An Exact Connection between two Solvable SDEs and a Non Linear Utility Stochastic PDEs *

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Abstract

The paper proposes a new approach, by stochastic flows, to the notion of dynamically consistent utilities also called forward dynamic utilities recently introduced by M. Musiela and T. Zariphopoulou. The market is incomplete and asset prices are modeled as Itô processes. First, we derive a stochastic PDEs that satisfies consistent stochastic utilities processes of Itô type and their dual convex conjugates. Second, under some regularities assumptions, using composite stochastic flows, we characterize all consistent utilities for a given increasing optimal wealth process and hence we establish the connection between two solvable SDEs and the utilities SPDEs. We also, express the volatility of consistent utilities as an operator of the first and the second order derivatives of the utility in terms of the optimal primal and dual policies. Finally, using stochastic change of variables techniques, we show for a given volatility vector of the utility the existence and uniqueness of a solution to the fully nonlinear second order utility SPDE from properties of the solutions of two SDEs.

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Introduction

Recently, the concept of forward dynamic utilities has been introduced by M. Musiela and T. Zariphopoulou [23, 22, 24, 26, 28], to model possible changes over the time of individual preferences of an agent, such a concept has also been studied by F. Berrier, M. Tehranchi and Rogers [11] and G. Zitkovic [39]. Other works are related to this problem as the papers of T. Choulli, C. Stricker and L. Jia [3], V. Henderson and D. Hobson [6].

The agent will adjust his preferences based on the information which will be revealed over time and will be represented by the filtration $(\mathcal{F}_t, t \geq 0)$ defined on the probability space $(\Omega, \mathbb{P}, (\mathcal{F}_t, t \geq 0))$.

In contrast to the classical literature, there is no pre-specified trading horizon at the end of which the utility datum is assigned. Rather, the agent starts with today’s specification of his utility, $U(0, x) = u(x)$, and then builds the process $U(t, x)$ for $t > 0$ in relation to the information flow given by $(\mathcal{F}_t, t \geq 0)$. It turns out that his utility choice for time $t > 0$ will be constrained by $(\mathcal{F}_t, t \geq 0)$. This, together with the choice of a normalization point, distinguishes the forward dynamic utility from the recursive utility for which the aggregator can be specified exogenously and the value function is recovered backwards in time.

Working in an incomplete market, we give the definition of a consistent progressive utility. However, in order to prove our main result, we restrict our attention to forward utilities which are Itô-semimartingales with spatial parameter $x$ satisfying the following dynamic

$$dU(t, x) = \beta(t, x)dt + \gamma(t, x)dW_t.$$

In Section 2, we begin by recalling the Itô-Ventzel formula. As in the classical Hamilton-Jacobi-Bellman framework, we proceed by verification to establish the dynamics of consistent utilities and give an explicit closed formula for the optimal policy. Assuming a sufficient drift constraint of HJB type we get the utility stochastic PDE that we investigate in this paper. From this HJB constraint on $\beta$, we study the role of risk prime of the utility $U$ defined by $\gamma_x/U_x$. Paragraph 3 go on details to the question of duality and the associated SPDE and gives a complete interpretation of the role of the volatility $\gamma$. Contrary to the backward case, the forward and the dual Hamilton-Jacobi-Bellman equations established in this work don’t give us a positive answer on the question of existence and uniqueness of solutions but show the important role of the volatility $\gamma$ of the $\mathcal{X}$-Consistent utility $U$ and the strong analogy between the primal and dual problem. The obstacles in the analysis come from the fact that the HJB equations are forward in time and therefore existing results of existence, uniqueness and regularity of weak (viscosity) solutions are not directly applicable. An additional difficulty comes from the fact that the volatility coefficient may depend on
higher order derivatives of $U$, in which case the SPDE cannot be turned into a regular PDE with random coefficients, using the method of stochastic characteristics. Moreover, the concavity property cannot be derived directly from the dynamics; that is still an open question in general, which we try to answer in Section 4. In paragraph 3.3, we show the stability of the notion of consistent utility by change of numeraire and then, without loss of generality, we can consider the market martingale where the portfolios are simple local martingales and the stochastic PDE’s are easy to deal with.

One of our main contributions is the new approach of consistent dynamic utilities using stochastic flows, which we discuss in Section 4. The idea is very simple and natural: Suppose that the optimal portfolio denoted by $X^*$ is strictly increasing with respect to the initial capital. Then, using the duality identity $U_x(t,X^*_t(x)) = Y^*_t(U_x(0,x))$, yields $U'(t,x) = Y(t,X(t,x))$ with $X$ denote the reverse flow of $X^*$ and $Y^*$ is the optimal state price density process. Finally we get $U$ by integration. We then, by stochastic flows techniques, construct all consistent utility generating $X^*$ as optimal portfolio. This is a new approach because, we construct the utility from its optimal processes. On the other hand, we establish that the volatility vector $\gamma$ of the consistent utility can be deduced from that of $X^*$ and of the optimal dual process. In fact, given optimal policies, the volatility vector $\gamma$ is interpreted as an operator $\Upsilon(x,U',U'')$ which is linear on $U_{xx}$ and the dependence on $U_x$ (resp. $x$) is identical to how volatility of flow $\mathcal{Y}$ (resp. $X^*$) depends on $\mathcal{Y}$ (resp. $X^*$).

Another main contribution of this paper is that, from the construction of utilities proposed, a connection between two solvable SDEs and the utilities SPDEs is easely established. In particular, given a volatility vector $\gamma$ such that $\gamma_x(t,x) = -xU_{xx}r^*(t,x) + \nu^*(t,U_x(t,x))$ we show the existence and uniqueness of a solution to the fully nonlinear second order SPDE from that of a pair of SDE’s. In any case this represents an interesting result in the theory of stochastic partial differential equations.

To the best of our knowledge, the stochastic partial differential equations and the Hamilton-Jacobi-Bellman equations established in this paper and satisfied by forward utilities and their dual have not been established in a general way. In [11] and [31] the authors study the case where the volatility vector of the utility is zero. In [28], the authors derive a stochastic PDE and study examples where the volatility of the utility is constant, proportional to $U$ (case of change of probability) and the case where the volatility is proportional to $xU_x$ which correspond to a change of numeraire.

Furthermore, as far as we know, there is no general consistent utilities construction proposed in the literature expect the case of power or exponential type.

The paper is organized as follows, we give the definition of consistent dynamic utilities Then, in order to study the HJB equation, we give more precisions on the market model
and we introduce the useful Itô-Ventzel formula. In Section 2, we provide the dynamic of consistent utilities and a closed form for the optimal policy. In Section 3, we study the dual process and establish a duality identity. In Section 3.3, we show the stability of the notion of consistent utility by a change of numeraire and we provide an example of consistent utility obtained by combining a standard utility function with some positive processes. In Section 4, we present our new approach and the main results of this work. We close the paper by a concrete example on how to construct consistent utilities, such method which generalizes the decreasing forward utilities studied by Berrier et al [11] and by M. Musiela et al [31].

1 Consistent Stochastic Utilities

We start by introducing the concept of a forward utility consistent with a given family of portfolios. All stochastic processes are defined on a filtered probability space \((\Omega, \mathcal{F}_{t \geq 0}, \mathbb{P})\) with filtration \((\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions. In general, \(\mathcal{F}_0\) is assumed to be trivial. In the Itô framework considered afterwards, \((\mathcal{F}_t)_{t \geq 0}\) will be usual Brownian augmented filtration.

1.1 Definition of Consistent Stochastic Utilities

A progressive utility \(U\) is a positive adapted continuous random field \(U(t, x)\), such that \(t \geq 0, x > 0 \mapsto U(t, x)\) is an increasing concave function, (in short utility function).

Obviously, this very general definition of progressive utility has to be constrained to represent, possibly changing over time, the individual preferences of an investor in a given financial market. The idea is to calibrate these utilities with regard to some convex class (in particular vector space) of wealth processes, denoted by \(\mathcal{X}\), on which utilities may have more properties.

As a utility function, a stochastic one measures the relative satisfaction of any portfolio and gives a selection criterion which allows to identify an optimal choice of investment at any time. Naturally, as the agent’s preferences are unique, we will impose below the uniqueness of the optimal process. Furthermore, at the optimal choice, the satisfaction is for its maximum and will be preserved at all futures times which explains the martingale property in the definition below. On the other hand if the strategy in \(\mathcal{X}\) fails to be optimal then it is better not to makes investment. It is necessary to understand this property as follows: the fact that an investor makes an investment, which is not optimal compared with what he would have been able to win if he had followed the optimal policy, constitutes a loss. From this, we will suppose that the utility of any strategy is a supermartingale. The optimum
represents the reference for the investor.

The class $\mathcal{X}$ is a test-class which only allows us to specify the stochastic utility. Once his utility is defined, an investor can then turn to a portfolio optimization problem on the general financial market to establish his optimal policy or to calculate indifference prices. Now we are able to define the $\mathcal{X}$-consistent stochastic utility as follows.

**Definition 1.1 ($\mathcal{X}$-consistent Utility).** A $\mathcal{X}$-consistent stochastic utility $U(t,x)$ process is a positive progressive utility, in particular $t \geq 0$, $x > 0 \mapsto U(t,x)$ is an increasing, strictly concave function with the following property:

- **Consistency with the test-class** For any admissible wealth process $X \in \mathcal{X}$, $\mathbb{E}(U(t, X_t)) < +\infty$ and 
  $$\mathbb{E}(U(t, X_t)/\mathcal{F}_s) \leq U(s, X_s), \ \forall s \leq t \ \text{a.s.}$$

- **Existence of optimal wealth** For any initial wealth $x > 0$, there exists an optimal wealth process $X^* \in \mathcal{X}$, such that $X^*_0 = x$, and $U(s, X^*_s) = \mathbb{E}(U(t, X^*_t)/\mathcal{F}_s) \ \forall s \leq t$. In short for any admissible wealth $X \in \mathcal{X}$, $U(t, X_t)$ is a positive supermartingale and a martingale for the optimal-benchmark wealth $X^*$.

Our definition of consistent dynamic utilities differs from the one introduced by Musiela and Zariphopoulou [23, 22, 24, 26, 28] or Barrier and al. [11] by the fact that we do not require that the wealth processes $X$ are discounted. This variation offers more options and allows us to study the invariance of the class of stochastic utilities by change of numéraire. In any case, there is no fixed horizon.

**Remark** A deterministic utility $u$ is a $\mathcal{X}$-consistent utility only when the test portfolios are local martingales. In this case, the optimal strategy is to do nothing.

**The Market Model** We consider a securities market which consists of $d + 1$ assets, one riskless asset, with price $S^0$ given by $dS^0_t = S^0_0r_t dt$ and $d$ risky assets. We model the price of the $d$ risky assets as a locally bounded positive semimartingale $S^i$, $i = 1, \ldots, d$ defined on the filtered probability space $(\Omega, \mathcal{F}_{t \geq 0}, \mathbb{P})$.

A (self–financing) portfolio is defined as a pair $(x, \phi)$, where the constant $x$ is the initial value of the portfolio and the column vector $\phi = (\phi^i)_{1 \leq i \leq d}$ is a predictable $S$-integrable process specifying the amount of each asset held in the portfolio. The value process $X^\phi = (X^\phi_t)_{t \geq 0}$ of such portfolio $\phi$ is given by

$$\frac{X^\phi_t}{S^0_t} = \frac{x}{S^0_0} + \int_0^t \phi^i d\langle S^i, S^0 \rangle^{\phi^i}, \ t \geq 0. \ \ (1)$$

Let us denote by $\mathbb{X}^+$ the set of positive wealth processes. To facilitate the exposition we only consider, in what follows, wealth processes in $\mathbb{X}^+$. This naturally leads us to characterize
portfolios by means of relative weights \( \pi \) in place of the amounts \( \phi \). The relation between these two notions is easy since \( \phi_t = (\pi_t^1 X_t^\phi(x), \ldots, \pi_t^d X_t^\phi(x))^T \), where the transpose operator is denoted by \( T \). The advantage of the second formulation is that the assumption of positive wealth is automatically satisfied, since the previous equation becomes with the notation \( X^\pi \) in place of \( X^\phi \),

\[
dX^\pi_t \over X^\pi_t = r_t dt + \pi_t \left( dS_t \over S_t - r_t dt \right), \quad t \geq 0
\]

where the \( d \)-dimensional vector such all components are equal to 1 is denoted by \( 1 \). Let us now recall that a probability measure \( Q \sim P \) is called an equivalent local martingale measure if, for any \( X \in \mathbb{X}^+ \), \( X \) is a local martingale under \( Q \). To ensure the absence of arbitrage opportunities, we postulate that the family of equivalent local martingale measures is not empty, (see [9], [7] for a precise statement and references). We stress that no assumption concerning completeness is made and in particular, many equivalent martingale measures may exist.

**Itô's Market:** Let \( W = (W_1, W_2, \ldots, W_n)^T \) be a \( n \)-standard Brownian motion \( (n \geq d) \), defined on the filtered probability space \((\Omega, \mathcal{F}, P)\). The filtration \((\mathcal{F}_t)_{t \geq 0}\) is the \( P \)-augmented filtration generated by the Brownian motion \( W \), i.e. \( \mathcal{F}_t = \sigma(W_s, 0 \leq s \leq t) \).

The risky asset prices are continuous Itô’s semimartingales with the dynamics:

\[
dS^i_t = b^i_t dt + \sigma^i_t dW_t, \quad 0 \leq i \leq d
\]

where the inner scalar product is denoted by “\( \cdot \)”. The coefficient \( b^i \) represents the appreciation rate by time unit of the asset \( i \) and \( \sigma^i \) its volatility vector in \( \mathbb{R}^n \), considered as a \( n \times 1 \) matrix. Denote by \( b \) the appreciation rate column vector \( n \times 1 \) \((b^i)_{i=1,\ldots,d}\), and by \( \sigma_t \) the volatility matrix \( n \times d \) (\( n \) lines \( d \) columns), whose \( i^{th} \) column is the vector \( \sigma^i_t \) for \( i = 1, \ldots, d \). The processes \( b, \sigma \) and \( r \) are \( \mathcal{F}_t \) non-anticipating processes and satisfy some minimal appropriate integrability conditions. Using vector and matrix notation, we have \( dS_t = S_t(b_t dt + \sigma^T_t dW_t) \). Moreover, equation (2) may be rewritten as, \( dX^\pi_t = X^\pi_t \left[ (r_t + \pi_t^i(b_i - r_t)1) dt + \sigma^T_t \pi_t dW_t \right] \).

As usual, the matrix \((\sigma\sigma^T)(t, \omega)\) is assumed to be **non singular**. This assumption is equivalent to suppose that, for any \( i \in 1..d \), the asset \( S^i_t \) cannot be replicated by an admissible portfolio.

The existence of an equivalent local martingale measure is equivalent, in this framework, to the fact that the excess of return vector belongs to the range of volatility matrix: in other words, there exists a \( \mathcal{F} \)-progressively measurable process \( \eta \in \mathbb{R}^n \) such that \( b_t - r_t1 = \sigma^T_t \eta_t \).

Additional integrability assumptions are necessary to ensure that the exponential martingale generated by \( \eta_t \) is the density of some probability measure.
We get that the dynamics of the portfolio becomes
\[ dX_t^\pi = X_t^\pi [r_t dt + \sigma_t \pi_t. (dW_t + \eta_t dt)]. \]

The key role is played by the volatility vector \( \sigma \pi \). For this and in order to facilitate the exposition we denote it by \( \kappa := \sigma \pi \). To fix the notation, we denote by \( \mathcal{R}_t^\sigma \subset \mathbb{R}^n \) the range of \( \sigma_t \), and by \( \mathcal{R}_t^\sigma, \perp \) the orthogonal vector subspace. By assumption, \( \kappa \) is required to lie at any time \( t \) in \( \mathcal{R}_t^\sigma \). Replacing \( X^\pi \) by \( X^\kappa \), the above equation becomes
\[ dX_t^\kappa = X_t^\kappa [r_t dt + \kappa_t. (dW_t + \eta_t dt)], \kappa_t \in \mathcal{R}_t^\sigma. \quad (4) \]

**Algebraic Notation** The following short notation will used intensively. Let \( \mathcal{R}^\sigma \) be a vector subspace of \( \mathbb{R}^n \). For any \( \alpha \in \mathbb{R}^n \), we denote by \( \alpha^\sigma \) the orthogonal projection of the vector \( \alpha \) onto \( \mathcal{R}^\sigma \) and by \( \alpha^\perp \) the orthogonal projection onto \( \mathcal{R}^\sigma, \perp \). Then, the following decomposition : \( \alpha = \alpha^\sigma + \alpha^\perp \) holds.

To close this section, let us introduce, as Musiela and Zariphopoulou ([28]), the generalized inverse of \( \sigma \) known as the Moore-Penrose inverse \( \sigma^+ \) and recall that \( \sigma^+ \) is the unique matrix \( d \times N \) satisfying the following four Penrose equations:
\[ \sigma \sigma^+ = (\sigma \sigma^+)^T, \quad \sigma^+ \sigma = (\sigma^+ \sigma)^T, \quad \sigma \sigma^+ \sigma = \sigma, \quad \sigma^+ \sigma \sigma^+ = \sigma^+. \quad (5) \]

Then \( \sigma \sigma^+ \) is the orthogonal projection matrix onto \( \mathcal{R}^\sigma \), and \( \alpha^\sigma = \sigma \sigma^+ \alpha \). Moreover, denoting by \( I_n \) the \( n \)-dimensional identity matrix, \( I_n - \sigma \sigma^+ \) is the orthogonal projection matrix onto \( \mathcal{R}^\sigma \). Moreover, under market assumptions \((\sigma^T \sigma \) non singular) there exists a unique vector \( \pi \) such that \( \kappa = \sigma \pi \) which is \( \pi = \sigma^+ \kappa \).

**Minimal Risk premium** The market incompleteness is described by the non-uniqueness of the risk premium vector \( \eta \) since for any \( \kappa \in \mathcal{R}^\sigma \), \( \kappa. \eta = \kappa. \eta^\sigma \). From this point we assume throughout this paper and without further mention that \( \eta \in \mathcal{R}^\sigma \). Recall that the use of the notation \( \eta^\sigma \) is often referred to as a minimal risk premium.

## 2 Stochastic Partial Differential Equation

In this section, under some additional regularity assumptions we will focus on the Hamilton-Jacobi-Bellman stochastic PDE satisfied by a \( \mathcal{F} \)-consistent stochastic utility using essentially Itô-Ventzel’s formula and techniques of dynamic programming established and developed in the classical theory of utility maximization (see for example H. Pham [13]). For
that purpose to be able to apply the dynamic programming principle some hypothesis are necessary. From now, \( X \)-consistent stochastic utilities \( U(t, x) \) are described as Itô’s semimartingales with spatial parameter \( x > 0 \), that is \( U(t, x) \) is a continuous random field with dynamics,

\[
dU(t, x) = \beta(t, x)dt + \gamma(t, x).dW_t,
\]

where, as in Kunita [21], the pair \((\beta, \gamma)\) is called the local characteristics of \( U \) and are assumed to be progressively random fields with values in \( \mathbb{R} \) and \( \mathbb{R}^n \) respectively.

We are concerned with the properties of the utility of admissible wealth processes. Before that, we want to give precise definition of the progressive utility, its derivatives and their dynamic properties.

### 2.1 Regular stochastic flows and Itô-Ventzel’s formula

**Regular Stochastic flows** There are several difficulties in the definition of semimartingales depending on a parameter, as explained in H. Kunita [21] and R.A. Carmona et al. [2], (see Appendix A).

First let us point out that in general equality (6) holds for any \( t \) except for a null set \( N_x \). Then the semimartingale \( U \) is well defined for \((t, x)\) if \( \omega \in (\cup_{x \in \mathbb{R}_+} N_x)^c \). However the exceptional set \( \cup_{x \in \mathbb{R}_+} N_x \) may not be a null set since it is an uncountable union of null sets.

However if we suppose that local characteristic \((\beta, \gamma)\) of \( U \) are \( \delta \)-Hölder, for some \( \delta > 0 \) (see appendix A), then according to H. Kunita [21] (Theorems 3.1.2 p.75) using Kolmogorov’s criterion, \( U(t, x) \) has a continuous modification for which (6) holds almost surely.

A detailed discussion about these difficulties and their consequences in terms of dynamic representation and differential rules are provided in H. Kunita [21] and R.A. Carmona et al. [2]. The main results are also recalled in Appendix A. Here we only give a self-contained definition of the regularity in the sense of Kunita [21]. In particular, albeit the process \( U \) and its local characteristics \((\beta, \gamma)\) are differentiable it is not enough as is showed in H. Kunita [21], to get that the dynamics of the derivative \( \frac{\partial}{\partial x} U(t, x) \) is the derivative term by term of that of \( U \).

Let \( m \) be a non-negative integer, \( \beta \) be a real function in \( C^m([0, +\infty[ \times [0, +\infty[) \) and \( \gamma \) be a \( C^m([0, +\infty[ \times [0, +\infty[) \) vector. We define the following seminorms for any compact \( K \),

\[
\|\beta\|_{m;K}(t) = \sup_{x \in K} \frac{|\beta(t, x)|}{1 + |x|} + \sum_{1 \leq \alpha \leq m} \sup_{x \in K} |\partial^\alpha_x \beta(t, x)|.
\]

\[
\|\gamma\|_{m;K}(t) = \sup_{x,y \in K} \frac{|\gamma^T(t, x) \cdot \gamma(t, y)|}{1 + |x|(1 + |y|)} + \sum_{1 \leq \alpha_1, \alpha_2 \leq m} \sup_{x,y \in K} |\partial^\alpha_x \partial^{\alpha_2}_y \gamma^T(t, x) \cdot \gamma(t, y)|.
\]
For simplicity if a random field \((G(t, x))_{t \geq 0, x \geq 0}\) is of class \(C^{0,2}([0, +\infty] \times [0, +\infty])\) we use the notation \(G_x\) for \(\frac{\partial}{\partial x} G\) and \(G_{xx}\) for \(\frac{\partial^2}{\partial x^2} G\).

**Definition 2.1.** Let \(m \geq 2\). A random field \(F\) is said to be \(C^{(m)}\) regular in the sense of Kunita if \(F : [0, +\infty] \times [0, +\infty] \rightarrow \mathbb{R}\) is of class \(C^{0,m}([0, +\infty] \times [0, +\infty])\) and satisfies
\[
F(t, x) = F(0, x) + \int_0^t \beta(s, x)ds + \int_0^t \gamma(s, x).dW_s.
\] (7)

In the sequel, according to H. Kunita [21] the pair of adapted random fields \((\beta, \gamma)\) is called the local characteristics of \(F\) and satisfy \(\beta : [0, +\infty] \times [0, +\infty] \rightarrow \mathbb{R}\) and the \(N\)-dimensional vector \(\gamma : [0, +\infty] \times [0, +\infty] \rightarrow \mathbb{R}^N\) are \(\mathcal{F}\)-adapted random field of class \(C^{0,m-1}([0, +\infty] \times [0, +\infty])\) such that \(||\beta||_{m-1,K}(t)\) and \(||\gamma||_{m-1,K}(t)\) are integrable with respect to \(t\), for any compact \(K \subset [0, +\infty]\).

Now we turn to the differential rules of semimartingales with spatial parameter. For this some other notations are needed. Let \(0 < \delta \leq 1\) and \(K\) a compact of \(\mathbb{R}_+\). For some random fields \(f(t, x)\) and \(g(t, x, y)\) we set
\[
||f||_{\delta : K} := \sup_{x,y \in K, x \neq y} \frac{|\partial_x^\delta f(x) - \partial_x^\delta f(y)|}{|x-y|^\delta}, \quad ||g||_{\delta : K} := \sup_{x,x',y,y' \in K, x \neq x', y \neq y'} \frac{|g(x, y) - g(x', y) - g(x, y') + g(x', y')|}{|x-x'|^\delta|y-y'|^\delta}.
\]

Using these notations and according to the results of H. Kunita [21], (Theorem 3.3.3 p.95, recalled in Appendix A), we have the following result.

**Theorem 2.1 (Differential Rules).** Let \(F\) be a random field of class \(C^{0,1}([0, +\infty] \times [0, +\infty])\) such that its local characteristics \((\beta, \gamma)\) are of class \(C^{0,1}([0, +\infty] \times [0, +\infty])\). Assume that the derivative \(\beta_x\) and \(\gamma_x\) are \(\delta\)-Hölder, with \(0 < \delta \leq 1\) such that for any compact \(K\) of \(\mathbb{R}_+\), \(||\beta||_{\delta : K}(t)\) and \(||\gamma||_{\delta : K}(t)\) are integrable with respect to \(t\), with \(a^\gamma(t, x, y) := \gamma(t, x)^T \gamma(t, y)\).

Then the derivative \(F_x\) of \(F\) with respect to the spatial parameter \(x\) satisfies, almost surely,
\[
F_x(t, x) = F_x(0, x) + \int_0^t \beta_x(s, x)ds + \int_0^t \gamma_x(s, x).dW_s\] (8)

Furthermore, if \(F\) is of class \(C^{(m)}\), \(m \geq 3\) then \(F_x\) is of class \(C^{(m-1)}\) with local characteristics \((\beta_x, \gamma_x)\) which are of class \(C^{0,m-2}([0, +\infty] \times [0, +\infty])\).

**Itô-Ventzel’s formula** Now, we need to study the dynamics of \((U(t, X^x_\cdot))\) \((X^x\) is a wealth process). The Itô-Ventzel’s formula is a generalization of classical Itô’s formula where the deterministic function is replaced by a stochastic process depending on a real or multivariate parameter. This enables us to carry out computations in a stochastically modulated dynamic framework.
Theorem 2.2 (Itô-Ventzel Formula). Consider a random field \( F : [0, +\infty[ \times [0, +\infty[ \to \mathbb{R} \) which is of class \( C^2 \), in the sense of Kunita,
\[
F(t, x) = F(0, x) + \int_0^t \beta(s, x) ds + \int_0^t \gamma(s, x) dW_s, \text{ a.s.} \tag{9}
\]
Furthermore, let \( X_t \) be a continuous semimartingale with decomposition
\[
X_t = X_0 + \int_0^t \mu_s^X ds + \int_0^t \sigma_s^X dW_s.
\]
Then \( F(t, X_t) \) is also a continuous semimartingale with decomposition
\[
F(t, X_t) = F(0, X_0) + \int_0^t \beta(s, X_s) ds + \int_0^t \gamma(s, X_s) dW_s + \int_0^t F_x(s, X_s) dX_s + \frac{1}{2} \int_0^t F_{xx}(s, X_s) d\langle dX_s \rangle
\]
Let us now comment the dynamics of \( F(t, X_t) \). The first line of the right hand side of this dynamic corresponds to the dynamic of the process \((F(t, x))_{t \geq 0}\) taken on \((X_t)_{t \geq 0}\) where the second one is none other than the classical Itô formula and the last one represents a term correction in particular we can write that \( \gamma_x(s, X_s) \sigma_s^X = \langle dF_x(s, x), dX_s \rangle|_{x = X_s} \). Finally we note that if \( F(t, x) \) is a function on the time \( t \), it follows that \( \gamma = 0 \) a.s. and \( \frac{\partial}{\partial t} F(t, x) = \beta(t, x) \). Injecting this identities we get the classical Itô formula.

We refer to H. Kunita [21], Theorem 3.3.1, p.92 for more details and the proof of this result.

We illustrate this result from the classical Itô’s formula.

Example: Itô’s Formula Let \( f(t, y, x) \) be a deterministic function \( \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) of class \( C^{1,2,2} \). Denote by \( \nabla_y \) the gradient with respect to \( y \) and by \( \Delta_{yy} \) the Hessian matrix with respect to \( y \) where \( t \) and \( x \) are fixed.

Let \( Y \in \mathbb{R}^n \) be a Itô’s semimartingale \( dY_t = \mu_t^Y dt + \sigma_t^Y dW_t \), with the diffusion generator
\[
L_t^Y = \mu_t^Y \nabla_y + \frac{1}{2} \text{trace}[\sigma_t^Y (\sigma_t^Y)^T \Delta_{yy}].
\]
Denote by \( F \) the stochastic random field \( F(t, x) \equiv f(t, x, Y_t) \). By the classical Itô’s formula
\[
dF(t, x) = f_t(t, x, Y_t) dt + \sigma_t^Y f(t, x, Y_t) dW_t + (\nabla_y f(t, x, Y_t), \sigma_t^Y) dW_t
\]
such that \( F \) is a stochastic random field with local characteristics \( \beta^F \) and \( \gamma^F \) given by
\[
\beta^F(t, x) = L_t^Y f(t, x, Y_t), \quad \gamma^F(t, x) = (\sigma_t^Y)^T \nabla_y f(t, x, Y_t).
\]
Let now $X$ be another real continuous semimartingale $dX_t = \mu_t^X dt + \sigma_t^X dW_t$ and $L^X$ its diffusion operator. We now compute the dynamics of $F(t, X_t) := f(t, Y_t, X_t)$ by the classical Itô’s formula applied to the vector $(Y_t, X_t)$ and compare it with the Itô-Ventzel’s formula. We obtain

$$dF(t, X_t) = L_t^{XY} f(t, X_t, Y_t) dt + \left((\sigma_t^Y)^T \nabla_y f(t, X_t, Y_t) + (\sigma_t^X)^T \nabla_x f(t, X_t, Y_t)\right) dW_t$$

$$= \beta(t, X_t) dt + \gamma(t, X_t). dW_t + \gamma_x(t, X_t). \sigma_t^X dt + L_t^X f(t, x) dt.$$

Denoting $\Delta_{y,x} f := \frac{\partial}{\partial x} (\nabla_y f)$, we have that $L_t^{XY} := \frac{\partial}{\partial t} + L_t^Y(t, x) + L_t^X(t, x) + \sigma_t^X \Delta_{y,x}(t, x)$, and then

$$\gamma_x(t, X_t). \sigma_t^X = \sigma_t^X \gamma_y(t, x).$$

### 2.2 Stochastic PDE of $\mathcal{R}$-consistent Dynamic Utilities

Using the same ideas as in interest rate modeling when studying the dynamics of the forward rates, or in the stochastic volatility models to characterize the drift of the stochastic implied volatility we show how the consistency property constraints the random fields $\beta(t, x)$ and $\gamma(t, x)$ in terms of the random field $U$, its derivatives and the market parameters $(r_t, \eta_t)$. 

**Lemma 2.3 (Drift Constraint).** Let $U$ be a progressive utility of class $C^{(2)}$ in the sense of Kunita with local characteristics $(\beta, \gamma)$ see (6). Then for any admissible portfolio $X^\kappa$,

$$dU(t, X_t^\kappa) = \left(U_x(t, X_t^\kappa) X_t^\kappa \beta(t, X_t^\kappa) + \gamma(t, X_t^\kappa)\right) dW_t$$

$$+ \left(\beta(t, X_t^\kappa) + U_x(t, X_t^\kappa) r_t X_t^\kappa + \frac{4}{2} U_{xx}(t, X_t^\kappa) Q(t, X_t^\kappa, \kappa_t)\right) dt,$$

where $Q(t, X_t^\kappa, \kappa_t) := \|\kappa_t\|^2 + 2 \kappa_t \left(\frac{U_x(t, x) \eta_t^\kappa + \gamma_x(t, x)}{x U_{xx}(t, x)}\right)$.

Let $\gamma^\kappa_x$ be the orthogonal projection of $\gamma_x$ on $\mathcal{R}^\kappa$; the previous expression is only depending on $\gamma^\kappa_x$ since $\kappa \in \mathcal{R}^\kappa$. Let $Q^*(t, x) = \inf_{\kappa \in \mathcal{R}^\kappa} Q(t, x, \kappa)$; the minimum of this quadratic form is achieved at the optimal policy $\kappa^*$ given by

$$\left\{\begin{array}{l}
\kappa_t^*(x) = - \frac{1}{U_{xx}(t, x)} (U_x(t, x) \eta_t^\kappa + \gamma_x^\kappa(t, x)) \\
x^2 Q^*(t, x) = - \frac{1}{U_{xx}(t, x)} ||U_x(t, x) \eta_t^\kappa + \gamma_x^\kappa(t, x)||^2 = - ||x \kappa_t^*(x)||^2.
\end{array}\right. \quad (10)$$

**Proof.** (i) The first assertion of the lemma is a direct implication of applied Itô-Ventzel formula to the composite process $(U(t, X_t^\kappa))_{t \geq 0}$ where $X_t^\kappa$ is an admissible wealth process with dynamics given by (4), $dX_t^\kappa = X_t^\kappa (r_t dt + \kappa_t (dW_t + \eta_t^\kappa dt))$.

(ii) Let us now check the second assertion of the lemma. In the minimization program, we can replace the vector $\gamma_x(t, x)$ by its orthogonal projection $\gamma^\kappa_x(t, x)$ on $\mathcal{R}^\kappa$ that yields to equation (10). Moreover, the minimum is given by $Q^*(t, x) = - ||\kappa_t^*(x)||^2$. \qed
This lemma suggests the constraint on the drift $\beta$ implying the consistency condition. The idea of the next theorem is to reformulate this constraint as a natural candidate for $\beta$.

**Theorem 2.4 (HJB-SPDE).** Let $U$ be a progressive utility of class $C^{(2)}$ in the sense of Kunita with local characteristics $(\beta, \gamma)$. As usual the risk tolerance coefficient of $U$ is $\alpha_U(t,x) = \frac{U_{xx}(t,x)}{U_x(t,x)}$.

Given the volatility vector $\gamma$ we define the utility risk premium $\eta^U(t,x) = \gamma_{\xi}(t,x)$, whose the orthogonal decomposition is

$$\eta^U(t,x) = \eta^{U,\sigma}(t,x) + \eta^{U,\perp}(t,x) \quad \text{with} \quad \eta^{U,\sigma}(t,x) \in \mathbb{R}^{\sigma}_t, \quad \eta^{U,\perp}(t,x) \in \mathbb{R}^{\sigma,\perp}_t.$$

With these notations, the quadratic form $x^2 Q(t,x,\kappa) = \|x\kappa_t\|^2 + 2\alpha_U(t,x)(x\kappa_t)(\eta^\sigma_t + \eta^{U,\sigma}(t,x))$ achieves it minimum at

$$x\kappa^*_t(x) = -\frac{1}{U_{xx}(t,x)}(U_x(t,x)\eta^\sigma_t + \gamma_{\xi}(t,x)) = -\frac{1}{\alpha_U(t,x)}(\eta^\sigma_t + \eta^{U,\sigma}(t,x))$$

a) Assume the drift constraint to be Hamilton-Jacobi-Bellman nonlinear type

$$\beta(t,x) = -U_x(t,x)r_t + \frac{1}{2}U_{xx}(t,x)\|x\kappa^*_t(t,x)\|^2 \tag{11}$$

Then the progressive utility is solution of the following HJB-SPDE

$$dU(t,x) = U_x(t,x)\left[-x r_t + \frac{1}{2} \alpha_U(t,x)(\|\eta^\sigma_t + \eta^{U,\sigma}(t,x)\|^2)\right] dt + \gamma(t,x) dW_t,$$

and for any admissible wealth $X^*_t$, the process $U(t,X^*_t)$ is a supermartingale.

b) Furthermore, if we assume that $\kappa^*_t(x) is sufficiently smooth so that for any initial wealth $x > 0$ the equation

$$dX^*_t = X^*_t\left[r_t dt + \kappa^*_t(X^*_t).dW_t + \eta^\sigma_t dt\right] \tag{12}$$

has at least one positive solution $X^*$, then $U(t,X^*_t)$ is a local martingale.

C) Moreover, if the local martingale $(U(t,X^*_t))_{t \geq 0}$ is a martingale then the progressive utility $U$ is a $\mathcal{F}$-consistent stochastic utility with optimal wealth process $X^*$.

This theorem proves that the pair consisting on the investment universe and the derivative with respect to $x$ of the volatility denoted by $\gamma_x$ describes completely the evolution of the stochastic utility $U$. The drift $\beta(t,x)$ may be interpreted as the best compromise between the investment universe and volatility of the utility represented by the random field $\gamma$. Hence $\beta(t,x)$ can also be interpreted as the best combination between the market risk premium $\eta^\sigma$ and utility risk premium represented by $\eta^{U,\sigma}(t,x) = \frac{\gamma_{\xi}}{U_x}$.

The assumption (11) on the drift $\beta$ is a sufficient condition under which the consistence with the investment universe of the second assertion of Definition 1.1 is satisfied. Nevertheless,
additional assumptions are needed on the existence of the wealth process \( X^* \) for which \( U(t,X^*_t) \) is a martingale. This explains the assumptions of the second part of the result.

The utility HJB-SPDE poses several challenges. It is fully nonlinear and not elliptic, the latter is a direct consequence of the "forward in time" nature of the involved stochastic optimization problem, there is no maximum principle. Thus, existing results of existence, uniqueness and regularity of weak (viscosity) solutions are not directly applicable. An additional difficulty comes from the fact that the volatility coefficient may depend on higher order derivatives of \( U \), in which case the SPDE cannot be turned into a regular PDE with random coefficients, using the method of stochastic characteristics. To overcome this difficulties we propose a new method based on stochastic change of variable; this method that we call "stochastic flow method" allows us to construct explicit solutions of this HJB-SPDE. This will be the subject of Section 4.

Proof. All assertions are simple consequences of the previous lemma, since by the assumption made on the drift \( \beta(t,x) \), \( \beta(t,x) + xU_x(t,x) r_t + \frac{x^2U_{xx}(t,x)}{2} Q(t,x,\kappa) \leq 0 \), a.s. \( \forall \kappa \in \mathbb{R}^\sigma \), with equality for \( \kappa^*(t,x) \). Therefore, \( U(t,X^*_t) \) is a positive supermartingale for any admissible strategy, moreover if equation (13) has a solution \( X^* \), then the process \( (U(t,X^*_t))_{t \geq 0} \) is a local martingale.

The additional assumption that \( U(t,X^*_t) \) is a true martingale yields the characterization of the \( U(t,x) \) as \( \mathcal{X} \)-consistent utility.

As in the classical theory of optimal choice of portfolio in expected utility framework the process \( U_x(t,X^*_t) \) has nice properties and a central place in the study of the dual problem that we will introduce in the next section.

Proposition 2.5. Let \( U \) be a progressive utility of class \( C^{(3)} \) in the sense of Kunita, with local characteristics \( (\beta, \gamma) \). Assume that all assumptions of Theorem 2.4 hold true, in particular that \( X^* \) is solution of

\[
    dX^*_t = X^*_t \left[ r_t dt + \kappa^*_t (X^*_t) \right].(dW_t + \eta_t^\sigma dt)
\]

Let \( L^* \) be the diffusion operator associated with \( X^* \), \( L^*_{t,x} = \frac{1}{2} \| x \kappa^*_t (x) \|^2 \frac{\partial^2}{\partial x^2} + \{ r_t + (x \kappa^*_t (x)) \eta_t^\sigma \} \frac{\partial}{\partial x} \).

i) Then, \( U_x \) is of class \( C^{(2)} \) in the sense of Kunita with local characteristics \( (\beta_x, \gamma_x) \) and satisfy

\[
    \left\{ \begin{array}{l}
    \gamma_x(t,x) + U_{xx}(t,x)(x \kappa^*_t (x)) = -U_x(t,x) \eta_t^\sigma + \gamma^*_x(t,x) \\
    \beta_x(t,x) = -U_x(t,x) r_t - L^*_{t,x} U_x(t,x) - (x \kappa^*_t (x)) \gamma^*_{x,x}(t,x)
    \end{array} \right.
\]

ii) The semimartingale \( U_x(t,X^*_t) \) has the following decomposition

\[
    dU_x(t,X^*_t) = U_x(t,X^*_t) \left[ - r_t dt + \left( \eta_t^{U,1} (t,X^*_t) - \eta_t^\sigma \right) \right].dW_t
\]
In particular, for any admissible wealth process \( X^\kappa \) (\( \kappa \in \mathcal{R}^\sigma \)), \( (X_t^\kappa U_x(t, X_t^\kappa)) \) is a local martingale and a martingale if \( X^\kappa = X^* \).

This result shows that \( U_x(t, X_t^\kappa) \) plays the role of a state price density process.

**Proof.** Theorem 2.1 shows that \( U_x \) is of class \( C^{(2)} \) in the sense of Kunita, with local characteristics \((\beta_x, \gamma_x)\). On the other hands, by Theorem 2.4, we have the identities \( \beta(x, t) = -xU_x(t, x) r_t + \frac{1}{2} x^2 U_{xx}(t, x) ||\kappa^*_t(x)||^2 \) and \( U_{xx}(t, x)(x\kappa^*_t(x)) = -(U_x(t, x)\eta^*_t + \gamma^*_t(t, x)). \)

The second identity is useful to calculate \( \frac{1}{2} U_{xx}(t, x)\partial_x(||\kappa^*_t(x)||^2) = U_{xx}(t, x)((x\kappa^*_t(x))\partial_x(x\kappa^*_t(x))). \)

Taking the derivative with respect to \( x \) in this second identity, its follows that

\[
U_{xxx}(t, x)(x\kappa^*_t(x)) + U_{xx}(t, x)\partial_x(x\kappa^*_t(x)) = -(U_{xx}(t, x)\eta^*_t + \gamma^*_t(t, x)).
\]

In fact we are interested in the inner product with the vector \( x\kappa^*_t(x) \) that yields to the following equality written in an appropriate form

\[
\frac{1}{2} U_{xxx}(t, x)||\kappa^*_t(x)||^2 + U_{xx}(t, x)((x\kappa^*_t(x))\partial_x(x\kappa^*_t(x))) = -(\frac{1}{2} U_{xxx}(t, x)||\kappa^*_t(x)||^2 + U_{xx}(t, x)\eta^*_t (x\kappa^*_t(x))) - \gamma^*_t(t, x). (x\kappa^*_t(x)).
\]

It is easy to recognize the first line as the derivative of \( \frac{1}{2} x^2 U_{xx}(t, x) ||\kappa^*_t(x)||^2 \) and the second line as related to the diffusion operator \( L^*_t x. \). In this form the relation \( \beta_x(t, x) = -U_x(t, x)r_t - L^*U_x(t, x) - (x\kappa^*_t(x))\gamma_{xx}(t, x) \) is easy to establish.

Then we have all the elements to calculate the dynamics of \( U_x(t, X_t^\kappa) \) using Itô-Ventzel formula

\[
dU_x(t, X_t^\kappa) = (\gamma_x(t, X_t^\kappa) + U_{xx}(t, X_t^\kappa)X_t^\kappa \kappa^*_t(X_t^\kappa)).dW_t
+ \{ \beta_x(t, X_t^\kappa) + L^*U_x(t, X_t^\kappa) + \gamma_{xx}(t, X_t^\kappa). (X_t^\kappa \kappa^*_t(X_t^\kappa)) \} dt.
\]

Note that in the last inner product, we can replace \( \gamma_{xx}(t, X_t^\kappa) \) by its orthogonal projection \( \gamma^*_t(t, X_t^\kappa) \) on the space \( \mathcal{R}^\sigma \). Thanks to the previous calculation, the expression in the brackets of the second line is exactly \( -r_tU_x(t, X_t^\kappa) \). The diffusion coefficient may also be simplified to \( -U_x(t, X_t^\kappa)\eta^*_t + \gamma^*_t(t, X_t^\kappa) \).

So that, we obtained the remarkably simple dynamics of \( U_x(t, X_t^\kappa) \)

\[
dU_x(t, X_t^\kappa) = U_x(t, X_t^\kappa)[ -r_tdt + (\eta^*_t(t, X_t^\kappa) - \eta^*_t).dW_t].
\]

Given an admissible wealth process, \( dX_t^\kappa = X_t^\kappa(r_t dt + (\kappa_t(dW_t + \eta^*_t dt))) \), standard Itô’s calculus provides an explicit form for the dynamics of \( Y_t^\kappa = X_t^\kappa U_x(t, X_t^\kappa) \) as

\[
\frac{dY_t^\kappa}{Y_t^\kappa} = \frac{dX_t^\kappa}{X_t^\kappa} + \frac{dU_x(t, X_t^\kappa)}{U_x(t, X_t^\kappa)} + <\frac{dX_t^\kappa}{X_t^\kappa}, dU_x(t, X_t^\kappa) >
= [\kappa_t - \eta^*_t + \eta^*_t(t, X_t^\kappa)].dW_t.
\]

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which implies that $X_t^\kappa U_x(t, X_t^\kappa)$ is a local martingale for any $\kappa \in \mathbb{R}$. In particular the diffusion coefficient $\alpha_t^* = X_t^\kappa U_x(t, X_t^\kappa)(\kappa^*(t, X_t^\kappa) - \eta_t^\kappa + \eta_t^\perp U_x(t, X_t^\kappa))$ of $Y_t^\kappa := Y_t^\perp = X_t^\kappa U_x(t, X_t^\kappa) := Y_t^\perp U_x(t, X_t^\kappa)$ provides us information on the derivative of the volatility vector $\gamma_x$ since $\alpha_t^* = X_t^\kappa \left[ \gamma_x(t, X_t^\kappa) + U_{xx}(t, X_t^\kappa) X_t^\kappa \kappa_t^*(X_t^\kappa) \right]$.

Note also that $Y_t^*$ is a positive local martingale with volatility process $\alpha^*$. Using the concavity of the utility $U$, one can easily see that $Y_t^* = X_t^* U_x(t, X_t^*) \leq U(t, X_t^*)$; then since $Y^*$ is a positive local martingale dominated by the martingale $U(., X^*)$, $Y^*$ is also a martingale. This conclude the proof.

\section{Duality}

After having introduced the consistent stochastic progressive utilities and establishes the associated SPDE, several questions remain open at this stage. Indeed, we have shown that the volatility $\gamma$ of these utilities plays a fundamental role since it, completely, describes the stochastic dynamics of utilities and the optimal policy. In particular, the projection of the derivative of the volatility $\gamma_x$ is proportional to the optimal policy. It now remains to give an interpretation of the orthogonal part $\gamma_x^\perp$. Moreover, the concavity of $U$ leads, naturally, to introduce the conjugate $\tilde{U}$ (also called the Legendre-Fenchel transform). We show that these conjugate is solution of a dual HJB-SPDE with optimal policy equal to $\eta_t^\perp = \gamma_t^\perp / U_x$.

In the classical theory of concave functions $f$ and its conjugate $\tilde{f}(y)$, the monotone functions $f_x$ and $-\tilde{f_y}$ are inverse of each other, $-\tilde{f_y}(y) = f_x^{-1}(y)$, in the stochastic framework, monotone functions are replaced by stochastic monotone flows and there inverse flows whose dynamics are given by the Itô-Ventzel formula. For simplicity, we present these results separately and in a appropriate form.

\subsection{Local characteristics of inverse flows}

Let $\phi$ and $\psi$ be two one-dimensional stochastic flows, with dynamics

$$
\begin{align*}
    d\phi(t, x) &= \mu^\phi(t, x) dt + \sigma^\phi(t, x) dW_t, \\
    d\psi(t, x) &= \mu^\psi(t, x) dt + \sigma^\psi(t, x) dW_t.
\end{align*}
$$

From Itô-Ventzel’s formula, under regularity assumption, the compound random field $\phi \circ \psi(t, x) = \phi(t, \psi(t, x))$ has a semimartingale decomposition whose characteristics are given explicitly from those of $\phi$ and $\psi$ and their derivatives.

\textbf{Theorem 3.1.} Suppose that $\phi$ is a random field, regular in the sense of Kunita, and $\psi(t, x)$ is a continuous semimartingale. Then the random field $\phi \circ \psi(t, x) = \phi(t, \psi(t, x))$ is a continuous
semimartingale with decomposition
\[
d(\phi \psi)(t,x) = \mu^\phi(t,\psi(t,x)) dt + \sigma^\phi(t,\psi(t,x)).dW_t \\
+ \phi_x(t,\psi(t,x))d\psi(t,x) + \frac{1}{2}\phi_{xx}(t,\psi(t,x))||\sigma^\psi(t,x)||^2 dt \\
+ \sigma_x^\phi(t,\psi(t,x)).\sigma^\psi(t,x)dt.
\]

(14)

The volatility of the compound process \(\phi \circ \psi\) is given by
\[
\sigma^{\phi \circ \psi}(t,x) = \sigma^\phi(t,\psi(t,x)) + \phi_x(t,\psi(t,x))\sigma^\psi(t,x).
\]

The next proposition gives the decomposition of the inverse of a strictly monotone stochastic flow. This proposition will be used several times throughout this paper.

**Proposition 3.2** (Inverse flow dynamics). Let \(\phi\) be a strictly monotone flow, regular in the sense of Kunita, with characteristics \((\mu^\phi(t,x),\sigma^\phi(t,x))\). The inverse process of \(\phi\) is denoted \(\xi, \xi(t,y) = \phi(t,.)^{-1}(y)\).

i) The inverse flow \(\xi(t,y)\) has a dynamics in the old variables
\[
d\xi(t,y) = -\xi_y(t,y)(\mu^\phi(t,\xi)dt + \sigma^\phi(t,\xi).dW_t) + \frac{1}{2}\partial_y(\xi_y(t,y)||\sigma^\phi(t,\xi)||^2)dt
\]

ii) In new variables, with \(\sigma^\xi(t,y) = -\xi_y(t,y)\sigma^\phi(t,\xi(t,y))\)
\[
d\xi(t,y) = \sigma^\xi(t,y).dW_t + \left(\frac{1}{2}\partial_y(\sigma^\xi(t,y)||^2) - \mu^\phi(t,\xi(t,y))\xi_y(t,y)\right)dt
\]

It is interesting to observe that the local characteristics of the inverse flow \(\xi\) can be easily interpreted as some derivatives. This point will play a crucial role in the sequel. The mathematical formulation of this remark is given in the following results, where the assumptions of Proposition 3.2 hold.

**Corollary 3.3.** Let \((\Phi(t,x), M^\phi(t,x), \Sigma^\phi(t,x))\) be the primitives, null at \(x = 0\), of \(\phi(t,x)\), \((\mu^\phi(t,x),\sigma^\phi(t,x))\) respectively. Then, the \(\Phi(t,x)\) dynamics is
\[
d\Phi(t,x) = M^\phi(t,x)dt + \Sigma^\phi(t,x).dW_t.
\]

Let \(\Xi\) denotes the primitive of \(-\xi(t,y)\) that is \(\xi(t,y) = -\Xi_y(t,y)\) with \(\Xi(t,0) \equiv 0\), the random field \(\Xi\) dynamics is,

- In old variables
  \[
d\Xi(t,y) = \Sigma^\phi(t,\xi(t,y)).dW_t + M^\phi(t,\xi(t,y))dt + \frac{1}{2}\Xi_{yy}(t,y)||\Sigma^\phi(t,\xi(t,y))||^2dt
  \]

- In new variables, denoting by \(M^\xi(t,y) = M^\phi(t,-\Xi_y(t,y))\) and \(\Sigma^\xi(t,y) = -\Sigma^\phi(t,-\Xi_y(t,y))\),
  \[
d\Xi(t,y) = \Sigma^\xi(t,y).dW_t + \left(M^\xi(t,y) + \frac{1}{2}\frac{||\Sigma^\xi(t,y)||^2}{\Xi_{yy}(t,y)}\right)dt.
\]
Proof. of the proposition and the corollary. The proof is essentially based on the generalized Itô’s formula established in the Appendix. For simplicity, we denote by \((\mu^\xi, \sigma^\xi)\) the local characteristic of \(\xi\) assumed to be regular. By Itô-Ventzel’s formula, we have

\[
\begin{align*}
\frac{d\phi}{d(t,\xi(t,y))} & = 0 \\
& = \mu^\phi(t,\xi(t,y))dt + \sigma^\phi(t,\xi(t,y))dW_t + \phi_{xx}(t,\xi(t,y))d\xi(t,y) \\
& + \frac{1}{2}\phi_{xx}(t,\xi(t,y))<d\xi(t,y) + \sigma^\phi(t,\xi(t,y))\sigma^\xi(t,y)dt
\end{align*}
\]

Recalling the following identities

\[
\phi_x(t,\xi(t,y)) = \frac{1}{\xi_y(t,y)}, \quad \phi_{xx}(t,\xi(t,y)) = -\frac{\xi_y(t,y)}{(\xi_y(t,y))^3},
\]

we can express the parameters of the decomposition in terms of \(\xi, \xi_y, \) and \(\xi_{yy}\) and the diffusion coefficient \(\sigma^\xi(t,y)\) of \(\xi\), since \(\sigma^\xi(t,y) = -\xi_y(t,y)\sigma^\phi(t,\xi(t,y))\).

It is immediate that

\[
\mu^\xi(t,y) = -\xi_y(t,y)\mu^\phi(t,\xi(t,y)) + \xi_y(t,y) <\sigma^\phi(t,\xi(t,y)), \sigma_y^\phi(t,\xi(t,y))> + \frac{1}{2}\xi_{yy}(t,y)||\sigma^\phi(t,\xi(t,y))||^2
\]

In terms of the stochastic random fields \(\mu^\phi\) and \(\sigma^\phi\), this may be written as

\[
\mu^\xi(t,y) = -\xi_y(t,y)\mu^\phi(t,\xi(t,y)) + \frac{1}{2}\partial_y[||\sigma^\phi(t,\xi(t,y))||^2].
\]

In terms of their own parameters, it follows from the strict monotonicity of \(\xi\) that

\[
\mu^\xi(t,y) = -\xi_y(t,y)\mu^\phi(t,\xi(t,y)) + \frac{1}{2}\partial_y[||\sigma^\xi(t,y)||^2/\xi_y(t,y)].
\]

The proof of Proposition 3.2 is now complete.

The proof of Corollary 3.3 is achieved, first by reconciling the results of the previous proposition and the following identities,

\[
(\Phi_x)^{-1}(t,y) = -\Xi_y(t,y), \quad \Phi_{xx}(t,-\Xi_y(t,y)) = -\frac{1}{\Xi_{yy}(t,y)}, \quad \text{and} \quad -C_x(t,-\Xi_y(t,y)) = \frac{D_y(t,y)}{\Xi_{yy}(t,y)}
\]

and second by integrating with respect to \(y\), using the initial condition \(\Xi(t,0) = 0\).

\[
\Box
\]

3.2 Convex Conjugate of Consistent Stochastic Utility, Dual SPDE’s

We now define the convex conjugate of a consistent stochastic utility.

**Definition 3.1.** Let \(\hat{U}\) be the convex conjugate random field of the \(\mathcal{X}\)-consistent stochastic utility \(U\), given by definition for \(t \geq 0\),

\[
\hat{U}(t,y) \overset{\text{def}}{=} \inf_{x>0} (U(t,x) - xy)
\]

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\( \hat{U}(t, y) \) is a progressive decreasing convex random field, with first derivative \( \hat{U}_y(t, y) = -I(t, y) \), where \( I(t, y) = U_x(t, .)^{-1}(y) \) is the inverse function of the decreasing function \( U_x(t, x) \).

The formula of inverse flows yields easily to the dynamics of \( \hat{U}(t, y) \) from the dynamics of \( U(t, x) \). Based on this remark and Corollary 3.3, in the following theorem we derive a stochastic partial differential equation whose a solution is the convex conjugate processes of consistent stochastic utility.

**Theorem 3.4.** Let \( U \) be a consistent progressive utility of class \( C^{(3)} \), in the sense of Kunita, satisfying the \( \beta \) constraint (11) with risk tolerance \( \alpha^U \) and utility risk premium \( \eta^U(t, x) = \frac{\gamma_x(t, x)}{U_x(t, x)} \). Let \( \hat{U}(t, y) \) be the dual convex conjugate, null if \( y = 0 \). Then

(i) In old variables, \( \hat{U} \) satisfies

\[
\frac{d\hat{U}(t, y)}{dt} = \beta^1(t, -\hat{U}_y(t, y))dt + \gamma(t, -\hat{U}_y(t, y)).dW_t,
\]

where

\[
\begin{align*}
\beta^1(t, x) &= \beta(t, x) - \frac{1}{2U_x(t, x)}\|\gamma_x(t, x)\|^2 \\
&= U_x(t, x)\left[-\nu_t + \frac{1}{2}\alpha^U(t, x)\left(\inf_{\theta_t \in R^n, \cdot^n} \|\theta_t - (\eta^\gamma + \eta^U(t, x))\|^2 - \|\eta^U(t, x)\|^2\right)\right].
\end{align*}
\]

The optimization program is achieved on \( \theta^*(t, x) = \eta^U, \cdot^n(t, x) \).

(ii) In new variables,

\[
\begin{align*}
d\hat{U}(t, y) &= \frac{1}{2\hat{U}_yy(t, y)}(\|\gamma_y(t, y)\|^2 - \|\gamma^*_y(t, y) + y\hat{U}_yy(t, y)\eta^\gamma\|^2 + y\hat{U}_y(t, y)r_t)dt + \hat{\gamma}(t, y).dW_t \\
\hat{\gamma}(t, y) &= \gamma(t, -\hat{U}_y(t, y)).
\end{align*}
\]

Furthermore the drift \( \begin{array}{c} \hat{\beta}(t, y) := \beta^1(t, -\hat{U}_y(t, y)) \end{array} \) is the solution an optimization program achieved on the optimal policy \( y^*(t, y) = \theta^*(t, -\hat{U}(t, y)) = -\gamma^*_y(t, y)/y\hat{U}_yy(t, y) \).

In particular, \( \hat{\beta} \) can be written as the solution of the following optimization program

\[
\hat{\beta}(t, y) = y\hat{U}_y(t, y)r_t - \frac{1}{2}y^2\hat{U}_yy(t, y)\inf_{\nu_t \in R^n, \cdot^n} \{\|\nu_t - \eta^\gamma\|^2 + 2(\nu_t - \eta^\gamma)(\gamma_y(t, y)/\hat{U}_yy(t, y))\}
\]

with \( -\gamma_y(t, y)/y\hat{U}_yy(t, y) = \eta^U(t, -\hat{U}(t, y)) = \gamma_x(t, -\hat{U}(t, y))/y \).

First, observe that as \( -\hat{U}_y \) is the inverse flows of \( U_x \), the dynamic of the convex conjugate \( \hat{U} \) of \( U \) becomes a simple consequence of Corollary 3.3. Second, the orthogonal part of the utility prime risk \( \eta^U, \cdot^n := \gamma^*_x/U_x \) is the optimal policy of the dual problem in (i). Third, given that \( \hat{\beta} \) is associated with an optimization program the dual drift \( \hat{\beta} \) is also constrained by HJB type relation in the new variables. This will allow us, simply, by analogy to establish the link between this optimization program and a dual optimization program. From this, we establish in next result, that the convex conjugate is consistent with some given family (the family of the State density processes) which we explicit in the next result.
Proof. Let us point out that by regularity assumptions, using Theorem 2.1, $U_x(t, x), \beta_x(t, x)$ and $\gamma_x(t, x)$ are regular enough to apply Itô-Ventzel formula. The assumptions of Proposition 3.2 and Corollary 3.2 are satisfied and hence the dynamics of the convex conjugate is a direct consequence of Corollary 3.3. Let us now recall that the drift $\beta(t, x)$ of $U(t, x)$ is given in Theorem 2.4 by

$$\beta(t, x) = U_x(t, x)\left(-xr_t + \frac{\alpha^U(t, x)}{2}\|\eta^U,\sigma(t, x) + \eta^\sigma\|^2\right).$$

To get the desired formula for $\beta^1(t, x)$ in (15), we use the following property of the orthogonal projection, the norm of the projection on $\mathcal{R}^\sigma$ is the distance to the orthogonal vector space $\mathcal{R}^\sigma, \perp$ that is, for any vector $a \in \mathcal{R}^N$, $\|a^\sigma\|$ is the distance between $a$ and the linear space $\mathcal{R}^\sigma, \perp:\]

$$\|a^\sigma\|^2 = \inf_{\nu \in \mathcal{R}^\sigma, \perp} \|\nu - a\|^2.$$ Replacing $a$ by $(\eta^\sigma + \eta^U,\sigma(t, x))$, yields the result. Now, we focus on the drift $\tilde{\beta}$ of $U(t, y)$ in new variables: $\tilde{\beta}(t, y) := \beta(t, -\tilde{U}_y(t, y)) + \frac{1}{2\tilde{U}_t(t, y)}\|\gamma_y(t, y)\|^2$.

Using essentially the following identities

$$\frac{U^2_x(t, -\tilde{U}_y(t, y))}{2U_{xx}(t, -\tilde{U}_y(t, y))} = \frac{1}{2}2\tilde{U}_{yy}(t, y), \quad \tilde{\gamma}_y(t, y) = -\gamma_x(t, -\tilde{U}_y(t, y)), \quad \tilde{\gamma}_y(t, y) = \frac{\tilde{\gamma}_x(t, y)}{\tilde{U}^2_x(t, y)}(t, -\tilde{U}_y(t, y))$$

we get the desired formula for $\tilde{\beta}$, i.e.

$$\tilde{\beta}(t, y) = y\tilde{U}_y(t, y)r_t - \frac{1}{2}y^2\tilde{U}_{yy}(t, y) \inf_{\nu_t \in \mathcal{R}^\sigma, \perp} \{\|\nu_t + \eta^\sigma\|^2 - 2(\nu_t - \eta^\sigma) \cdot (\tilde{\gamma}_y(t, y))\}$$

On the other hand by orthogonal projection and using the fact that $\eta^\sigma \in \mathcal{R}^\sigma_t$ there exist one and only one optimal process $\nu^*$ given by

$$\nu^*_t(y) = \frac{-\tilde{\gamma}_y(t, y)}{y\tilde{U}_{yy}(t, y)} = \eta^U, \perp(t, -\tilde{U}_y(t, y))$$

which achieves the proof.

Let us now focus on the dual optimization problem.

Definition 3.2 (State price density process). A random field $Y$ is called a state price density process if for any wealth process $X^\kappa, \kappa \in \mathcal{R}^\sigma, YX^\kappa$ is a local martingale. It follows that $Y$ satisfies the following dynamics

$$\frac{dY_t}{Y_t} = -r_t dt + (\nu_t - \eta^\sigma_t).dW_t, \quad \nu_t \in \mathcal{R}^\sigma, \perp. \quad (17)$$

From this we denote by $\mathcal{Y}$ the family of all state density process and we index it by $\nu \in \mathcal{R}^\sigma, \perp$, that is

$$\mathcal{Y} := \{Y^\nu, \nu \in \mathcal{R}^\sigma, \perp : Y^\nu \text{ satisfies } (17)\}.$$
Where we recall that $R^*_{\sigma, \perp}$ denotes the orthogonal space of the range $R^*_{\sigma}$ of $\sigma_t$.

One of the main results of this paper is the following theorem, based on the interpretation of the second assertion of the previous theorem.

**Theorem 3.5.** Let $U$ be a consistent progressive utility of class $C^{(3)}$, in the sense of Kunita, satisfying the $\beta$ HJB constraint. Then, its conjugate process $\tilde{U}(t,y)$ (convex decreasing stochastic flows) is consistent with the family of state density processes $\Psi$. That is for any $Y^* \in \Psi$, $\tilde{U}(t,Y^*_t)$ is a submartingale and there exists a dual optimal choice given by

$$\nu^*(t,y) = \frac{\gamma_y(t,y)}{yU_{yy}(t,y)} = \frac{\gamma_y(t, -\tilde{U}_y(t,y))}{y} = y^{1, \perp}(t, -\tilde{U}_y(t,y)).$$

Furthermore, for any $y > 0$ the equation

$$\frac{dY_t^{\nu^*}}{Y_t^{\nu^*}} = -r_t dt + \left(\nu^*(t,Y_t^{\nu^*}) - \eta^*_t\right) dW_t,$$

admits at least one solution which is $Y_t^*(y) := U_x(t, X_t^*((U_x)^{-1}(0,y))$. 

**Remark.** Let $\Psi(t,x) := U_x(t, X_t^*(x))$, if $X_t^*(x)$ is strictly monotone in $x$, by taking the inverse $X(t,x)$, we can obtain $U_x(t,x)$ in terms of $\Psi(t,x)$ and $X(t,x)$.

**Proof.** The first assertion of this result is essentially obtained by analogy to the primal problem. Indeed, using the $\tilde{\beta}$ expression’s (16), which is

$$\tilde{\beta}(t,y) = y\tilde{U}_y(t,y)r_t + \frac{1}{2}y^2\tilde{U}_{yy}(t,y) \sup_{\nu_t \in R^*_{\sigma, \perp}} \left\{ -||\nu_t - \eta^*_t||^2 - 2(\nu_t - \eta^*_t) \left(\frac{\tilde{\gamma}_y(t,y)}{yU_{yy}(t,y)}\right) \right\}$$

One can easily remark, by analogy to expression of $Q$ in Lemma 2.3 and that of $\beta$, equation (11), Theorem 2.4, that $\tilde{U}$ is consistent with the family of processes $\Psi$, that is, $\tilde{U}(t,Y^*_t)$ is a submartingale for any $Y^*_t \in \Psi$ and a local martingale for the optimal choice (Theorem 3.4) $\nu^*(t,y) = -\frac{\gamma_y(t,y)}{yU_{yy}(t,y)} \gamma^*_x(t, -\tilde{U}(t,y))/y$. In particular, we note that

$$\frac{dY_t^{\nu^*}}{Y_t^{\nu^*}} = -r_t dt + \left(\eta^{1, \perp}(t, -\tilde{U}_y(t,Y_t^{\nu^*})) - \eta^*_t\right) dW_t.$$ 

On the other hand we recall that according to Proposition 2.5 assertion ii) that $U_x(t, X_t^*)$ satisfies

$$\frac{dU_x(t,X_t^*)}{U_x(t,X_t^*)} = -r_t dt + \left(\eta^{1, \perp}(t, X_t^*) - \eta^*_t\right) dW_t,$$

Denoting $Y_t^*(y) = (U_x(t, X_t^*((U_x)^{-1}(0,y))))_{t \geq 0}$ and using $-\tilde{U}_y(t,Y_t^*(y)) = X_t^*((U_x)^{-1}(0,y))$ yield that $Y^*$ and $Y^{\nu^*}$ satisfy the same dynamics which in turn implies the optimality of $Y^*$.

$\square$
3.3 Transformation by Change of Numeraire

Stability of the Notion of Consistent Progressives Utilities by Numeraire Change

One of the main reasons that we are interested in progressive utilities is the fact they are consistent with the financial market in contrast to the classical utilities functions which are not stable by change of numeraire and thus the value function of the portfolio optimization problem, in the classical sense, is highly dependent on market parameters \((r, \eta)\). Moreover, because it is more convenient and simpler to work with local martingales then semimartingales and because the change of numeraire (as we will see below) modifies the risk premium as well as the interest rate process, the goal of this section is then to deepen the question of the change of numeraire in order to simplify the utility stochastic PDEs and to get an intuitive interpretation of its parameters.

**Theorem 3.6** (Stability by change of numeraire).

Let \( U(t, x) \) be a stochastic field, \( N \) be a positive semimartingale and denote by \( \mathcal{X}^N \) the class of process defined by \( \mathcal{X}^N = \{ X/N, \ X \in \mathcal{X} \} \), then the process \( V \) defined by

\[
V(t, x) = U(t, xN_t)
\]

is a \( \mathcal{X}^N \)-Consistent stochastic utility in the market of numeraire \( N \) if and only if \( U \) is an \( \mathcal{X} \)-Consistent stochastic utility.

Roughly speaking the theorem says that the notion of \( \mathcal{X} \)-Consistent utilities is preserved by change of numeraire. An interpretation of this stability is that if the agent decides to invest in a second market (foreign market) his preferences (risk aversion) are still unchanged, which is very natural view the uniqueness representation of his preferences.

Furthermore, as the set of equivalent martingale measures is nonempty, for any equivalent martingale measure \( \mathcal{M} \) this theorem shows that studying \( \mathcal{X} \)-Consistent stochastic utilities is equivalent to study the \( \mathcal{X}^\mathcal{M} \)-Consistent utilities. The advantage here is that the new wealth processes in \( \mathcal{X}^\mathcal{M} \) are positive local martingales (in particular a supermartingales). From this point we will deepen the study of our utilities in the new market \( \mathcal{X}^\mathcal{M}, M \in \mathcal{M}(\mathcal{X}) \).

To show this result it is enough to verify the assertions of definition 1.1 using identity \( V(t, \bar{X}_t) = U(t, X_t) \).

Now, we turn to more quantitative aspects of the change of numeraire. The idea is to proof how, by change of numeraire techniques, we simplify the stochastic PDE’s of Consistent stochastic utilities. To this end some hypothesis and some regularities are needed.

**Assumption 3.1.** The new market-numeraire \( N \), is assumed to satisfy:

\[
\frac{dN_t}{N_t} = \mu_t^N dt + \delta_t^N dW_t, \quad N_0 = y.
\]
The wealth process $\tilde{X}$ is defined by $\tilde{X}_t := X_t / N_t$ where $X$ denotes the wealth process in the initial market. By Itô’s formula, we can easily write the following dynamics satisfied by those processes

$$
\frac{d\tilde{X}_t(\tilde{x})}{\tilde{X}_t(\tilde{x})} = (r_t - \mu_t^N + \delta_t^N . \eta_t^2)(dt) + (\kappa_t - \delta_t^N)(dW_t) dt.
$$

Denoting $\tilde{r} = r - \mu^N - \delta^N \cdot \eta^\sigma$ and $\tilde{\eta} = \eta^\sigma - \delta^N$, the new market price of risk we get

$$
\frac{d\tilde{X}_t(\tilde{x})}{\tilde{X}_t(\tilde{x})} = \tilde{r}_t dt + (\kappa_t - \delta_t^N)(dW_t + \tilde{\eta}_t dt).
$$

Let us now stress the fact that if $\delta^Y \in \mathcal{R}^\sigma$ then, as $\mathcal{R}^\sigma$ is vector space, we can replace $\kappa - \delta^Y \in \mathcal{R}^\sigma$ by $\tilde{\kappa} \in \mathcal{R}^\sigma$ and the optimization problem still unchanged. Hence we get that a consistent utility $V$ in this new market satisfies the same dynamics as in the initial market only by replacing $r$, $\eta$ by $\tilde{r}$, $\tilde{\eta}$. Else if $\delta^Y \perp \neq 0$ the optimization problem is quite different and the dynamics of consistent utilities are modified, as we will see in the next result.

We denote by $H^r, \eta^\sigma$ the state price density process given by

$$
H_t^r, \eta^\sigma = \exp(-\int_0^t r_s ds - \int_0^t \eta_s^2 dW_s - \frac{1}{2} \int_0^t ||\eta^\sigma_s||^2 ds). \quad (18)
$$

Now we have the following theorem, which generalizes the $\mathcal{X}$-Consistent utility SPDE’s.

**Theorem 3.7.** Let $U(t, x)$ be a $\mathcal{X}$-Consistent stochastic utility satisfying the assumptions of Theorem 2.4. The random field $V(t, x) := U(t, xN_t)$ is a solution of the following stochastic partial differential equation

$$
dV(t, \tilde{x}) = V_x(t, x) \left\{ \frac{1}{2\alpha^V(t, x)} \left( ||\tilde{\eta}_t + \eta^V_x||^2 - ||\tilde{\eta}_t - \eta^V_x||^2 \right) - \tilde{\kappa}^N_t \right\} dt + \alpha^V(t, \tilde{x}).dW_t
$$

with $\alpha^V(t, x) := V_{xx}(t, x)/V_x(t, x)$ and $\eta^V(t, x) = \gamma^V(t, x)/V_x(t, x)$ denote the risk tolerance and the utility premium of $V$. $\gamma^V(t, \tilde{x}) = \gamma^U(t, \tilde{x}N_t)$ and the optimal policy given by

$$
\tilde{x}\kappa^N_t(\tilde{x}N_t) = -\frac{1}{\alpha^V(t, \tilde{x})} \left( \tilde{\eta}_t^2 + \eta^V_x \right) (t, \tilde{x}) + \tilde{x}\delta^N_t \eta^V_x. \quad (19)
$$

Furthermore, taking $N = H^r, \eta^\sigma$ (where $H^r, \eta^\sigma$ is defined as in (18)) the market has no risk premium and the ratio $\frac{\gamma^V}{\gamma^U}$ has the same impact as a risk premium, but depending on the level of the wealth $x$ at time $t$. In particular, the previous dynamic of $V$ is simpler

$$
dV(t, \tilde{x}) = \frac{(V_x)^2(t, \tilde{x})}{2V_{xx}(t, \tilde{x})} \| \gamma^V_x(t, \tilde{x}) \|^2 dt + \gamma^V(t, \tilde{x}).dW_t.
$$

and the convex conjugate $\tilde{V}$ of $V$ satisfies

$$
d\tilde{V}(t, y) = \frac{1}{2\tilde{U}_{yy}(t, y)} \| \tilde{\gamma}_y^+(t, y) \|^2 dt + \tilde{\gamma}(t, y).dW_t.
$$

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Let us comment on the content of this theorem and its relation to the previous results. The dynamic (19) of consistent utilities and the optimal policy (19) are more complicated than ones in the initial framework. We recognize in the optimal policy formula a first term (very similar to that of the initial market) which corresponds to an optimization program without $\delta^N$ added to a second one which correspond to a translation by $\delta^N$. This is due to the fact that the dynamic of new wealth processes are a kind of combination of that of the initial market and the dynamic of the state price density processes in the dual problem studied in previous section. It suffices to take $\delta^N$ in the range of the matrix $\sigma$, to get the SPDE’s of the old market and to take $\delta = 0$ to get SPDE similar to the dual HJB-SPDE. Finally, in the second part of the result, taking a numeraire with good properties this HJB-SPDE is more simplified.

We close this section, by the the following corollary which is a consequence of the above theorem.

**Corollary 3.8.** Under the assumptions of Theorem 3.7, taking $N = H^{\tau, \eta}$, we have

1. $\gamma_x \in \mathcal{R}^\sigma$ implies $\tilde{V}$ is a local martingale and the optimal dual process is constant: $Y^* \equiv 1$.
2. $\gamma_x \in \mathcal{R}^{\sigma, \perp}$ implies $V$ is a local martingale and the optimal wealth $X^*(x) \equiv x$.

The new market defined from the first one by change of numeraire $1/H^{\tau, \eta}$ (the market numeraire) is called a martingale market because new wealths are local martingales.

**Proof.** The proof of this result is based on the Itô-Ventzel’s lemma applied to $U(t, xN_t)$ and the fact that the optimal wealth process $\tilde{X}^* = X^*/N$ by definition of $V$, where $X^*$ denote the optimal portfolio process associated to $U$. 

---

**Example: Change of probability and numeraire in standard utility function**

In this paragraph, we provide an example of a $\mathcal{F}$-Consistent stochastic utilities obtained by combining a standard utility function $v$ with some positive processes $N$ and $Z$ satisfying

$$\frac{dN_t}{N_t} = \mu^N_t dt + \delta^N_t dW_t, \quad \frac{dZ_t}{Z_t} = \mu^Z_t dt + \delta^Z_t dW_t, \quad Z_0 = 1.$$ 

Our goal here is to study the conditions that satisfy the triplet $(v, N, Z)$ we have that the drift $\beta$ of the process $(U(t, x))_{t \geq 0}$ defined by $U(t, x) = Z_t v(x/N_t)$ satisfies the HJB constraint (11) of Theorem 2.4.

**Theorem 3.9.** Let $v$ be an utility function.
- Except in the case where \( v \) is a power or exponential utility the process \( U \) defined by \( U(t, x) = Z_t v(x/N_t) \) is a \( \mathcal{F}_t \)-Consistent stochastic utility iff \( Z \) is a martingale, \( ZX^{\kappa}/N, \kappa \in \mathbb{R}^\sigma \) are positive local martingales and \( \delta \in \mathbb{R}^\sigma \). In this case the optimal policy is given by \( \kappa_t^* = \delta_t^N \).

- If \( v \) is a power or exponential utility, then Condition "\( Z \) is martingale, \( ZX^{\kappa}/N \) is a supermartingale for any \( \kappa \in \mathbb{R}^\sigma \)" is not a necessary condition.
  - If \( v \) is a power utility with risk aversion \( a \), it suffices to take \( Z \) and \( N \) such that the following equation is satisfied
    \[
    \frac{1}{a} \mu_t^Z + r_t - \mu_t^N + \delta_t^N . \eta_t - \delta_t^N . \delta_t^Z \perp + \frac{1}{2(1-a)} \| \eta_t^\sigma - \delta_t^N . \delta_t^Z \| \geq 1 + \frac{a}{2} \| \delta_t^N \| = 0.
    \]
  - If \( U \) is an exponential utility it suffices to take \( Z \) and \( N \) such that the following conditions are satisfied
    \[
    \mu_t^N = r + \delta_t^N . \eta_t^\sigma, \quad \mu_t^Z = \frac{1}{2} \| \eta_t^\sigma - \delta_t^N . \delta_t^Z \|, \quad \delta_t^N \in \mathbb{R}^\sigma.
    \]

This result gives necessary and sufficient conditions under which \( U \), defined above, is a \( \mathcal{F}_t \)-Consistent stochastic utilities. Note also that this example generalize that in [29] where the authors consider the case where \( u \) is an exponential utility and provides only a sufficient condition.

**Proof.** To facilitate the exposition, let us denote by \( \tilde{r} = r - \mu_t^N + \delta_t^N . \eta_t^\sigma, \quad \tilde{\eta} = \eta_t^\sigma - \delta_t^N \).

By Itô formula for all \((t, \omega)\), the consistence with the universe of investment is equivalent to
\[
U(t, X_t^\kappa)\mu_t^Z + U_x(t, X_t^\kappa)X_t^\kappa \left( r_t + (\kappa_t - \delta_t^N . (\eta_t^\sigma + \delta_t^Z) ) \right) + \frac{X_t^\kappa}{2} U_{xx}(t, X_t^\kappa)\| \kappa_t \|^2 \leq 0.
\]

Assume that the drift of the utility process \( U \) satisfies the HJB constraint (11) then we have
\[
U(t, X_t^\kappa) \mu_t^Z + U_x(t, X_t^\kappa) X_t^\kappa \left( xU_x(\kappa_t - \delta_t^N) . (\tilde{\eta}_t + \delta_t^Z) \right) + \frac{x^2 U_{xx}}{2} \| \kappa_t - \delta_t^N \|^2 = 0. \tag{20}
\]

One easily sees that by Lemma 2.3, the optimal policy is given by
\[
\kappa_t^*(x) = -\frac{1}{x U_{xx}} \left( -x U_{xx}(t, x) \delta_t^N . \eta_t^\sigma + U_x(t, x) (\tilde{\eta}_t^\sigma + \delta_t^Z) \right). \tag{21}
\]

Injecting this identity in (20) yields
\[
U(t, X_t^\kappa) \mu_t^Z + U_x(t, X_t^\kappa) x U_x(\kappa_t - \delta_t^N) . (\tilde{\eta}_t + \delta_t^Z) - \frac{(U_x)^2}{2 U_{xx}} \| \tilde{\eta}_t^\sigma + \delta_t^Z \| \geq 1 + \frac{x^2 U_{xx}}{2} \| \delta_t^N \|^2 (t, x) = 0.
\]

Next replace \( U(t, x) \) by \( Z_t v(x/N_t) \) and simplify by \( x v(x) Z_t \). It follows using the definition of \( \tilde{\eta} \) and \( \tilde{r} \) that \( \forall t \geq 0, x > 0 \)
\[
\frac{v}{x v_x} \mu_t^Z + r_t - \mu_t^N + \delta_t^N . \eta_t^\sigma - \delta_t^N . \delta_t^Z \perp + \frac{v_x}{2 x v_{xx}} \| \tilde{\eta}_t^\sigma + \delta_t^Z \| \geq 1 + \frac{x^2 U_{xx}}{2 v_x} \| \delta_t^N \|^2 (t, x) = 0. \tag{22}
\]
The case: $v/xv$ and $v_x/xv_{xx}$ are proportional, in turn $v$ is a power or exponential utility.

- $v(x) = x^a/a$. Then equation (22) becomes, $\forall t \geq 0$
  \[ \frac{1}{a} \mu_t^Z + r_t - \mu_t^N + \frac{\delta_t^N \cdot \eta_t^\sigma - \delta_t^N \cdot \delta_t^Z \cdot \eta_t^\sigma}{2(1 - a)} \||^{2} + \frac{1 + a}{2} \||^{2} = 0. \]

- $v(x) = -\frac{1}{c} e^{-cx}$, $c > 0$. Then $\forall t \geq 0$, $x > 0$,
  \[ \mu_t^Z - \frac{1}{2} \|| - \delta_t^N \cdot \eta_t^\sigma + \delta_t^Z \cdot \eta_t^\sigma = 2 - cx \left(r_t - \mu_t^N + \frac{\delta_t^N \cdot \eta_t^\sigma - \delta_t^N \cdot \delta_t^Z \cdot \eta_t^\sigma}{2(1 - a)} \right) + \frac{(cx^2}{2} - cx)\||^{2} = 0. \]

Evidently this is a second order polynomial identically null, consequently all coefficients are nulls, i.e., $\tilde{r} = r - \mu^N + \delta^N \cdot \eta^\sigma = 0$, $\mu_t^Z = \frac{1}{2} \|| - \delta^N \cdot \eta^\sigma + \delta^Z \cdot \eta^\sigma$, $\delta \in \mathcal{R}^\sigma$

Second case : $v/xv'$ and $v'/xv''$ are not proportional, then it is immediate that all terms of (22) are equal to zero, in turn $\tilde{r} = 0$, $\mu_t^Z = 0$, $\delta \in \mathcal{R}^\sigma$, $\eta^\sigma - \delta^N + \delta^Z \in \mathcal{R}^\sigma$ and hence the optimal strategy $\kappa^*$ in (21) is simply $\delta^N$.

We resume the situation: $Z$ is a martingale, $X^\kappa/N$ is a martingale under the probability $Q^Z$ defined by $dQ^Z/dP = Z$ and $\delta \in \mathcal{R}^\sigma$.

\[ \Box \]

4 Utility Characterization and Stochastic Flows Method for Solving Stochastic PDE’s

As mentioned in paragraph 2.2, contrary to the deterministic case, the method of stochastic characteristics might not be applicable to solve the utility stochastic PDEs. To overcome this difficulty we present here a new approach, by stochastic change of variable, to the stochastic partial differential equations satisfied by progressive utilities. This new approach that we call “stochastic flow method” is based on the properties of the optimal wealth $X^*$ and the state price density process $Y^*_t$. Before presenting this new technique, let us start with brief analysis of some results established in previous sections: Let $U$ be a consistent utility with optimal wealth $X^*$ then, according to Theorem 3.5 and Proposition 2.5, the process $Y^*$ defined by $Y^*_t(U_x(0, x)) = U_x(t, X^*_t(x))$ is optimal for the dual problem and such that $Y^*X^*$ is a martingale.

The starting point of our approach is now the following: knowing the behavior of the derivative of the utility $U$ along the optimal path is it enough to characterize $U$? This may seem too much to ask but the answer is simple and natural: Assume that the optimal portfolio denoted by $X^*$ is strictly increasing with respect to the initial capital and denote by $X$ the reverse flow, then using the duality identity $Y^*_t(U_x(0, x)) = U_x(t, X^*_t(x))$ we have $U_x(t, x) = Y^*_t(U_x(0, X(t, x)))$, integrating with respect to $x$ we get $U$.  

From this we assume for the rest of the paper the following main assumption.

**Assumption 4.1.** The wealth process $X^*_t(x)$ is assumed to be continuous and increasing in $x$ from 0 to $+\infty$ with $X^*_t(0) = 0$, $X^*_t(+\infty) = +\infty$ for any $t$ and satisfies

$$
\frac{dX^*_t(x)}{X^*_t(x)} = r_t dt + \kappa^*(t, X^*_t(x)). (dW_t + \eta^*_t dt), \quad \kappa^*(t, x) \in \mathbb{R}^\sigma, \forall x > 0, \text{ a.s.}
$$

Denote by $\mathcal{X}(t, z)$ the inverse of the flow $X(t, z) = (X^*_t(.))^{-1}(z)$.

Financially speaking this hypothesis which may be a consequence of no arbitrage opportunity says that: we do not invest more to earn less. On the other hand, this monotony assumption is true in many examples and, according to the classical results on the the stochastic differential equations (SDE), is satisfied as soon as $x\kappa^*$ is locally bounded, see Kunita [21].

Note also, by duality identity, that monotonicity of $X^*$ imply that the dual process $Y^*_t(y)$ is, in turn, strictly increasing and therefore invertible with respect to its initial condition $y$ for any date $t$. Consequently, we also do the following hypothesis.

**Assumption 4.2.** $Y^*(t, y)$ is continuous and increasing in $y$ from $+\infty$ to 0 satisfying

$$
\frac{dY^*_t(y)}{Y^*_t(y)} = -r_t dt + \left(\nu^*_t(Y^*(t, y)) - \eta^*_t\right).dW_t, \quad \nu(t, y) \in \mathbb{R}^{n, r}, \forall y > 0, \text{ a.s.} \quad (23)
$$

**Remark.** Note also that if assumption 4.2 hold we have by duality identity the monotonicity of $X^*$.

Starting from the idea above, the assumptions 4.1 and 4.2 allows us to compound $X^*$ with the inverse of $Y^*$ and $Y^*$ with the inverse flow of $X^*$ and thus, under some additional regularities assumptions, we establish one of our main contributions which consist on the characterization of all consistent utilities generating $X^*$ as an optimal portfolio. In particular we give the decomposition of the derivative $\gamma$ of the volatility vector as an operator of $U_x$ and $U_{xx}$. The second main result of this paper introduce a new method solve the utility stochastic PDE. The idea is to transform SPDE to a system of two stochastic differential equation (SDE). Herein, the method proposed can be used for a large class of SPDE.

There are two different messages on our approach hence we decide to present the associated results separately. Note that the results of these section can be obtained first on the martingale market and, simply, by using results of Theorem 3.6 we get the simillar ones on the initial market.
4.1 Utility Characterization

To fix the idea we consider a given wealth process $X^*$, a state density price process $Y^*$ and an utility function $u(x)$. The objective is to construct a consistent utility $U$ starting from the function $u(x)$ ($U(0,x) = u(x)$), generating $X^*$ as optimal wealth and $Y^*$ as optimal dual process. According to the necessary analysis above the constructed utility process may satisfy $U_x(t,X^*(x)) = Y^*(u_x(t))$.

To illustrate our approach we first start by proving the main result in a special case where we assume that the process $Y^*(y) = yH_{r,\eta}\sigma$ which is equivalent to $\nu^* = 0$ a.s. The advantage of this case is that we can find all messages of our construction and a complete overview of main calculations. This allows the reader to understand better the proof of the general case where there are some more calculations and notations.

Assuming $Y^*(y) = yH_{r,\eta}\sigma$, the identity (??) (Theorem 3.5) suggests a very simple way to associate a progressive utility $U(t,x)$ to the wealth process $X^*$. Indeed, if $\mathcal{X}(t,z)$ is the reverse flow of $X^*(x)$, then the increasing process $U(t,x)$ satisfying $U_x(t,x) = u_x(\mathcal{X}(t,x))H_{r,\eta}\sigma$ is a good candidate to be a consistent stochastic utility. Another remarkable property of this random field is that $U(t,x) = U_x(0,x)H_{r,\eta}\sigma$, which is another way to express that the optimal dual process $\nu^*$ is null. We are then in measure to state one of the important results of this section.

**Theorem 4.1.** In addition to the monotony assumption 4.1, assume $X^*_t(x)$ to be $C^{(2)}$ in the sense of Kunita and to satisfy $H_{r,\eta}\sigma X^*$ is martingale, $\forall x > 0$. Moreover, let $u$ be an utility function such that $x \mapsto u_x(x)$ is integrable near to infinity. Then, according to the results of Theorem 3.5, putting $\mathcal{Y}(t,x) = u_x(\mathcal{X}(t,x))H_{r,\eta}\sigma$, the process $U$ defined by

$$U(t,x) = H_{r,\eta}\sigma \int_0^x u_x(\mathcal{X}(t,z))dz$$

is a $\mathcal{F}$-Consistent stochastic utility satisfying the following dynamic

$$dU(t,x) = \left(-U(t,x)r_t + \frac{1}{2U_{xx}(t,x)}||\gamma^x_t(t,x) + U_x(t,x)\eta^x_t||^2\right)dt + \gamma(t,x).dW_t$$

with $\gamma(t,x) = -\int_0^x U_{xx}(t,z)z\kappa^x(t,z)dz - U(t,x)\eta^x_t$ and a dual convex conjugate given by

$$\bar{U}(t,y) = \int_y^{+\infty} X^*_t(\frac{z}{H_{r,\eta}\sigma})dz.$$
Proof. First by definition $U$ is strictly increasing concave random field and of class $C^{(3)}$ in the sense of Kunita. Let us now focus on the dynamic of this progressive utility. To get started, we introduce the intermediate process $\tilde{U}(t,x) := \int_0^x u_x(X(t,z))dz$ with a simpler convex conjugate $\tilde{U}(t,y) = \int_0^{+\infty} X^*_t((u_x)^{-1}(z))dz$. Denoting by $(\beta^U, \gamma^U)$ and $(\beta^{\tilde{U}}, \gamma^{\tilde{U}})$ the local characteristic of $\tilde{U}$ and $\tilde{U}$, it follows using the dynamic of $X^*$ and the identity $\tilde{U}_y(t,y) = -X^*_t((u_x)^{-1}(y))$ that

$$
\begin{align*}
\beta^{\tilde{U}}(t,y) &= -r_t \tilde{U}_y(t,y) - \tilde{U}_y(t,y)\kappa^*(t,\tilde{U}_y(t,y))\eta^*_t \\
\gamma^{\tilde{U}}(t,y) &= -\tilde{U}_y(t,y)\kappa^*(t,\tilde{U}_y(t,y))
\end{align*}
$$

(24)

On the other hand, using the correspondence between the parameters diffusions of $U_x$ and $\tilde{U}_y$ given in Proposition 3.2 (or equivalently Corollary 3.3), we have

$$
\begin{align*}
\gamma_x^U(t,x) &= -U_{xx}(t,x)\kappa^*(t,x) \\
\beta_x^U(t,x) &= \beta^{\tilde{U}}(t,U_x(t,x)) + \frac{1}{2U_{xx}(t,x)}||\gamma_x^U(t,x)||^2
\end{align*}
$$

By this and (24), it is straightforward to check that

$$
\begin{align*}
\gamma_x^U(t,x) &= -x\tilde{U}_{xx}(t,x)\kappa^*(t,x) \\
\beta_x^U(t,x) &= -x\tilde{U}_{xx}(t,x)r_t + \gamma_x^U(t,x)r_t + \gamma_x^U(t,x)\eta^*_t + \frac{\partial}{\partial x} \left( \frac{1}{2U_{xx}(t,x)}||\gamma_x^U(t,x)||^2 \right)
\end{align*}
$$

In turn we get that $\tilde{U}(t,x)$ satisfies

$$
d\tilde{U}(t,x) = \beta^U(t,x)dt + \gamma^U(t,x).dW_t, \quad (25)
$$

with

$$
\begin{align*}
\gamma^U(t,x) &= -r_t \tilde{U}_x(t,x)z\kappa^*(t,x)dz \\
\beta^U(t,x) &= -x\tilde{U}_{xx}(t,x)r_t + \tilde{U}(t,x)r_t + \gamma^U(t,x)\eta^*_t
\end{align*}
$$

As $U(t,x) = H^{r,\eta_R(t,x)}_t \tilde{U}(t,x)$, Itô’s formula leads to

$$
dU(t,x) = H^{r,\eta_R(t,x)}_t \left( \beta^U(t,x) - \tilde{U}(t,x)r_t - \gamma^U(t,x)\eta^*_t + \frac{1}{2U_{xx}(t,x)}||\gamma_x^U(t,x)||^2 \right)dt \\
+ \left( H^{r,\eta_R(t,x)}_t \gamma^U(t,x) - U(t,x)\eta^*_t \right).dW_t.
$$

Denote by $\gamma(t,x) := H^{r,\eta_R(t,x)}_t \gamma^U(t,x) - U(t,x)\eta^*_t = -\int_0^x U_{xx}(t,z)z\kappa^*(t,z)dz - U(t,x)\eta^*_t$, we obtain using $\beta^U$ formula and identities $U = H^{r,\eta_R(t,x)}_t \tilde{U}$, $U_x = H^{r,\eta_R(t,x)}_t \tilde{U}_x$, $U_{xx} = H^{r,\eta_R(t,x)}_t \tilde{U}_{xx}$ that $U$ satisfies the desired dynamic given by

$$
dU(t,x) = \left( -U(t,x)r_t + \frac{1}{2U_{xx}(t,x)}||\gamma_x^U(t,x)||^2 \right)dt + \gamma(t,x).dW_t
$$
and $\gamma_x(t, x) = -U_{xx}(t, x)\kappa^*(t, x) - U_x(t, x)\eta_t^\sigma$. Then $U(t, X_t^\kappa), \kappa \in \mathbb{R}^\kappa$ is a supermartingale and a local martingale for $\kappa = \kappa^*$. To conclude it suffices to prove that $U(t, X_t^\kappa)$ ($X^* := X^{\kappa^*}$) is a martingale. For this we recall that by definition we have $U(t, X_t^\kappa) = H_t^{r, \eta^\sigma} \int_{X_t^\kappa} u_x(X(t, z))dz$. By a change of variables $z = X_t^\kappa(z)$ and integration by part, we get

$$U(t, X_t^\kappa) = u_x(x)H_t^{r, \eta^\sigma} X_t^\kappa(x) - \int_0^x (H_t^{r, \eta^\sigma} X_t^\kappa(z'))u_{xx}(z')dz'.$$

Integrability assumptions and martingale property of $H_t^{r, \eta^\sigma} X_t^\kappa$ give the desired property of $U(t, X_t^\kappa)$. The proof is now complete.

We showed in Theorem 4.1 that for a monotone optimal wealth $X^*$, assumption $X^*H^{r, \eta^\sigma}$ is a martingale is sufficient assumption to require in order to construct at least a consistent utility of optimal wealth $X^*$ and of optimal dual process $Y^*(y) = y H^{r, \eta^\sigma}$. We ask now the question how to determine the progressive utilities associated with more general processes $Y^*$. As we saw it, in the Theorem 3.5, the intuition is to look for $U$ such that $U_x(t, x) = \mathcal{Y}o\mathcal{X}(t, x)$ where $\mathcal{Y}(t, x) = Y_t^*(u_x(x))$.

We now state the general consistent utility characterization theorem.

**Theorem 4.2.** Let $(X_t^\kappa(x))$ and $(Y^*(t, x))$ be two stochastic flows satisfying, in addition to assumptions 4.1 and 4.2, that $X^*Y^*$ is a martingale. Let $u$ an utility function, put $Y(t, x) = Y^*(t, u_x(x))$, $X(t, z) = (X^*(t, .))^{-1}$ and assume that $x \mapsto \mathcal{Y}(t, X(t, z))$ is integrable near to zero. Then the process $U$ defined by

$$U(t, x) = \int_0^x \mathcal{Y}(t, X(t, z))dz$$

is a $\mathcal{X}$-Consistent stochastic utility satisfying the HJB type SPDE,

$$dU(t, x) = \left( -xU_x(t, x)r_t + \frac{1}{2U_{xx}(t, x)}||\gamma_x^\sigma(t, x) + U_x(t, x)\eta_t^\sigma||^2 \right)dt + \gamma(t, x).dW_t,$$

with volatility vector $\gamma$ given by

$$\gamma(t, x) = -U(t, x)\eta_t^\sigma - \int_0^x \left( zU_{xx}(t, z)\kappa^*(t, z) - \nu_t^\sigma(U_x(t, z)) \right)dz.$$

The associated optimal portfolio and the minimal martingale measure are $X^*$ and $Y^*$ and the convex conjugate is given by

$$\tilde{U}(t, y) = \int_y^{+\infty} X_t^\kappa((\mathcal{Y})^{-1}(t, z))dz.$$
In the first Theorem of this paragraph we build for a given initial utility function a consistent progressive utility of a given wealth. The extension which we give here which up to technical points characterizes all consistent progressive utilities equivalent to the previous one (in the sense that they are the same optimal wealth process). This result expresses only how we must diffuse the function \( U_x(0,x) = u_x(x) \) to stay within the framework of consistent progressive utilities. The answer is intuitive because it expresses that it is enough to keep a monotone field \( Y^* \) which does not influence the reference market. On the other hand it is important to remark that the derivative with respect to \( x \) of the volatility vector \( \gamma \) is the sum of two orthogonal vectors and is given by

\[
\gamma_x(t,x) = \nu^*_t(U_x(t,x)) - xU_{xx}(t,x)\kappa^*(t,x) - U_x(t,x)\eta^*_t \]

and consequently, given \( \kappa^* \) and \( \nu^* \), it is interpreted as an operator \( \Upsilon(t,x,U_x,U_{xx}) \) which is linear on \( U_{xx} \), that depends on \( U_x \) through the volatility \( \nu^* \) of \( Y^* \) and an affine term on \( \eta^* \). Depends on \( x \) only through the optimal policy \( \kappa^* \). We also emphasizes that the term \( xU_{xx}/U_x \) in this formula is the relative risk aversion of an investor with utility process \( U \).

Remark. After giving the proof of this result, we want to draw the attention to the fact that this Theorem can be showed first in the case of the martingale market \( (r = 0, \eta = 0) \). This allows us to simplify the calculus and we can always comeback to this initial market by a technique of change of numeraire.

Proof. Under assumption 4.1 the inverse \( \mathcal{X} \) of \( X^* \) with respect to \( x \) satisfies by Proposition 3.2

\[
d\mathcal{X}(t,x) = -x\mathcal{X}_x(t,x)\kappa^*(t,x).dW_t + \frac{1}{2}\partial_x(\mathcal{X}_x(t,x)||x\kappa^*(t,x)||^2)dt.
\]

The hypothesis made on \( X^* \) and \( Y^* \) entail that we can apply the Itô-Ventzel formula to the compound flow \( \mathcal{Y}_o\mathcal{X} \). To study \( U_x(t,x) \) we are first interested on the coefficient of \( dW_t \) of the dynamics of \( \mathcal{Y}_o\mathcal{X} \) because it represents the derivative of the volatility of the utility \( \gamma \). As \( (\mathcal{Y}_o\mathcal{X})_x = U_{xx} \) and \( U_x = \mathcal{Y}_o\mathcal{X}_x \), formula (3.1) gives us that

\[
\gamma_x(t,x) = \nu^*_t(U_x(t,x)) - xU_{xx}(t,x)\kappa^*(t,x) - U_x(t,x)\eta^*_t.
\]

This identity shows that the vector \( \gamma_x \) is the sum of two orthogonal vectors since the first term \( \nu^*_t(U_x(t,x)) \) belong by hypothesis to the orthogonal of the second which is \(-U_x(t,x)(\eta^*_t + \ldots)\).
(x_{xx}/U_x)(t,x)\kappa^s(t,x)) that belongs by hypothesis to the space \( \mathcal{R}_x^q \). Throughout the projection of \( \gamma_x \) on \( \mathcal{R}_x^q \) is the vector \( \gamma_x^q(t,x) = -xU_{xx}(t,x)\kappa^s(t,x) - U(t,x)x\eta_x^q. \)

As \( U_x = \mathcal{Y}_o\mathcal{X}, \gamma_x \) is the volatility process of \( U_x \), it is enough to integrate it with respect to \( x \) to obtain the result.

We now focus our interest on the drift \( \mu^{U_x} \) of the derivative \( \mathcal{Y}_o\mathcal{X} \) of \( U \). The idea and calculations are exactly identical to those of the proof of Corollary ?? . Indeed by the assumptions and equation (14) we have

\[
\mu^{U_x}(t,x) = -(x\mathcal{X}_x(t,x)\mathcal{Y}_x(t,x) + \mathcal{Y}_o\mathcal{X}(t,x))r_t \\
+ \frac{1}{2}(\mathcal{Y}_x(t,x)\partial_x(\mathcal{X}_x(t,x)||x\kappa^s(t,x)||^2) + \frac{1}{2}(\mathcal{Y}_x(t,x)||\kappa^s(t,x)||^2) \\
- x\mathcal{X}_x(t,x)\partial_x(\mathcal{Y}_o(t,x)(\nu^*_t(\mathcal{Y}_o(t,x)) - \eta^*_t)/\kappa^s(t,x) - x\mathcal{X}_x(t,x)\mathcal{Y}_o(t,x)\kappa^s(t,x).\eta^*_t.
\]

Note that in the last line the term \( -x\mathcal{X}_x(t,x)\partial_x(\mathcal{Y}_o(t,x)(\nu^*_t(\mathcal{Y}_o(t,x)) - \eta^*_t)/\kappa^s(t,x) \) comes from Itô-Ventzel formula and corresponds to \( <d\mathcal{Y},d\mathcal{X}> \).

To lead the proof we proceed by analyzing line by line the above equality. Using the identities \( \mathcal{Y}_x(\mathcal{X}(t,x)) = \mathcal{Y}_{xx}(\mathcal{X}(t,x))\mathcal{X}_x(t,x) \) and \( U_{xx}(t,x) = \mathcal{Y}_x(\mathcal{X}(t,x))\mathcal{X}_x(t,x) \), the first line becomes

\[
-(x\mathcal{X}_x(t,x)\mathcal{Y}_x(t,x) + \mathcal{Y}_o\mathcal{X}(t,x))r_t = -(xU_{xx}(t,x) + U_x(t,x))r_t = -\partial_x(xU_x)(t,x)r_t. \tag{26}
\]

Rewriting the second we obtain

\[
\frac{1}{2} \left[ (\mathcal{Y}_x(t,x)\mathcal{X}_x(t,x)||\kappa^s(t,x)||^2) + (\mathcal{Y}_{xx}(t,x)\mathcal{X}_x(t,x)||\kappa^s(t,x)||^2) \right] \\
= \frac{1}{2} \partial_x[\mathcal{Y}_x(t,x)\kappa^s(t,x)||\kappa^s(t,x)||^2]. \tag{27}
\]

Finally, from the assumption that \( \nu^*_t(\mathcal{Y}(t,x))\kappa^s(t,x^*_t(x)) = 0 \) we deduce \( \nu^*_t(\mathcal{Y}_o(t,x))\kappa^s(t,x) = 0 \) and \( \partial_x(\nu^*_t(\mathcal{Y}_o(t,x)))\kappa^s(t,x) = 0 \). This yields in the last line

\[
-x\mathcal{X}_x(t,x)\partial_x(\mathcal{Y}_o(t,x)(\nu^*_t(\mathcal{Y}_o(t,x)) - \eta^*_t))\kappa^s(t,x) - x\mathcal{X}_x(t,x)\mathcal{Y}_o(t,x)\kappa^s(t,x).\eta^*_t = 0. \tag{28}
\]

Identities (26), (27) and (28) combined with the expression of \( \mu^{U_x} \) and that of \( \gamma_x \) yields

\[
\mu^{U_x}(t,x) = \partial_x \left( -xU(t,x)r_t + \frac{1}{2U_{xx}(t,x)||\kappa^s(t,x)||^2} \right).
\]

As \( U(t,0) \equiv 0 \) we get by integration that \( U \) satisfies

\[
dU(t,x) = \{ -xU_x(t,x)r_t + \frac{1}{2U_{xx}(t,x)||\kappa^s(t,x)||^2} \} dt + \gamma(t,x).dW_t.
\]
Using \( \gamma_x^\sigma(t, x) + U_x(t, x)\eta_{\sigma}^\tau = -xU_{xx}(t, x)\kappa^\ast(t, x) \) one easily sees that
\[
dU(t, x) = \{-xU_x(t, x)r_t + \frac{1}{2}\left\|\gamma_x^\sigma(t, x) + U_x(t, x)\eta_{\sigma}^\tau\right\|^2_{U_{xx}(t, x)}\}dt + \gamma(t, x).dW_t.
\]
The proof is complete. \( \square \)

Before proceeding to the second major contribution let us stress the fact that this new approach that uses stochastic change of variable is a trajectorial method which build the utilities process, in contrast with the traditional caracteristics method, from its optimal processes.

4.2 Stochastic Flows Method for Solving Stochastic PDE’s

In the previous paragraph using two invertible stochastic flows \( X^\ast \) and \( Y^\ast \) we construct a consistent utility with the desired dynamic. Naturally, the question of the converse point of view is required. Starting from a stochastic PDE that satisfy consistent utilities, the question is then under which assumptions we have existence and uniqueness of the solution? What can we deduce about the monotony and the concavity of a possible solution? Answering these questions is the purpose of this paragraph. In the following theorem we propose a new method that allows us to address the issue of such resolution of fully nonlinear second order stochastic PDEs.

**Theorem 4.3** (Conversely). Let \( U \) be a semimartingale with spatial parameter s.t. \( U(0, .) = u(, .) \) satisfying the following SPDE
\[
dU(t, x) = \left\{-xU_x(t, x)r_t + \frac{1}{2}\left\|\gamma_x^\sigma(t, x) + U_x(t, x)\eta_{\sigma}^\tau\right\|^2_{U_{xx}(t, x)}\right\}dt + \gamma(t, x).dW_t, \tag{29}
\]

Where \( \gamma_x(t, x) + U_x\eta_{\sigma}^\tau = \left[\nu^\ast_t(U_x) - xU_{xx}\kappa^\ast\right](t, x) \) with \( \kappa^\ast(t, x) \in \mathbb{R}_t^\sigma \) and \( \nu^\ast_t(U_x(t, x)) \in \mathbb{R}^{\sigma, \perp} \forall t \geq 0, x > 0 \ a.s. \)

If the solution of the system
\[
\begin{align*}
\frac{dX^\ast_t(x)}{X^\ast_t(x)} &= r_t dt + \kappa^\ast(t, X^\ast_t(x)).(dW_t + \eta_t dt) \\
\frac{dY^\ast(t, y)}{Y^\ast(t, y)} &= -r_t dt + (\nu^\ast_t(Y^\ast(t, y)) - \eta_t^\ast).dW_t
\end{align*}
\]
exists, monotone and are a regular flows (true if \( \pi \) and \( \nu^\ast \) are locally bounded: see H. Kunita [21] Theorem 3.4.1 p.101), then the solution \( U \) of SPDE (29) exists, is unique and it is given by
\[
U(t, x) = \int_0^x Y(t, X(t, z))dz
\]
with \( Y(t, x) = Y^\ast(t, U_x(0, x)) \) and \( X(t, z) = (X^\ast(t, .))^{-1}. \)
Roughly speaking the theorem says that we can transform the problem of stochastic partial differential equations which is generally more difficult to study and for which no general results of existence or comparison. The SDEs are the ones corresponding to the decomposition onto the sum of two orthogonal vectors of the vector volatility of this SPDE. There are several advantages of this transformation. Firstly, the partial differential equations are more intuitive than the fully non-linear stochastic SPDE. Secondly, SDEs are simpler to study as seen in the existence of diverse works in this domain and seen in the multitude of results on the existence, uniqueness and on the integrability of solutions. Third, if the solutions of these SDEs are monotonic stochastic flows then we can deduce several properties of the solution $U$ of the SPDE which to the best of our knowledge there are no or few results that assert the monotonicity or the convexity of such solutions. Also, there may be other advantages in numerical methods and simulations of the SDE then of SPDE. Finally we stress however that this method which is a logical consequence of our approach is not specific to the case of the stochastic partial differential equation associated to consistent progressive utilities and can be extended to a more general framework.

5 Random risk aversion and decreasing consistent utilities

In the previous sections, we have studied various aspects of stochastic utilities and we give a fairly complete characterization of these utilities in a Brownian market. We also give some simple examples of change of probability and numeraire and which illustrate the various difficulties we encounter in our study. The purpose of this paragraph is now to build, from simple Consistent utilities indexed by a parameter $\alpha$ more general consistent utilities by reasoning only in terms of the optimal wealth and by considering random editions of the parameter $\alpha$. The powers utilities in what follows are concrete examples to illustrate the proposed method.

Let $Z^\alpha$ satisfy the following dynamics

$$
\frac{dZ_t^\alpha}{Z_t^\alpha} = -(1 - \alpha) r_t - \frac{1 - \alpha}{2\alpha} \|\eta_t^\sigma + \delta_t^Z\|^2 dt + \delta_t^Z dW_t, \quad Z_0 = 1, \quad \delta_t^Z \in \mathcal{R}_t^\sigma,
$$

where we assume that the volatility process $\delta^Z$ and the minimal risk prime $\eta^\sigma$ are bounded.

According to Theorem 3.9 taking $N \equiv 1$ the process

$$
v^\alpha(t, x) = Z_t^\alpha \frac{x^{1-\alpha}}{1-\alpha},
$$

(30)
is a $\mathcal{X}$-Consistent dynamic utility such that the optimal wealth process $X^{*,\alpha}$ and the optimal dual process $\mathcal{Y}^\alpha$ are given by,

$$X_t^{*,\alpha}(x) = x X_t^{*,\alpha}(1) = x e^{\frac{1}{2} \int_0^t \left( \eta_s^2 + \delta_s^2 \right) dW_s + \int_0^t \left( r_s - \frac{1}{2} \|\eta_s^2 + \delta_s^2\|^2 + \frac{1}{2} \eta_s^2 \right) ds} \left( 1 + \int_0^t \left( \eta_s^2 + \delta_s^2 \right) ds \right)$$

$$Y_t^{*,\alpha}(y) = u^\alpha_x(t, X_t^{*,\alpha}(u^\alpha_x(0, y))) = y H_t^{r,\sigma^\alpha}, \quad \mathcal{Y}^\alpha(t, x) := Y_t^{*,\alpha}(u^\alpha_x(0, y)) = x^{-\alpha} H_t^{r,\sigma^\alpha}$$

where $H^{r,\sigma^\alpha}$ is the state price density process defined in (18). Note that the optimal wealth is linear with respect to its initial condition and the optimal dual process $Y^{*,\alpha}$ is independent on the relative risk aversion $\alpha$ and finally $X^{*,\alpha}Y^{*,\alpha}$ is martingale.

**Random risk aversion:** At this stage the coefficient $\alpha$, which is the relative risk aversion, was supposed constant, it is about the simplest case of the powers $\mathcal{X}$-Consistent utilities. But it is completely conceivable that this risk aversion is in general random. Indeed we can imagine at date $t = 0$ that the investor pulls at random the value of this coefficient. For every value $\alpha$ he associates:

- a weight $\nu(\alpha)$ ($\nu$ is a finite positive measure),
- a proportion $x_\alpha(x)$ of its initial wealth (strictly increasing $x, x_\alpha(x) \to \infty$ if $x \to \infty$) that he is going to invest on the financial market by considering $u^\alpha$ as utility, he will so realize $X^{*,\alpha}(x_\alpha(x))$ as wealth (associated with this edition).

His final wealth is consequently the sum of the processes $X^{*,\alpha}(x_\alpha(x)) = x_\alpha(x)X^{*,\alpha}(1)$ weighted by the measure $\nu$

$$X_t^\ast(x) = \int_{\mathbb{R}_+} x_\alpha(x) X_t^{*,\alpha}(1) \nu(d\alpha)$$

By analogy, we suppose that $\mathcal{Y}$ defined below is integrable near to zero

$$\mathcal{Y}(t, x) = H_t^{r,\sigma^\alpha} \int_{\mathbb{R}_+} (x_\alpha(x))^{-\alpha} d\nu(\alpha)$$

which implies the following result.

**Proposition 5.1.** Let the measure $\nu$ with support in $\mathcal{I}$, by boundary assumption of $\delta^Z$, $X^*$ is an admissible wealth process strictly increasing with respect to $x$. Moreover, $X^*\mathcal{Y}$ is martingale.

**Proof.** Indeed, by definition $X^*$ is a positive local martingale satisfying the following dynamics

$$\frac{dX_t^*}{X_t^*} = r_t dt + \left[ \int_{\mathbb{R}_+} x_\alpha(x) X_t^{*,\alpha}(1) \nu(d\alpha) \right] \left( \eta_t^\sigma + \delta_t^Z \right) \left( dW_t + \eta_t^\sigma dt \right)$$

34
where we note that $\kappa^*$ given by

$$
\kappa^*(t, X_t^*) = \left[ \int_0^t x_\alpha(x) X_t^{*, \alpha}(1) \frac{2(\nu(\alpha))}{\alpha} \right] \big( \eta^* + \delta_Z^* \big)$$

is in $\mathcal{R}_t^\alpha$, $t \geq 0$.

By boundary condition on $\delta_Z$ and $\eta^*$, $X^* H^{r, \eta^*}$ is martingale. Finally that $X^*$ is strictly increasing with respect to $x$ is a simple consequence of the strict monotony of the proportions $x_\alpha(.)$ and the fact that $X^*, \alpha(1)$ is independent of $x$.

By Theorem 4.1, for any utility function $u$ such that $u_x(\mathcal{X}(t, x))$ is integrable near to 0 the process $U$ defined by

$$U(t, x) = H_t^{r, \eta^*} \int_0^x u_x(\mathcal{X}(t, z))dz,$$

is a $\mathcal{X}$-Consistent utility with $X^*$ as optimal wealth and $H^{r, \eta^*}$ as optimal dual process.

**Case of decreasing $\mathcal{X}$-Consistent utilities**

Let now $u_x(x) = \mathcal{Y}(0, x) = \int_{\mathcal{R}_t^\alpha}(x_\alpha(x))^{-\alpha}d\nu(\alpha)$ be the function given by (33). In terms of the dual of the consistent utility $U$, we know according to Theorem 4.1, that $\tilde{U}$ is given by

$$\tilde{U}(t, y) = \int_y^{+\infty} X_t^*(\mathcal{Y}^{-1}(t, z))dz = \int_{\mathcal{R}_t^\alpha} \left( \int_y^{+\infty} x_\alpha((u_x)^{-1}(\frac{z}{H_t^{r, \eta^*}}))dz \right) X_t^{*, \alpha}(1)d\nu(\alpha).$$

Furthermore, if the functions $x_\alpha$ satisfy $x_\alpha(x) = (\mathcal{Y}(0, x))^{-\frac{1}{\alpha}} = (u_x)^{-1}(x)$, then integrating yields

$$\tilde{u}(t, y) = \int_{\mathcal{R}_t^\alpha} \frac{1}{1 - \frac{x}{\alpha}} \left( 1 - y^{1 - \frac{1}{\alpha}} X_t^{*, \alpha}(1)(H_t^{r, \eta^*})^{\frac{1}{\alpha}} \right) d\nu(\alpha)$$

$$= \int_{\mathcal{R}_t^\alpha} \frac{1}{1 - \frac{x}{\alpha}} \left( 1 - y^{1 - \frac{1}{\alpha}} Z_t^\alpha \right) d\nu(\alpha).$$

Where in the last line we have used according to equation (32) the identity $X_t^{*, \alpha}(1)(H_t^{r, \eta^*})^{\frac{1}{\alpha}} = Z_t^\alpha$. Now taking $\delta_Z = 0$ and $r = 0$ we get at first that $Z_t^\alpha = exp(-\frac{1 - \alpha}{2\alpha} \int_0^t ||\eta_s||^2ds)$ and finally

$$\tilde{u}(t, y) = \int_{\mathcal{R}_t^\alpha} \frac{1}{1 - \frac{x}{\alpha}} \left( 1 - y^{1 - \frac{1}{\alpha}} e^{-\frac{1 - \alpha}{2\alpha} \int_0^t ||\eta_s||^2ds} \right) d\nu(\alpha).$$

Which is the representation of decreasing consistent (forward) utilities in time. These utilities were essentially studied by Zariphopoulou and al. [31] and by M. Tehranchi and al. [11]. In [31], M. Musiela and T. Zariphopoulou developed several examples with different measures $\nu$ as well as properties of the associated optimal wealth.

Note that in the construction above the family $\{X^{*, \alpha}, \alpha \in \mathbb{R}_t^\alpha\}$ constitutes the family of optimal wealths associated to consistent utilities $\{u_\alpha, \alpha \in \mathbb{R}_t^\alpha\}$ of a power type given by
It is important to note then that the process of construction which we proposed in this paragraph can be easily generalized and is not specific to powers utilities. Indeed, we can consider a family of wealths \( \{ X^{*,\alpha}, \alpha \in \mathbb{R}_+^* \} \) indexed by a parameter \( \alpha \) associated with consistent utilities which are not necessarily the same type. So by the same reasoning it is easy to build the wealth \( X^* \). On the other hand, we note that the fact that \( Y^{\alpha} \) in (32) was independent on \( \alpha \) and proportional to \( H^{r,\eta} \) played a fundamental role in satisfying the optimality conditions. Indeed, it implies that \( X^{*,\alpha} H^{r,\eta} \) are martingales for any \( \alpha \) which simplifies considerably the construction of the process \( Y^* \) being proportional to \( H^{r,\eta} \). This property in a more general case can be then replaced by \( X^{*,\alpha}, \alpha \in \mathbb{R}_+^* \) are martingales under a same probability measure \( Q \).

NB: The key point of this section is to argue directly in terms of the optimal wealth and dual process and not in terms of consistent utilities for the simple reason that the sum of two consistent utilities is not a consistent utility, except in the very particular case where both optimal wealths and optimal dual processes are identical. On the other hand the sum of two acceptable wealths is always an acceptable wealth.

A  Itô-Ventzel’s formula

The Itô-Ventzel’s formula is a generalization of classical Itô’s formula where the deterministic function is replaced by a stochastic process depending on a real or multivariate parameter. There are several difficulties in the definition of semimartingale depending on a parameter, as explained in H. Kunita [?]. For instance, let us consider the Itô integral of a predictable process \( f_t(x) \) with parameter \( x \) in some domain \( D \) of \( \mathbb{R}^+ \) with respect to some Brownian motion \( B \). Suppose that \( \int_0^T f_s(x)^2 ds < +\infty \) holds for each \( x \in D \). Then the Itô integral \( M_t(x) = \int_0^t f_s(x) dB_s \) is well defined for any \( t \) except for a null set \( N_x \). It is a continuous local martingale with parameter \( x \in D \). Then \( M_t(x) \) is well defined for \( (t,x) \) if \( \omega \in (\bigcup_{x \in T} N_x)^c \). However the exceptional set \( \bigcup_{x \in T} N_x \) may not be a null set since it is an uncountable union of null sets. To overcome this technical problem we must take a good modification of the random field \( M_t(x) \) so that it is well defined for all \( (t,x) \) a.s. and is continuous or continuously differentiable with respect to \( x \) for all \( t \) almost surely.

A.1  Notation and Definition

Functional spaces  We shall first introduce some notations. Let \( D \) be a domain in \( \mathbb{R}^+ \), \( m \) an non-negative integer and denote by \( C^m(D, \mathbb{R}) \) the set of all functions \( g : D \rightarrow \mathbb{R} \) which are \( m \)-times continuously differentiable. Using the notation \( g^{(m)} \) for the derivative of order
of some function $g$, we introduce the seminorms defined on some compact subset of $D$ by

$$||g||_{m,K} = \sup_{x \in K} \frac{|g(x)|}{1 + |x|} + \sum_{1 \leq \alpha \leq m} \sup_{x \in K} |g^{(\alpha)}(x)|.$$ 

Equipped with these seminorms, $C^m(D, \mathbb{R})$ is a Frechet space. When $D$ itself is a compact space we drop out the reference to $K$.

We sometimes need to refer to more regular functions whose derivatives of order $m$ are $\delta$-Hölder continuous ($0 < \delta \leq 1$). Then we introduce a new family of seminorms,

$$||g||_{m+\delta,K} = ||g||_{m,K} + \sup_{x,y \in K, x \neq y} \frac{|g^{(m)}(x) - g^{(m)}(y)|}{|x - y|\delta}.$$ 

on the set of $C^m(D, \mathbb{R})$ whose last derivative is $\delta$-Hölder continuous.

**Definition A.1.** A continuous function $g(t, x)$, $x \in I$, $t \geq 0$ is said to belong to $C^{m,\delta}$, $\delta \in [0,1]$ if for every $t$, $f(t) = f(t, \cdot)$ belongs to $C^{m,\delta}$ and $||f(t)||_{m+\delta,K}$ is integrable with respect to $t$ for any compact subset $K$ of $I$. If the set $K$ is $I$, the function $f$ is said to belong to the class $C^{m,\delta}_{ub}$. Furthermore, if $||f(t)||_{m+\delta}$ is bounded in $t$ it is said to belongs to $C^{m,\delta}_{ub}$

We also need to introduce the same kind of definition for functions depending on two parameters

$$||g||_{m+\delta,K} = ||g||_{m,K} + \sum_{\alpha = m} ||\partial_x^\alpha \partial_y^\alpha g(x, y)||_{\delta,K}$$

$$||g||_{\delta,K} = \sup_{x, x', y, y' \in K, x \neq x', y \neq y'} \frac{|g(x, y) - g(x', y) - g(x, y') + g(x', y')|}{|x - x'|\delta |y - y'|\delta}.$$ 

**Definition A.2.** A continuous function $g(t, x, y)$, $x, y \in I$, $t \in [0, T]$ is said to belong to $\tilde{C}^{m,\delta}$, $\delta \in [0,1]$ if for every $t$, $g(t, \cdot, \cdot)$ belongs to $\tilde{C}^{m,\delta}$ and $||g(t)||_{m+\delta,K}$ is integrable on $[0, T]$ with respect to $t$ for any compact subset $K$ of $I$. If the set $K$ is $I$, the function $g$ is said to belong to the class $\tilde{C}^{m,\delta}_{ub}$. Furthermore, if $||g(t)||_{m+\delta}$ is bounded in $t$ it is said to belong to $\tilde{C}^{m,\delta}_{ub}$

$C^{m,\delta}$-process: Let $U(t, x)$ a family of real valued process with parameter $x \in I$. We can regard it as a random field with double parameter $t$ and $x$. If $U(t, x, \omega)$ is a continuous function of $x$ for almost all $\omega$ for any $t$, we can regard $U(t, \cdot)$ as a stochastic process with values in $C = C(I, \mathbb{R})$ or a $C$-valued process. If $U(t, x, \omega)$ is $m$-times continuously differentiable with respect to $x$ for almost all $\omega$ for any $t$, it can be regarded as a stochastic process with values in $C^m = C^m(I, \mathbb{R})$ or a $C^m$-valued process. If $U(t, x)$ is a continuous process with value in $C^m$, it is called a continuous $C^m$-process. A $C^{m,\delta}$-valued process and continuous $C^{m,\delta}$-processes are defined similarly.
\textbf{\(\hat{C}^m,\delta\)-process :} Let \(G(t, x, y)\) be a stochastic process with parameter \(x, y \in \mathcal{I}\). If it is \(m\)-times continuously differentiable with respect to each \(x\) and \(y\) a.s. for any \(t\), it is called a stochastic process with values in \(\hat{C}^m\) or a \(\hat{C}^m\)-valued process. The \(\hat{C}^m,\delta\)-valued process and continuous \(\hat{C}^m,\delta\)-valued process are defined similarly.

\textbf{Theorem A.1.} Let \(M_t(x), x \in \mathcal{I}\) be a family of continuous local martingales such that \(M_0(x) \equiv 0\). Assume the joint quadratic variation \(< M_t(x), M_t(y) >\) has a modification \(A(t, x, y)\) of a continuous \(\hat{C}^m,\delta\)-process for some \(m \geq 1\) and \(\delta \in (0, 1]\). Then \(M_t(x)\) has a modification of a continuous \(C^m,\epsilon\)-process for any \(\epsilon < \delta\). Furthermore, for each \(n \geq m\), \(\partial^n_x M_t(x), x \in \mathcal{I}\) is a family of continuous local martingales with joint quadratic variation \(\partial^n_x \partial^n_y A(t, x, y)\).

\textbf{Definition A.3.} We shall call the random field \(M_t(x)\) with the property of the previous Theorem a continuous local martingale with values in \(C^m,\epsilon\) or a continuous \(C^m,\epsilon\)-local martingale.

\textbf{Regular Itô’s random fields \(C^{m,\delta}\)-semimartingale :} Suppose \(U(t, x), x \in \mathcal{I}\) is a family of continuous semimartingale decomposed as \(U(t, x) = B(t, x) + M(t, x)\), where \(M(t, x)\) is a local martingale and \(B(t, x)\) is a continuous process of bounded variation. \(U(t, x), x \in \mathcal{I}\) is said to belong to the class \(C^{m,\delta}\) or simply to be \(C^{m,\delta}\)-semimartingale if \(M(t, x)\) is a continuous \(C^{m,\delta}\)-local martingale and \(B(t, x)\) is a continuous \(C^{m,\delta}\)-process such that \(D^\alpha B(t, x), \alpha \leq m\) are all process of bounded variation. Further if \(\delta = 0\) it is called a \(C^m\)-semimartingale.

Let \(U\) be a semimartingale satisfying

\[dU(t, x) = \beta(t, x)dt + \gamma(t, x)dW_t, \quad U(r, x) = U(x),\]

where \(\beta\) and \(\gamma\) are predictable process.

\textbf{Definition A.4} (Kunita).

- The pair \((\beta, \gamma)\) is called the local characteristic of \(U\).
- Let \(m\) be a non-negative integer and \(a(t, x, y) := \gamma(t, x)^* \gamma(t, y)\). The local characteristic \((\beta, \gamma)\) is said to be in the class \(\mathcal{B}^{m,0}\) if both \(\beta\) and \(a\) are predictable process with value \(C^m\) and if for any compact subset \(K_1 \subset \mathbb{R}_+^+\) and \(K_2 \subset \mathbb{R}_+^+ \times \mathbb{R}_+^+\),

\[||\beta(t, .)||_{m; K_1}, \ ||a(t, .,.)||_{m; K_2} \in L^1.\]

Where

\[||f||_{m; K_1} = \sup_{x \in K_1} \frac{|f(x)|}{1 + |x|} + \sum_{1 \leq |\alpha| \leq m} \sup_{x \in K_1} |D^\alpha f(x)|.\]
\[ \| g \|_{m,K_2} = \sup_{x,y \in K_2} \frac{|g(x,y)|}{(1 + |x|)(1 + |y|)} + \sum_{1 \leq |\alpha| \leq m} \sup_{x,y \in K_2} |D_2^\alpha D_y g(x,y)| \]

**Definition A.5 (Itô-Ventzel Regularity).** A semimartingale random field \( U \) is said to be Itô-Ventzel regular if \( U \) is a continuous \( C^2 \)-process and continuous \( C^1 \)-semimartingale with local characteristic satisfying previous assumption.

**Theorem A.2 (Itô-Ventzel’s Formula (Kunita)).** Let \( (U(t,x)) \) be an Itô-Ventzel regular semimartingale random field and let \( X_t \) be a continuous semimartingale with values in \( \mathcal{I} \) and volatility \( \sigma^X \), then \( U(t,X_t) \) is a continuous semimartingale and

\[
U(t, X_t) = U(0, X_0) + \int_0^t \beta(s, X_s) ds + \int_0^t \gamma(s, X_s) dW_s \\
+ \int_0^t \frac{\partial U}{\partial x}(s, X_s) dX_s + \int_0^t \frac{\partial^2 U}{\partial x^2}(s, X_s) < X >_s ds \\
+ \int_0^t \langle \frac{\partial \gamma}{\partial x}(s, X_s), \sigma^X_s \rangle ds.
\]

Furthermore, according to H. Kunita [21] Theorem 3.3.3 p.94 we have the following differential rules for stochastic integrals.

**Theorem A.3 (Differential rules for stochastic integrals).** (i) Let \( F(t,x) \) be a continuous \( C^{m,\delta} \)-semimartingale with local characteristic belonging to the class \( \mathcal{B}^{m,\delta} \) where \( \delta > 0 \). Let \( X(t,x) \), \( x \in \Lambda \), \( t \in [0,T] \) be a continuous predictable process with values in \( C^{k,\gamma}(\Lambda, \mathcal{I}) \) where \( \gamma > 0 \) and \( \Lambda \subset \mathbb{R}^e \). Set

\[
M(t,x) = \int_0^t F(X(s,x), ds).
\]

Then \( M(t,x) \) has a modification of continuous \( C^{m,\delta} \)-semimartingale with values in \( C^{m\wedge k,\varepsilon}(\Lambda, \mathbb{R}) \) with local characteristic belonging to the class \( \mathcal{B}^{m\wedge k,\gamma\delta} \) with \( \varepsilon < \gamma\delta \).

Further if \( g_t \) is a continuous predictable process with values in \( \Lambda \), then we have the equality:

\[
\int_0^t M(ds, g_s) = \int_0^t F(X(s,g_s), ds). \tag{34}
\]

(ii) If \( m \geq 1 \) and \( k \geq 1 \), then we have the equality:

\[
\frac{\partial}{\partial x^i} M(t, x) = \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x^j} X^i(s,x) \frac{\partial}{\partial x^l} F(X(s,x), ds). \tag{35}
\]
References


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