# Modèles stochastiques pour la propagation dans les fibres optiques

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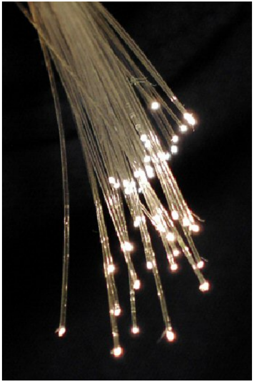
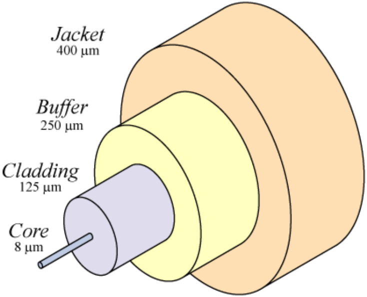
# Nonlinear fiber optics in communications

- A very efficient way to transmit information : high data rates (up to six time faster than satellites)
- Explosion of the field in the 1990's, due to development of Erbium-doped amplifiers and lasers
- Wavelength division multiplexing $\rightsquigarrow$ capacities of more than 1 Tb/s

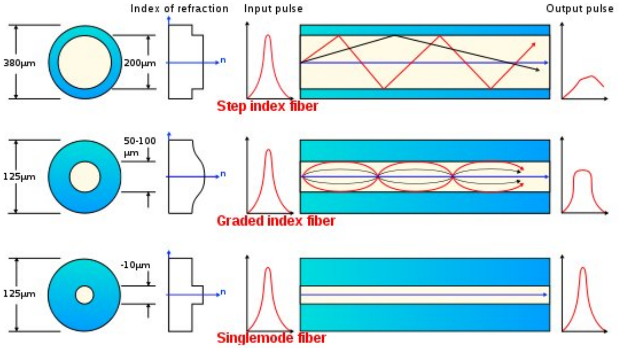


# Optical fibers




Glass or plastic fiber that carries light along its length ;
Light is kept in the core by total internal reflection : the fiber acts as a waveguide

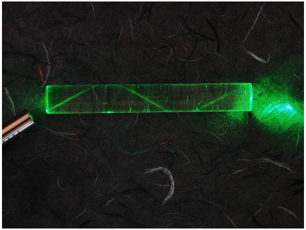
# Single mode/multimode fibers




Core diameter may be less than 10 times the wavelength of the propagation light
$\rightsquigarrow$ use of Maxwell equations

Nonlinear fiber optics plays an important role in the design of such high capacity systems

Agrawall : Applications of nonlinear fiber optics



Limiting factors to be taken into account :

- Chromatic dispersion (monomode fibers)
  $\rightsquigarrow$ use of nonlinearity, or more efficiently, of dispersion managed fibers
- Polarization mode dispersion (PMD) : linked to birefringence

# The NLS equation in fiber optics

Maxwell equations for the electromagnetic field $\rightsquigarrow$

$$-\frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} + \Delta \vec{E} - \nabla(\mathrm{div}\vec{E}) = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2 \vec{P}}{\partial t^2}$$

$\vec{E}$ : electric field

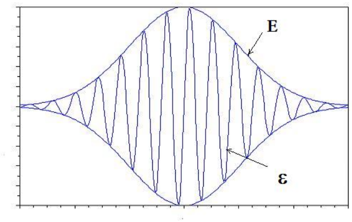
$\vec{P}$ : polarization (depends nonlinearly on $\vec{E}$)

**Properties of the material :**
- isotropic
- centrosymmetric

$$\frac{1}{\varepsilon_0}\vec{P}(\vec{x},t) = \int_{-\infty}^{t} \chi^{(1)}(x^{\perp}, t-t_1)\vec{E}(t_1)dt_1$$
$$+\iiint_{-\infty}^{t} \chi^{(3)}(x^{\perp}, t-t_1, t-t_2, t-t_3)(\vec{E}(t_1).\vec{E}(t_2))\vec{E}(t_3)dt_1 dt_2 dt_3$$

## Weakly nonlinear WKB expansion :

$$\vec{E} = \varepsilon \vec{E}_0 + \varepsilon^2 \vec{E}_1 + \varepsilon^3 \vec{E}_2 + \ldots$$




$$E_j(x^{\perp}, z, t) = U_j(x^{\perp}, t) \star \left[ \mathcal{E}_j(\varepsilon z, \varepsilon^2 z, \varepsilon t) e^{i(k_0 z - \omega_0 t)} \right] + cc$$

(convolution in time)

Newell, Moloney, Nonlinear Optics

### Assumptions :

- monomode fiber : only one mode $\widehat{U}(x^{\perp}, \omega)$ is confined
- polarization preserving

Then compatibility conditions for $\vec{E}_1$ and $\vec{E}_2$ give the NLS equation for $\mathcal{E}_0$ :

$$i\partial_{\zeta}\mathcal{E}_0 - \frac{1}{2}k_0" \frac{\partial^2 \mathcal{E}_0}{\partial \tau^2} + \frac{\omega_0}{c} n_2 |\mathcal{E}_0|^2 \mathcal{E}_0 = 0$$

with $\tau = \varepsilon(t - k_0' z)$, $\zeta = \varepsilon^2 z$.

Many effects have been neglected...

The sign of $k_0$" (GVD) determines whether the medium is "focusing" or "defocusing".

# NLS : mathematical properties

One dimensional cubic nonlinear Schrödinger equation (dimensionless form) :

$$i\partial_t u + \lambda \partial_x^2 u + |u|^2 u = 0, \ x \in \mathbf{R}, \ \lambda = \pm 1$$

Integrable by inverse scattering (Zakharov-Shabat, 1972)
$\rightsquigarrow$ explicit soliton solutions

- **Focusing case ($\lambda$=1) :** "bright solitons"

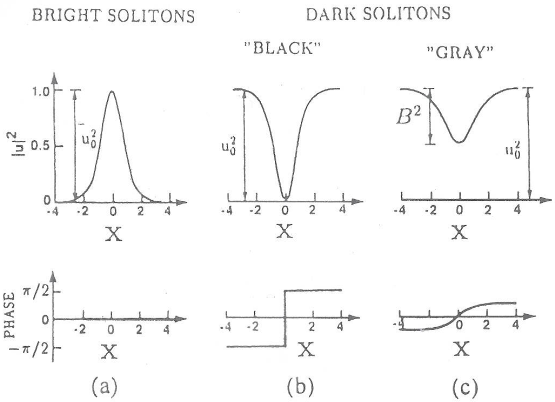
$$u(t,x) = A_\omega(x - 2vt - s)e^{i(vx - v^2 t + \omega t + \theta)}$$

with

$$A_\omega(x) = \sqrt{2\omega}\,\mathrm{sech}(\sqrt{\omega}x)$$

- **Defocusing case ($\lambda$=-1) :** No localized solution ;
  "dark solitons" with $|v| \leq \rho$, $\lim_{|x|\to\infty} |u(t,x)| = \rho$.
  $v = 0$ : "black soliton"

## Soliton profiles



**Conserved quantities :**

- Energy :

$$N(t) = \int |u(t)|^2 dx = N(0)$$

- Hamiltonian :

$$H(t) = \int |\nabla u(t)|^2 dx - \frac{\lambda}{2} \int |u(t)|^4 dx = H(0)$$

Allow to show propagation of $H^1$ solutions ($N(t)$ and $H(t)$ finite) for all time, or along all the fiber

**Remark :** Global propagation also holds if only $N(0)$ finite (Strichartz estimates : dispersion inequalities in space + convolution inequalities in time)

**Remark :** not always true ; 1-D focusing quintic NLS equation (not physical)

$$i\partial_t u + \partial_x^2 u + |u|^4 u = 0$$

Explicit solution which blows up at $t = 1$ :

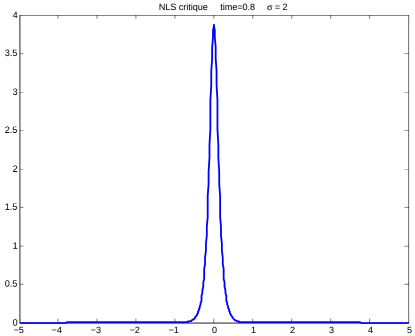
$$u(t,x) = \frac{e^{-i\frac{x^2}{4(1-t)}} e^{\frac{i}{1-t}}}{\sqrt{1-t}} \frac{3^{1/4}}{\sqrt{\cosh \frac{2x}{1-t}}}$$

"Critical $L^2$ case" : the scale change

$$u_\mu(t,x) = \mu^{-1/2} u(\frac{t}{\mu^2}, \frac{x}{\mu})$$

preserves both the equation and the $L^2$ norm.

Same phenomenon for the 2-D cubic NLS equation (but no explicit solution)


NLS critique time=0.8 σ = 2
4
3.5
3
2.5
2
1.5
1
0.5
0
−5
−4
−3
−2
−1
0
1
2
3
4
5

# Dispersion managed fibers

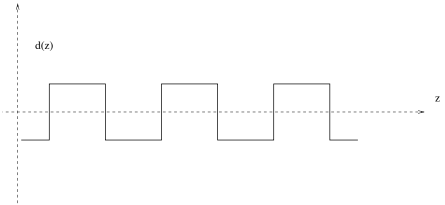
In DM fibers, the group velocity dispersion varies with the distance along the fiber $\rightsquigarrow$ NLS equation is modified by (dimensionless form)

$$i\partial_t u + \frac{1}{2}D(t)\partial_x^2 u + |u|^2 u = 0$$

with

$$D(t) = d(t) + d_m + m(t)$$

$d_m$ : residual dispersion (small) ; $m$ : random process with zero mean



## Case m=0 (no random perturbation) :

Lot of physical/numerical studies

Zharnitsky, Grenier, Jones, Turitsyn, Ph. D 2001 :
On large propagation distances, rescaled solution close to solution of an averaged equation :

$$i\partial_t v + d_m \partial_x^2 v + \langle Q \rangle (v, v, v) = 0$$

with

$$Q(v_1, v_2, v_3, t) = T^{-1}(t)(T(t)v_1 . T(t)v_2 . \overline{T(t)v_3})$$

and $T(t)u_0$ solution of

$$i\partial_t u + d(t)\partial_x^2 u = 0$$

with $u(0) = u_0$.

DM soliton = ground state of the averaged system $(d_m \geq 0)$

## **Asymptotic evolution of the pulse** from Turitsyn et. al. Opt. Comm. 1999

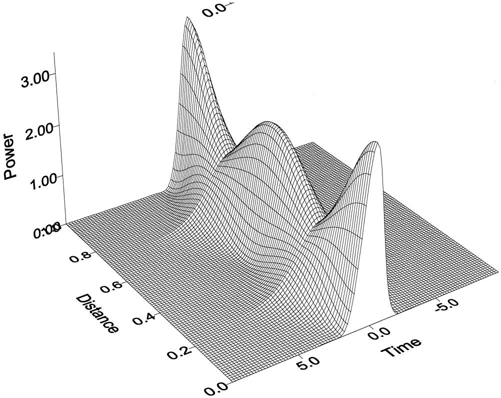



Fig. 1. Evolution over one compensation period of DM soliton shown in normal (bottom) and logarithmic (upper picture) scales. In the leading order the dynamics is self-similar (bottom) and is given by Eqs. (37). It is seen that the dips appear at the beginning (end) and in the middle of the periodic cell. Dispersion map $d(z) = \tilde{d} + \langle d \rangle = \pm 5 + 0.15$, $c(z) = 1$.

## Case of random perturbations :

Garnier, Opt. Comm., 2002 : collective coordinate approach on the complete model

We consider :

- $d_m = d(t) = 0$
- $m(t) = \dot{\beta}(t)$ is a white noise i.e $\dot{\beta}$ is Gaussian and $\langle \dot{\beta}(t)\dot{\beta}(s) \rangle = \sigma_0 \delta(t-s)$

  natural if the correlation length of $m$ is much less than the deterministic characteristic length

## Mathematical notations :

$$idu + \partial_x^2 u \circ d\beta + |u|^2 u dt = 0, \; x \in \mathbf{R}, \; t > 0$$

- $\beta = \beta(t)$ : real valued Brownian motion
- $\partial_x^2 u \circ d\beta$ : Stratonovich product

## Global propagation of solutions :

Marty, PhD Thesis, 2005 : case $|u|^2u$ replaced by $f(|u|^2)u$, $f$ smooth and bounded ; regular solutions.

AdB, A. Debussche, JFA 2010 : Global propagation of solutions if $N(0)$ is finite

- No Hamiltonian conservation $\rightsquigarrow$ use of "Strichartz" estimates
- Representation of the solution :

$$u(t,x) = S(t,0)u_0 + i\int_0^t S(t,s)|u|^2u(s)ds$$

with $S(t,s) = e^{i(\beta(t)-\beta(s))\Delta}$ given by

$$S(t,s)\varphi(x) = \left(\frac{1}{4\pi i(\beta(t)-\beta(s))}\right)^{1/2}\int e^{i\frac{|x-y|^2}{4(\beta(t)-\beta(s))}}\varphi(y)dy$$

not a convolution in time.

## Diffusion-approximation :

Continuity of the solution w.r.t. to the Brownian paths $\rightsquigarrow$

- $m$ centered stationary process + classical ergodic assumptions
- $v^\varepsilon$ solution of

$$i\partial_t v + \varepsilon m(t)\partial_x^2 v + \varepsilon^2 |v|^2 v = 0$$

then $u^\varepsilon(t,x) = v(\frac{t}{\varepsilon^2}, x)$ converges in law as $\varepsilon$ goes to zero to the solution of

$$idu + \sigma_0 \partial_x^2 u \circ d\beta + |u|^2 u \; dt = 0$$

with the same initial state, and with

$$\sigma_0^2 = 2\int_0^{+\infty} \mathbf{E}[m(0)m(t)]dt$$

Also true with the addition of (small) residual dispersion $\varepsilon^2 d_m$.

## Remarks :

- A. Debussche, Y. Tsutsumi, JMPA 2011 :
  Global propagation (for finite $N(0)$) still true for

  $$idu + \partial_x^2 u \circ d\beta + |u|^4 u \, dt = 0$$

  contrary to the deterministic case. Make use of "local smoothing properties" of the linear equation.
- scaling argument : $\beta(t)$ Brownian Motion $\rightsquigarrow \lambda^{1/2}\beta(t/\lambda)$ is also a B.M. $\rightsquigarrow |u|^8 u$ should be the critical nonlinearity in $L^2$ (invariant by the $L^2$ scaling)
- Global propagation in the cubic case still true with $d_m \neq 0$, hence $d(t) \neq 0$.
- Long distance behavior ?

# Numerical simulations

R. Belaouar, AdB, A. Debussche : $u_j^n$ approximation of $u(n\delta t, j\delta x)$

- Splitting + spectral
- Finite differences in time and space

Crank-Nicolson :

$$\frac{i}{\delta t}(u_j^{n+1} - u_j^n) + \frac{\chi_n}{\sqrt{\delta t}}(u_{j+1}^{n+1/2} - 2u_j^{n+1/2} + u_{j-1}^{n+1/2})$$
$$= \frac{1}{2}(|u_j^n|^2 + |u_j^{n+1}|^2)u_j^{n+1/2}$$

$(\chi_n)$ : family of independent $\mathcal{N}(0,1)$

Relaxation scheme C. Besse, 1998

$$\frac{i}{\delta t}(u_j^{n+1} - u_j^n) + \frac{\chi_n}{\sqrt{\delta t}}(u_{j+1}^{n+1/2} - 2u_j^{n+1/2} + u_{j-1}^{n+1/2}) = \phi_j^{n+1/2}u_j^{n+1/2}$$
$$\frac{1}{2}(\phi_j^{n-1/2} + \phi_j^{n+1/2}) = |u_j^n|^2$$

**Semi-discrete Crank-Nicolson :**

$$i\frac{u_{n+1}-u_n}{\delta t}+\frac{\chi_n}{\sqrt{\delta t}}\Delta u_{n+1/2}+\frac{1}{2}(|u_n|^2+|u_{n+1}|^2)u_{n+1/2}=0$$

has order one in time :

$$\lim_{C\to\infty}\mathbf{P}\Big(\max_{n=0,\cdots,\frac{T}{\delta t}}|u_n-u(t_n)|_{H^1}\geq C\delta t\Big)=0$$

uniformly in $\delta t$, provided $u_0\in H^7$ ; in addition, for any $\delta<1$ there is a r.v. $K_\delta$ such that

$$\max_{n=0,\cdots,\frac{T}{\delta t}}|u_n-u(t_n)|_{H^1}\leq K_\delta(\delta t)^\delta$$

R. Marty, 2011 : Same result for Strang splitting and Lipschitz nonlinearity

## Ideas of proof :

- convergence of the scheme : compactness method
- use of a cut-off $\rightsquigarrow$ Lipschitz nonlinearity
- linear case :

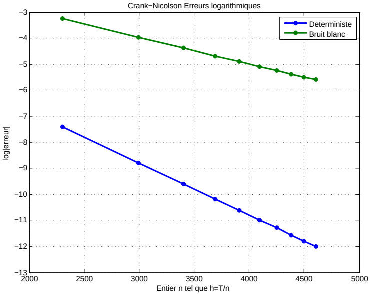
$$i\frac{v_{n+1}-v_n}{\delta t}+\frac{\chi_n}{\sqrt{\delta t}}\Delta v_{n+1/2}=0$$

$\rightsquigarrow |\hat{v}(n\delta t,\xi)-\hat{v}_n(\xi)|^2=|1-e^{iM_n(\xi)}|^2|\hat{v}_0(\xi)|^2$ with

$$M_n(\xi)=\sum_{k=0}^{n}[\sqrt{\delta t}\chi_k|\xi|^2-2\arctan(\frac{\sqrt{\delta t}}{2}\chi_k|\xi|^2)]$$

which implies (martingale inequalities)

$$\mathbf{E}\Big(\max_{n=0,\cdots,\frac{T}{\delta t}}|v_n-v(t_n)|_{L^2}^2\Big)\le C_\alpha(\delta t)^{2\alpha}\mathbf{E}(|v_0|_{H^{6\alpha}}^2)$$


Crank−Nicolson Erreurs logarithmiques
Deterministe
Bruit blanc
log|erreur|
Entier n tel que h=T/n
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2000
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3000
3500
4000
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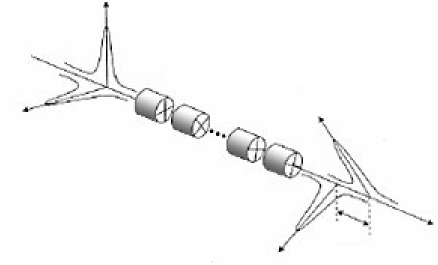
5000

# Polarization mode dispersion

A phenomenon due to birefringence : the light is a vector field and components of $\vec{E}$ travel with different group velocities

In realistic situations, birefringence parameters non uniform ; random variations in orientations on a length scale of $\sim 100m$ with zero average
Residual effects $\rightsquigarrow$ pulse spreading, called polarization mode dispersion (PMD)



## Model for random variations of birefringence :

Way-Menyuk, 1994, 1996 :

$$i\frac{\partial \mathbf{A}}{\partial z} + b\Sigma\mathbf{A} + ib'\Sigma\frac{\partial \mathbf{A}}{\partial t} + \frac{d_0}{2}\frac{\partial^2 \mathbf{A}}{\partial t^2}$$
$$+\tfrac{5}{6}\left|\mathbf{A}\right|^2\mathbf{A} + \left(\mathbf{A}^*\sigma_3\mathbf{A}\right)\sigma_3\mathbf{A} + \tfrac{1}{3}\begin{pmatrix} A_1^*A_2^2 \\ A_2^*A_1^2 \end{pmatrix} = 0,$$

$b = \Delta n$ : birefringence strength, $b' = \frac{db}{d\omega}$ and $\Sigma$ is a $2\times 2$ matrix with random coefficients depending on $z$

Assume (random) orientation angle $\theta$ such that

$$\frac{d\theta}{dz} = 2\varepsilon\alpha(z), \qquad \varepsilon^2 = \frac{\ell_c}{\ell} \ll b^2$$

$\alpha$ : real valued centered Markov process, with unique ergodic invariant measure

$\ell_c$ correlation length of $\alpha$

$+$ averaging over fast oscillations $\rightsquigarrow$ Manakov PMD equation

## Manakov PMD equation

Garnier-Marty, 2006 :

$$i\frac{\partial \mathbf{X}}{\partial t}(t,x)+\frac{d_0}{2}\frac{\partial^2 \mathbf{X}}{\partial x^2}(t,x)+\frac{8}{9}\left|\mathbf{X}\right|^2\mathbf{X} = -ib'\boldsymbol{\sigma}(t)\frac{\partial \mathbf{X}}{\partial x}-\frac{1}{6}\left(N-\langle N\rangle\right),$$

with

$$\boldsymbol{\sigma} = \begin{pmatrix} |\nu_1|^2-|\nu_2|^2 & 2\bar{\nu}_1\bar{\nu}_2 \\ 2\nu_1\nu_2 & -|\nu_1|^2+|\nu_2|^2 \end{pmatrix} = \sigma_1 m_1(t)+\sigma_2 m_2(t)+\sigma_3 m_3(t)$$

and $\nu$ is obtained as

$$d\nu = i\sqrt{\gamma_c}(\sigma_1\nu(t)\circ dW_1(t)+\sigma_2\nu(t)\circ dW_2(t))+i\gamma_s\sigma_3\nu(t)dt$$

$\sigma_i$ : Pauli matrices

$N$ : cubic term with random coefficients depending on $\nu$

$\langle N\rangle$ : same with averaged coefficients w.r.t. invariant measure of $\nu$

PMD correlation length much smaller than deterministic length $z_0$
$\rightsquigarrow$ small parameter $\varepsilon$ such that

- correlation length of order one
- fiber length of order $\varepsilon^{-2}$

Then, setting $\mathbf{X}^{\varepsilon}(t,x) = \frac{1}{\varepsilon}\mathbf{X}(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$ and $\boldsymbol{\sigma}_{\varepsilon}(t) = \boldsymbol{\sigma}(\frac{t}{\varepsilon^2})$, we get the evolution

$$i\frac{\partial \mathbf{X}^{\varepsilon}}{\partial t}(t,x) + \frac{d_0}{2}\frac{\partial^2 \mathbf{X}^{\varepsilon}}{\partial x^2}(t,x) + \frac{ib'}{\epsilon}\boldsymbol{\sigma}_{\varepsilon}(t)\frac{\partial \mathbf{X}^{\varepsilon}}{\partial x}(t,x) + F_{\nu_{\varepsilon}(t)}(\mathbf{X}^{\varepsilon}) = 0$$

with

$$F_{\nu_{\varepsilon}(t)}(\mathbf{X}^{\varepsilon}) = \frac{8}{9}|\mathbf{X}|^2\mathbf{X} + \frac{1}{6}(N^{\varepsilon}(\mathbf{X}) - \langle N(\mathbf{X})\rangle)$$

and $N^{\varepsilon}$ is the same as $N$ with $\nu(t)$ replaced by $\nu^{\varepsilon}(t) = \nu(\frac{t}{\varepsilon^2})$

## Diffusion approximation :

Garnier-Marty, Wave Motion, 2006, linear case

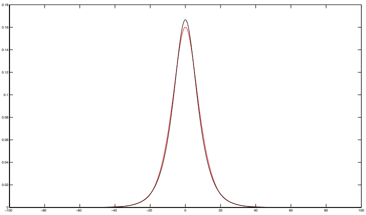
AdB, M. Gazeau, to appear in Ann. Appl. Prob., full model

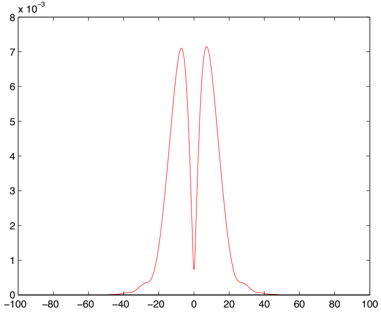
Under standard assumptions on the original ($\mathbf{C}^2$ valued) process $\nu$ defining $(m_1, m_2, m_3)$ :

- Existence of a unique global adapted square solution in $L^2$ for all positive $\varepsilon$
- Moreover, if $\mathbf{X}_0 \in H^3$, then for any stopping time $\tau$ with $\tau < \tau^*$ a.s., the process $\mathbf{X}_\tau^\varepsilon$ (stopped at $\tau$) converges to $\mathbf{X}_\tau$ in distribution in $C([0,T], H^1)$, where $\mathbf{X}_\tau$ is the unique local solution of

$$id\mathbf{X} + \left(\frac{d_0}{2}\frac{\partial^2 \mathbf{X}}{\partial x^2} + \frac{8}{9}|\mathbf{X}|^2\mathbf{X}\right) dt + i\sqrt{\gamma}\sum_{k=1}^{3} \sigma_k \partial_x \mathbf{X} \circ dW_k = 0$$

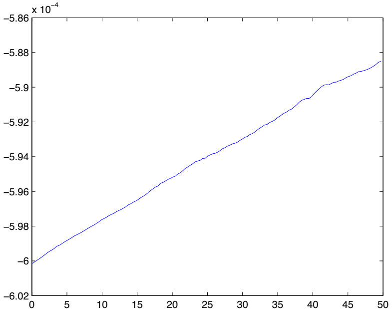
## Mean amplitude of each component

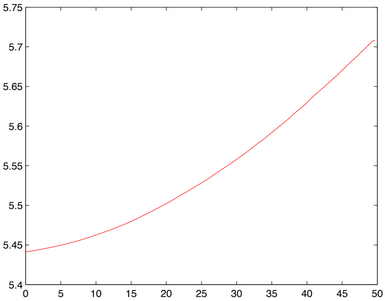




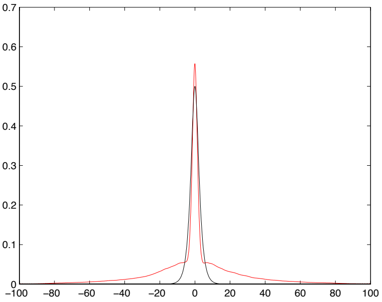

Initial state : Manakov soliton

## Evolution of the energy /pulse width (in mean)



# Same simulation with different initial state



## Evolution of the energy /pulse width (in mean)