon \to 0} \|g^{\epsilon}_{z,\tau}\|_{H^2_\sigma(\mathbb{R}, H^{-1/2}(\Gamma))} = \infty, \qquad \text{and}$$

$$\lim_{\epsilon \to 0} \|\mathrm{SL}^{\chi}_{\Gamma}\, g^{\epsilon}_{z,\tau}\|_{H^1_\sigma(\mathbb{R}, H^1(D))} = \infty.$$

Algorithmic Aspects

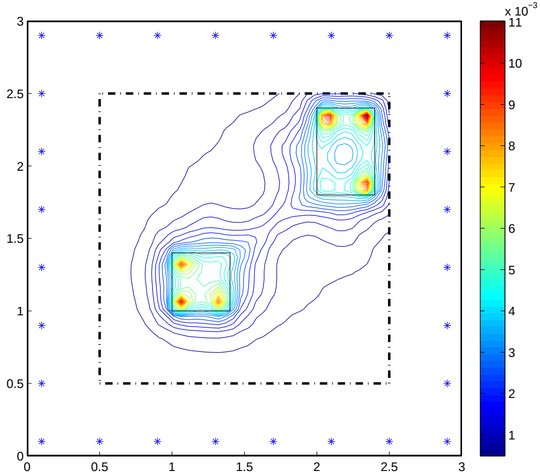
- A regularization is needed to solve the near field equation, e.g.

$$(\epsilon + \mathcal{N}_d^* \mathcal{N}_d) g_{z,\tau}^{\epsilon} = \mathcal{N}_d^* \phi_{z,\tau}$$

- Dimension of discretized matrix is huge
  Example: 10 sources/receivers, 100 time steps yield unknown $g_{z,\tau}^{\epsilon}$ of dimension $10 * 10 * 100 = 10^4$
  - Convolution structure of the kernel saves some memory
  - System matrix $\mathcal{N}_d$ has a large kernel
  - Compute a sufficient number of the first singular values/vectors of $\mathcal{N}_d$ (only necessitates evaluation of matrix-vector products) to approximate $\mathcal{N}_d$
- The choice of τ in $g_{z,\tau}^{\epsilon}$ in the latter theorem seems arbitrary. For numerical implementation it is not arbitrary since support in time of the density $g_{z,\tau}^{\epsilon}$ has to be truncated.

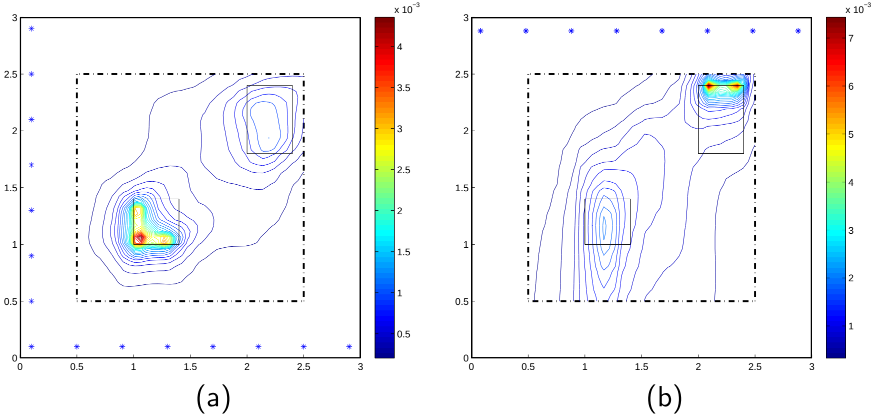
Numerical Examples I

wave speed=1, source $\sim \sin(4t)e^{-1.6(t-3)^2}$, $\lambda_c = 2\pi/4 \approx 1.6$, full aperture, 1% added random noise



Numerical Examples II

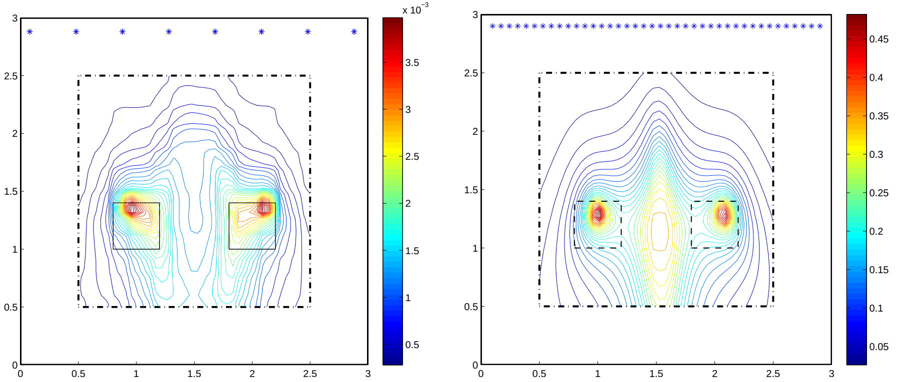
wave speed=1, source $\sim \sin(4t)e^{-1.6(t-3)^2}$, $\lambda_c = 2\pi/4 \approx 1.6$, full aperture, 1% added random noise



(a) (b)

Numerical Examples III

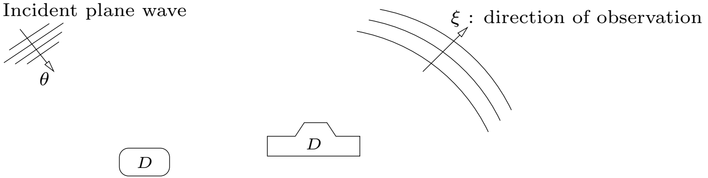
(a) wave speed=1, source $\sim \sin(4t)e^{-1.6(t-3)^2}$, $\lambda_c = 2\pi/4 \approx 1.6$, full aperture, 1% added random noise. (b) Frequency domain reconstruction at central wave number $k_c = 4$ using standard frequency domain linear sampling method



(a) (b)

Sampling Methods with Far Field data

Physical Setting – Radar, Sonar, Microwave applications



- Incident travelling wave front $u_{\text{inc}}(x,t;\theta) := \chi(t - \theta \cdot x)$
  The function χ has compact support and is zero for $t \leq T = \sup_{x\in D} |x|$.
- **Scattered field**: $u_{\text{sca}}(\cdot,\cdot;\theta)$ solves

$$\begin{aligned}
&\partial_{tt} u_{\text{sca}} - \Delta u_{\text{sca}} = 0 \quad (\Omega \times \mathbb{R}),\\
&u_{\text{sca}} = -u_{\text{inc}} \quad (\partial D \times \mathbb{R}),\\
&u_{\text{sca}} = 0 \quad (\Omega \times (-\infty, 0))
\end{aligned}$$

- **Far field**: $u_\infty(\xi,t;\theta) = \lim_{r\to\infty} u_{\text{sca}}(r\xi, r+t;\theta)$ for $\xi \in \mathbb{S}$, $t \in \mathbb{R}$
- **Thm**: $u_{\text{sca}}(r\xi,t;\theta) = u_\infty(\xi, t-r;\theta)/r + O(1/r^2)$ as $r \to \infty$

Mathematical Setting – Inverse problem

- **Measured Data**: $u_\infty(\xi, s, \theta)$ for all $\xi, \theta \in \mathbb{S}$ and all $s \in \mathbb{R}$.
- **Far-field operator**:

$$(\mathcal{F}g)(\xi, s) := \int_{\mathbb{R}} \int_{\mathbb{S}} u_\infty(\xi, s - s_0, \theta) g(\theta, s_0) ds_0 d\theta$$

- $\mathcal{F}g$ is the farfield associated with

$$(\mathcal{H}_\chi g)(x, t) := \int_{\mathbb{R}} \int_{\mathbb{S}} \chi(t - s_0 - \theta \cdot x) g(\theta, s_0) d\theta ds_0$$

  $\mathcal{H}_\chi$ is the time domain Herglotz operator.

- **Thm**: For all $a > 0$, the operator
  $\mathcal{H}_\chi : H_0^{5/2}(0, a; L^2(\mathbb{S})) \to H^{3/2}(-T, T + a; H^{1/2}(\partial D))$ is bounded and injective.

Factorizing the Far Field Operator

- For a single layer potential $u = \mathrm{SL}_{\partial D}\psi$ the far field is given by

$$u_\infty(\xi, t) = (R\psi)(\xi, t) = \frac{1}{4\pi}\int_{\partial D} \psi(x_0, t + \xi \cdot x_0) dx_0$$

Thus, $\mathcal{F}g = -R\, S_{\partial D}^{-1}\, \mathcal{H}_\chi\, g$

- **Thm 1**: Setting $\mathcal{F}_\chi := -\partial_t \chi \star \mathcal{F}$

$$\boxed{\mathcal{F}_\chi = \mathcal{H}_\chi^* \circ \partial_t S_{\partial D}^{-1} \circ \mathcal{H}_\chi}$$

- **Thm 2**: $\partial_t S_{\partial D}^{-1}$ possesses the following coercivity property: Let $a > 0$ then

$$\int_0^a \int_{\partial D} \partial_t S_{\partial D}^{-1}(\psi)\psi dx dt \geq C\|\psi\|^2_{H^{-3/2}(0,a;H^{-1/2}(\Gamma))}$$

for all $\psi \in H^{3/2}(0, a; H^{1/2}(\partial D))$

Range Inclusions

Thm: $\mathcal{F}_\chi : H_0^{5/2}(0,a;L^2(\mathbb{S})) \to H^{-5/2}(0,a;L^2(\mathbb{S}))$ is a positive and selfadjoint operator that has a "square root" $B : H_0^{5/2}(0,a;L^2(\mathbb{S})) \to L^2(0,a;L^2(\mathbb{S}))$ such that

$$\boxed{\mathcal{F}_\chi = B^* B}$$

Moreover the following inclusions hold:

$$\mathrm{Rg}\left(\mathcal{H}_\chi^* : H_0^{3/2}(-T,a+T;H^{-1/2}(\partial D)) \to H^{-5/2}(0,a;L^2(\mathbb{S}))\right)$$

$$\cap$$

$$\mathrm{Rg}\left(B^* : L^2(0,a;L^2(\mathbb{S})) \to H^{-5/2}(0,a;L^2(\mathbb{S}))\right)$$

$$\cap$$

$$\mathrm{Rg}\left(\mathcal{H}_\chi^* : H^{-3/2}(-T,T+a;H^{-1/2}(\partial D)) \to H^{-5/2}(0,a;H^{1/2}(\partial D))\right)$$

Characterization of D

Test functions: we use the far fields associated with the point sources

$$\phi_{z,\tau}(x,t) = \frac{\chi(t-\tau-|x-z|)}{4\pi|x-z|}, \quad x \in \mathbb{R}^3 \setminus \{z\}, t \in \mathbb{R}.$$

$$\boxed{\phi^\infty_{z,\tau}(\xi,t) := \chi(t-\tau+\xi\cdot z)/(4\pi), \quad \xi \in \mathbb{S}, t \in \mathbb{R}}$$

Main Thm: Let $\tau > 0$ et let $a > 0$ such that the far field $\phi^\infty_{z,\tau}$ is supported in $\mathbb{S} \times [0,a]$ for all sampling points $z \in \Omega \supset D$. Then

$$\phi^\infty_{z,\tau} \in \mathrm{Rg}(B^*) \quad \Longleftrightarrow \quad z \in D.$$

Remark: Numerically $B^* \equiv (\mathcal{F}_\chi)^{1/2}$.

Conclusion and Outlook

Conclusion:

- Inverse scattering in the time domain
- Factorization of near field and far field operators
- Linear sampling method in the time domain domain for near field data
- Factorization method in the time domain for far field data
- Both methods use measurements of causal waves
- Implementation difficult - huge dimension

Outlook:

- Penetrable media, Electromagnetic problem
- Exploit sampling in time with the parameter τ
- Implementation of suitable data structures and faster SVD
- Other regularizations