Tutorial:
Black-Box Complexity:
From Complexity Theory to Playing Mastermind

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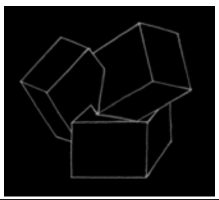


GECCO'12 Companion, July 7–11, 2012, Philadelphia, PA, USA.
ACM 978-1-4503-1178-6/12/07.

Bio-Sketch

- Benjamin Doerr is a senior researcher at the Max Planck Institute for Informatics and a professor at Saarland University.
- He received his diploma (1998), PhD (2000) and habilitation (2005) in mathematics from Kiel University.
- Together with Frank Neumann and Ingo Wegener, he founded the theory track at GECCO and served as its co-chair 2007-2009.
- He is a member of the editorial boards of Evolutionary Computation and Information Processing Letters.
- His research area includes theoretical aspects of randomized search heuristics, in particular, run-time analysis and complexity theory.

Objectives of the Tutorial

- This is a tutorial on black-box complexity. This is currently one of the hottest topics in the theory of randomized search heuristics.
- I shall try my best to..
  - tell you on an elementary level what black-box complexity is and how it shapes our understanding of randomized search heuristics
  - give an in-depth coverage of two topics that received much attention in the last few years
    - stronger upper bounds and the connection to guessing games
    - alternative black-box models
  - sketch several open problems
- Don't hesitate to ask questions whenever they come up!

Agenda

- Part 1: Black-box complexity: A complexity theory for randomized search heuristics (RSH)
  - Introduction/definition
  - Lower bounds for all RSH (example: needle functions)
  - Thorn in the flesh: Are there better RSH out there? (example onemax)
  - Different black-box models – what is the right difficulty measure?
- Part 2: Tools and techniques (in the language of guessing games)
  - From black-box to guessing games
  - A general lower bound
  - How to play Mastermind
  - A new game
- Summary, open problems

Timeline

2002
Droste, Jansen, Tinnefeld, Wegener. **A new framework for the valuation of algorithms for black-box optimization.** FOGA

2006
Droste, Jansen, Wegener. **Upper and Lower Bounds for Randomized Search Heuristics in Black-Box Optimization.** Theory Comput. Syst. 39

2009
Anil, Wiegand. **Black-box search by elimination of fitness functions.** FOGA

2010
Lehre, Witt. **Black-box search by unbiased variation.** GECCO

2011
Doerr, Johannsen, Kötzing, Lehre, Wagner, Winzen. **Faster Black-Box Algorithms Through Higher Arity Operators.** FOGA

Doerr, Winzen. **Towards a Complexity Theory of Randomized Search Heuristics: Ranking-Based Black-Box Complexity.** CSR

Rowe, Vose. **Unbiased black box search algorithms.** GECCO

Doerr, Kötzing, Lengler, Winzen. **Black-Box Complexities of Combinatorial Problems.** GECCO

Doerr, Kötzing, Winzen. **Too Fast Unbiased Black-Box Algorithms.** GECCO

Doerr, Winzen. **Black-Box Complexity: Breaking the O(n log n) Barrier of LeadingOnes.** EA

2012
Doerr, Winzen. **Playing Mastermind with constant-size memory.** STACS

Doerr, Winzen. **Reducing the arity in unbiased black-box complexity**. GECCO

Part 1: Complexity Theory for RSH

- Why a complexity theory for RSH?
  - Understand problem difficulty!
- How?
  - Black-box complexity!
- What can we do with that?
  - General lower bounds, thorn in the flesh
- Different notions of black-box complexity

Why a Complexity Theory for RSH?

- Understand problem difficulty!
  - Randomized search heuristics (RSH) like evolutionary algorithms, genetic algorithms, ant colony optimization, simulated annealing, … are very successful for a variety of problems.
  - Little general advice which problems are suitable for such general methods
  - Solution: Complexity theory for RSH
- Take a similar successful route as classical algorithmics!
  - Algorithmics: Design good algorithms and analyze their performance
  - Complexity theory: Show that certain things are just not possible
  - The interplay between the two areas proved to be very fruitful for the research on classic algorithms

Algorithms vs. Complexity Theory for RSH – An Example

Algorithm Analysis: Prove how a certain algorithm solves a particular problem.

The (1+1) EA finds a minimum spanning tree with an expected number of $O(m^2 \log(m\, w_{\max}))$ fitness evaluations.

Complexity Theory: What can the best possible algorithm for this problem do **or not**.

No RSH can solve the Needle problem in an expected number of less than $(2^n+1)/2$ fitness evaluations.

- Bottom line: Spanning tree is easy for RSH, the Needle problem not.

Reminder: Classic Complexity Theory

- General approach: Complexity (difficulty) of a problem := Performance of the best algorithm on the hardest problem instance
- Example: "Sorting n numbers needs $\Theta(n \log(n))$ pair-wise comparisons."
  - Problem: "Sorting an array of n numbers"
  - Instance (input to algorithm): An (unsorted) array of n numbers
  - Algorithms: All that run on a Turing machine
  - Performance (cost) measure: Number of pair-wise comparisons
    - $T(A,I)$ = number of comparisons performed when algorithm A runs on instance I
  - Theorem: "Complexity of sorting = $\min_A \max_I T(A,I) = \Theta(n \log(n))$."
- How does this work for RSH?
  - Algorithms = RSHs, Performance = number of fitness evaluations, …

Complexity Theory for RSH

- Algorithms: Randomized search heuristics (RSH)
  - may generate solutions and query their fitness
  - no explicit access to the problem description
  - → *black-box optimization algorithm*
- Performance measure $T(A,I)$ = expected number of *fitness evaluations* until algorithm A running on instance I queries an optimum of I
- Black-box complexity: Expected number of fitness evaluations the best black-box algorithm needs to query the optimum of the hardest instance.
  - $\min_A \max_I T(A,I)$

"How many search point have to be evaluated to find the optimum."

BBC: What Can We Do With It?

- Black-box complexity: Expected number of fitness evaluations the best black-box algorithm needs to query the optimum of the hardest instance.
  - $\min_A \max_I T(A,I)$
- 3 uses:
  - Measure for problem difficulty [that's how we designed the definition]
  - Universal lower bounds [next slide]
  - A thorn in the flesh [next to next slide]

BBC: Universal Lower Bounds

- Black-box complexity: Expected number of fitness evaluations the best black-box algorithm needs to query the optimum of the hardest instance.
  - $\min_A \max_I T(A,I)$
- Follows right from the definition: The black-box complexity is a lower bound on the performance of any RSH!
  - BBC := $\min_A \max_I T(A,I) \le \max_I T(B,I)$ = performance of B
- Example:
  - Theorem [DJTW'02]: The black-box complexity of the needle function class is $(2^n+1)/2$.
  - Consequence: No RSH can solve the needle problem in sub-exponential time.
  - One simple proof replaces several proofs for particular RSH ☺

BBC: A Thorn in the Flesh

- If the black-box complexity is lower than what current best RSH achieve, you should wonder if there are better RSH for this problem!
- Example: OneMax functions
  - for all "bit-strings" $z \in \{0,1\}^n$ let f_z: $\{0,1\}^n \to \{0,\ldots,n\}$; $x \mapsto$ "number of positions in which x and z agree"
    - all f_z have a fitness landscape equivalent to the classic OneMax function (counting the number of ones in a bit-string).
  - Theorem [many, see later]: The black-box complexity of the class of all OneMax functions is $\Theta(n / \log(n))$.
  - But: All standard RSH need at least $\Omega(n \log(n))$ time!
  - Are there better RSH that we overlooked?
- Same motive as in classical theory: $n \times n$ matrix multiplication can be done in time $O(n^{2.3727})$, only lower bound is $\Omega(n^2)$.

Alternative Black-box Models

- Previous slide: "Are there better RSH?"
- Alternative answer: The black-box model allows too powerful (unnatural) algorithms.
- Next x slides: Discuss alternative black-box models
  - very active research area in the last 3 years
  - no definitive answer
- Common theme: Instead of allowing all black-box optimization algorithms, only regard a restricted class!
  - restricted class should include most classic RSH

Alternative 1: Unbiased BBC

- Lehre&Witt (GECCO'10 theory track best paper award):
  - allow only unbiased variation operators: treat all bit-positions $(1, \ldots, n)$ and the two bit-values (0, 1) equally!
  - equivalent: if σ is an automorphism of the hypercube, then the probability that y is an offspring of $x_1, \ldots, x_k$ must be equal to the probability that $\sigma(y)$ is an offspring of $\sigma(x_1), \ldots \sigma(x_k)$
  - Observation: Most RSH are unbiased
    - exception: one-point crossover
  - Result: The unbiased, mutation-only BBC of OneMax is $\Theta(n \log(n))$
    - as observed for random local search, (1+1) EA, …
- Anti-result [DKW'11]: Also the TRAP_k function has an unbiased, mutation-only BBC of $\Theta(n \log(n))$.
  - contrasts the $\Omega(n^k)$ performance of all classic RSH
- Interesting [DJKLW'11]: Unbiased 2-ary BBC of OneMax: $O(n)$.

Crossover helps?

Alternative 2: Ranking-Based BBC

- D&Winzen (CSR'11), suggested by Niko Hansen: ranking-based
  - do not regard the absolute fitness values, but make all decisions dependent only on how fitnesses of search points compare!
  - Observation: Many RSH follow this scheme
    - exception: fitness-proportionate selection
  - Bad news: OneMax has a ranking-based BBC of $\Theta(n / \log(n))$ ☹
  - Good news: For BinaryValue…
    - BBC: $\log(n)$
    - ranking-based BBC: $\Omega(n)$
    - many RSH: $\Theta(n \log n)$
  - Open problem: Partition…
    - BBC: $O(n)$, heavily exploints absolute fitness values
    - Unbiased BBC: Maybe exponential?

Alternative 3: Memory-Restricted BBC

- Droste, Jansen, Wegener (Theor. Comput. Syst. 2006):
  - suggest to restrict the memory: store only a fixed number of search points and their fitness
  - inspired by bounded population size
  - conjecture: with memory one, the BBC of OneMax becomes the desired $\Theta(n \log(n))$
- D&Winzen (STACS'12): Disprove conjecture.
  - Even with memory one, the BBC of OneMax is $\Theta(n / \log(n))$. [I'll give a proof in the second part of the tutorial]

Summary Alternative BBC Models

- Different models:
  - unrestricted (classic)
  - unbiased
  - ranking-based
  - memory-restricted
- None is yet "the ultimate complexity notion" for RSH
- Each expanded our understanding
  - what makes a problem hard
  - what makes a RSH powerful
- Many open problems…

Summary Part 1

- Black-box complexity (BBC): "Minimum number of search points that have to be evaluated to find the optimum"
  - Expected number of fitness evaluations the best black-box algorithm needs to query the optimum of the hardest instance.
  - $\min_A \max_I T(A,I)$
- Uses:
  - Measure of problem difficulty
  - Universal lower bounds
  - Thorn in the flesh
- Particular problem: What is the most useful class of black-box algorithms to be regarded?

Part 2: Tools and Techniques

Plan for the 2nd part of this tutorial:

- Explain, why BBC and guessing games are almost the same
- Use the language of guessing games to demonstrate some techniques
  - Random guessing: The BBC of OneMax or "how to play Mastermind with two colors?"
  - A simple "information theoretic" lower bound
  - Clever guessing:
    - Mastermind with n colors [intermediate summary "tools and techniques"]
    - Memory-restricted BBC of OneMax = Mastermind with 2 rows
- A game derived from BBC studies ☺

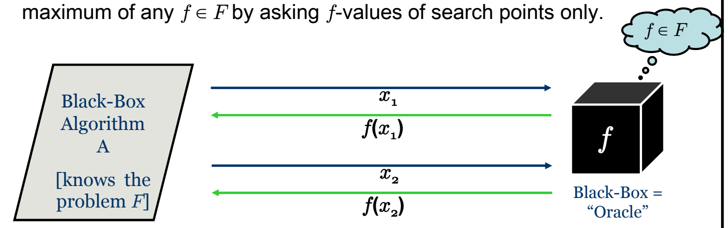
A Formal Definition of BBC

- Optimization problem: A set F of functions f: $\{0,1\}^n \to \mathbb{R}$
  - Aim is to find the maximum of a given $f \in F$.
  - Language:
    - An $f \in F$ is called an "instance of F"
    - $\{0,1\}^n$ "search space"
    - $x \in \{0,1\}^n$ "search point"
  - Example "Maximum Clique": For each graph G on the vertex set $\{1,\dots,n\}$, $f_G(x)$ is the size of the vertex set represented by x, if this is a clique in G, and 0 otherwise. $F := \{f_G \mid G$ a graph with vertices $1,\dots,n\}$.

- A black-box algorithm for F: A randomized algorithm that finds the maximum of any $f \in F$ by asking f-values of search points only (no explicit access to the instance, e.g., the graph G in the clique example).

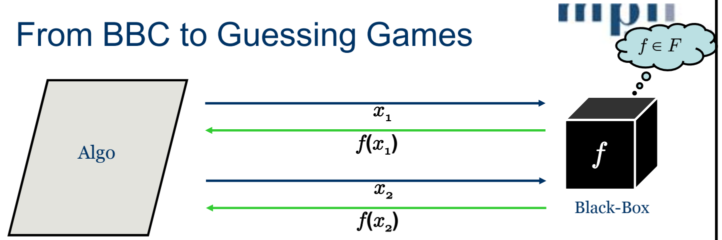
A Formal Definition of BBC

- A black-box algorithm for F: A randomized algorithm A that finds the maximum of any $f \in F$ by asking f-values of search points only.



- Performance $T(A,f)$ of A for $f \in F$: Expected time until an x with $f(x)$ = OPT(f) is queried
- Performance $T(A,F)$ of A on F: $\max_{f \in F} T(A,f)$
- BBC of F: $\min_A T(A,F)$, where A runs over all black-box algorithms for F

From BBC to Guessing Games



- Guessing game:
  - BlackBox chooses a hidden $f \in F$.
  - Algo tries to guess an x with $f(x)$ maximal
  - For each incorrect guess, BlackBox tells $f(x)$ to Algo
- Optimal strategy for Algo = optimal black-box algorithm
- Optimal strategy for black-box = "most difficult" $f \in F$
- Optimal number of rounds in the game = BBC(F)

Classic Guessing Game: Mastermind

- 2-player game
  - *CodeMaker* hides a ~~4~~ n-digit ~~6~~ k-color code C.
  - *CodeBreaker* tries to guess it using few guesses
- Guess: Some color code G
- Answer:
  - Number of positions in which C and G agree ("black answer-pegs" [here: red])
  - ~~Number of additional code letters that occur in a wrong position ("white pegs")~~

2-Color Mastermind = BBC(OneMax)

- OneMax test function: f: $\{0,1\}^n \to \{0,\ldots,n\}$; $x \mapsto$ "number of ones in x"
  - easy to find the unique global optimum (1,…,1).
  - RLS, (1+1) EA, … do this in $\Theta(n \log n)$ time.
- (Generalized) OneMax function, OneMax problem:
  - For each $z \in \{0,1\}^n$, let
    f_z: $\{0,1\}^n \to \{0,\ldots,n\}$; $x \mapsto$ "number of bits in which x and z agree"
  - All f_z have isomorphic fitness landscapes
  - OneMax problem: $F := \{f_z \mid z \in \{0,1\}^n\}$, the set of all OneMax functions
- Observation: Mastermind with the two "colors" 0 and 1 corresponds to the black-box complexity BBC(F)

Mastermind: 3 (?) Results

- $\Theta(n / \log n)$ guesses sufficient&necessary for $k = 2$ (BBC of OneMax)
  - Anil, Wiegand: "Black-box search by elimination of fitness functions". *Foundations of Genetic Algorithms* (FOGA) (2009)
- $\Theta(n \log k / \log n)$ for $k \leq n^{1-\varepsilon}$
  - Chvátal: "Mastermind". *Combinatorica* (1983)
- $\Theta(n / \log n)$ for $k = 2$
  - Erdős, Rényi: "On two problems in information theory". *Magyar Tud. Akad. Mat. Kutató Int. Közl* (1963)

Proof: Random Guessing

- CodeBreaker's strategy:
  - Guess $\Theta(n / \log n)$ random codes.
  - Look at all answers.
  - With high probability, no secret code other than the true one leads to these answers [elementary, straight-forward computation]
- Comments:
  - *Erdős probabilistic method* at its best.
  - Best possible (apart from constant factors hidden in $\Theta(\ldots)$)
  - Note: Non-adaptive strategy – questions do not depend on previous questions and answers.

A General Lower Bound

- [DJW'06, in the language of games] Consider a guessing game such that
  - there are s different secrets
  - each query has at most $k \geq 2$ different answers.

  Then the expected number Q of queries necessary to find the secret is at least $(\log_2(s) / \log_2(k)) - 1 = \log_k(s) - 1$.
- Information theoretic view: To encode the secret in binary, you need $\log_2(s)$ bits. Each answer can be encoded in $\log_2(k)$ bits. If Q rounds suffice, $Q \log_2(k)$ bits could encode the secret. [1)]
- Game-theoretic view: In the game tree, each node has at most k children. Hence at height Q, there are at most k^Q nodes. If s is bigger, then at some nodes, more secrets are possible. [1)]

[1)] Argument correct for deterministic strategies. For randomized ones, in addition, Yao's minimax principle is needed.

Back to 2-Color Mastermind…

- Lower bound: $(1 + o(1))\, n / \log_2(n)$
  - Argument: 2^n possible secrets, n+1 possible answers
    → general lower bound: $\log_2 (2^n) / \log_2 (n+1) = (1+o(1))n / \log_2 (n)$
  - Information theoretic view: “learn at most $\log_2 (n)$ bits per question”
- Upper bound computed precisely: $(2 + o(1))\, n / \log_2 (n)$
  - Weaker by a factor of 2
  - Reason (informal): Typically, a random question yields an answer between $n/2 - \Theta(\sqrt{n})$and $n/2 + \Theta(\sqrt{n})$.
    - “learn $\log_2 (\Theta(\sqrt{n})) \approx (1/2) \log_2 (n)$ bits per question”
- Big open problem (already mentioned in the Erdős-Rényi paper): What is the correct bound? Can you ask better questions?

Clever Guessing: Mastermind for $k = n$?

- Random guessing (Chvátal): $\Theta(n \log(n))$ needed and sufficient.
- Informal justification:
  - The expected answer to a random question is 1.
  - “learn only a constant number of bits per question”.
  - Information theory: $\log(n^n)/\log(\text{constant}) = n \log(n)$ questions
- Can we ask better questions?
  - Info-theory argument: We need to “learn more bits per question”
  - Problem: For the first question, the expected answer is 1, no matter what we ask (→ learn constant number of bits ☹)
  - If something works, it must be adaptive: Current question uses previous answers!

Clever Guessing: First Step

- Story-line so far: Adaptively ask clever questions!
  - Difficulty: How to use previous answers?
- One idea (inspired by Goodrich (IPL 2009)):
  - If you get the answer “0”, then for each position you know one color that does not appear there
  - basically reduces the number of colors by one
  - future questions: only use possible colors
  - good news: the answer “0” is not too rare
    - for $k = n$ colors, the probability that a random guess gets a “0”-answer, is $(1 - (1/n))^n \approx 1/e \approx 0.37$

Clever Guessing: Reduce the Colors

- Story-line: Adaptively ask clever questions!
  - Plan: Get a “0”-answer and get rid of one color per position.
- Lemma: For k colors and n positions, the probability that a random guess is answered “0”, is $(1 - (1/k))^n \geq 4^{-n/k}$.
- Rough estimate: Reducing the number of colors from n to $8n / \log\log(n)$ takes time n $4^{-n/(8n / \log\log(n))} = n (\log n)^{1/2}$.
- With only $8n / \log\log(n)$ colors possible at each position, a random guess has an expected answer of $\log\log(n)/8$
  → “learn $\Theta(\log\log\log(n))$ bits” [can be made precise]
- “Theorem”: $O(n \log(n) / \log\log\log(n))$ questions suffice!

Clever Guessing: Reduce the Colors (2)

- Story-line: Adaptively ask clever questions by reducing the number of colors (by getting a “0”-answer)
  - gains so far: a logloglog(n) factor ☹
- Reducing the number of colors from k to k-1 per position:
  - so far: get a “0”-answer after at most $4^{n/k}$ random guesses
  - Example: k = n/100.
    - Random guess has an expected answer of 100.
    - Time to wait for a “0” is (1+o(1)) e^{100}.
    - → Waiting for something quite rare ☹
  - Better: Partition the n positions into blocks of size n/100 and ask randomly in each block (fill up the rest with dummy colors)
    - expected contribution per block: 1
    - waiting time for a “0” in a block: constant

Clever Guessing: Reduce the Colors (3)

- Story-line: Adaptively ask clever questions by reducing the number of colors.
  - So far: Ask randomly and hope for a “0”
- Improved reducing the number of colors from k to k-1:
  - Partition the n positions into n/k blocks of roughly equal size.
  - For each block:
    - Ask random colors in the block, put a dummy color in the rest
    - expected waiting time for a “0”: at most 4
  - Total expected waiting time: 4 n/k [previously: $4^{n/k}$] ☺☺☺
- Total time to reduce the number of colors from k to k/2:
  - at most (k/2) 4 n / (k/2) = 4n

Clever Guessing: Reduce the Colors (4)

- Story-line: Adaptively ask clever questions.
  - Clever color reducing: From k to k/2 colors in 4n queries
- Goodrich 2009: log(n) times halving the colors finds the secret code in O(n log n) questions [apart from constants, the same bound as Chvátal]
- We [D., Spöhel, Thomas, Winzen]:
  - Do the halving trick $\sqrt{\log n}$ times
    → n / $2^{\sqrt{\log n}}$ colors possible at each position
  - Then do random guesses (using only possible colors)
    - expected answer: $2^{\sqrt{\log n}}$
      → “learn log($2^{\sqrt{\log n}}$) = $\sqrt{\log n}$ bits per question”
  - Theorem: Solve Mastermind with k=n colors in O(n $\sqrt{\log n}$) questions ☺

Intermediate Summary: Methods

- Information theoretic argument:
  - If for each query only k different answers exist and if F contains s functions with distinct unique optima, then the black-box complexity of F is at least ($\log_2(s)$ / $\log_2(k)$) – 1.
- Random guessing:
  - Often, a small number of random guesses together with the answers received uniquely determine the solution.
  - “Information theoretic hand-waiving”: If a random query typically sees k answers each with probability at least Θ(1/k), then around $\log_2(s)$/ $\log_2(k)$ question might suffice.
- Clever guessing: To get a better bound, you have to ask questions that reveal more information (example: reducing the colors in MasterMind).

A Second Example of "Clever Guessing"

- Original problem: Memory-restricted BBC of OneMax
  - Memory-restriction: From one iteration to the next, the BB-algorithm may only store k search points together with their fitness.
  - Conjecture [DJW'06]: For k = 1, the BBC becomes the $\Theta(n \log n)$ we know from the (1+1) EA.
- Transfer to guessing games [easy to see]:
  - This BBC problem is equivalent to Mastermind with two rows only.
- Theorem [DW'12]: You can win 2-row Mastermind with $O(n / \log n)$ queries.
  - Details: next few slides.
- Corollary: The memory-1 restricted BBC of OneMax is $\Theta(n / \log n)$.

Details: Two Rows Suffice!

- Result: On a board with two rows, you can still find the secret code with $O(n / \log n)$ guesses!

- Precise rules:
  - We start the game with an empty board
  - If there is an empty row, CodeBreaker can enter a guess, which will be answered by CodeMaker
  - If there is no empty row, CodeBreaker must empty one of the two rows and *forget the content*.
- Theorem: CodeBreaker has a strategy that
  - finds the secret code in $O(n / \log n)$ rounds
  - uses two rows only (all actions depend solely on these rows).

Fewer Rows: Proof Ideas

- Original Mastermind: Guess $\Theta(n / \log n)$ random codes. Store all guesses and answers on the board. Think.
  - Needs $\Theta(n / \log n)$ rows.
- 3 ingredients of our proof:
  - Find parts of the code: Determine $\Theta(n^\varepsilon)$ code letters with $\Theta(n^\varepsilon / \log n)$ relatively random guesses (ε constant)
    - Do this $n^{1-\varepsilon}$ times: find the code with $\Theta(n^\varepsilon / \log n)$ rows.
  - Determine such a part with *constant* number of rows
    - Do this $n^{1-\varepsilon}$ times: find the code with $\Theta(1)$ rows.
  - Do everything in *two* rows

Proof Idea (1): Find Parts of the Code

- Lemma:
  - Let $B \subseteq [n]$, $|B| = n^\varepsilon$. "part"
  - Let $G_1, G_2, \ldots$ be $\Theta(n^\varepsilon / \log n)$ guesses such that
    - G_i is random in positions in B
    - All G_i are equal in positions in $[n] \setminus B$
  - Then with high probability these guesses and answers determine the secret code in B.
- Argument:
  - Basically, we play the game in B (and use the previous proof)
  - Only difficulty: The answers we get "are not for B only", but for the whole guess
    - Same deviation for all guesses
  - Some maths: Not a problem, guesses also determine deviation ☺

Proof Idea (2): Same with $O(1)$ Rows

- Plan: Simulate the previous slide in $O(1)$ rows
- Example: Find the first $L = \Theta(n^\varepsilon / \log n)$ code letters
  - $B_1 := L$ random letters.
  - Guess B_1 1…1 in row 1 and learn answer A_1.
  - Guess B_1 A_1 1….1 in row 2 and ignore answer
  - $B_2 := L$ random letters
  - Guess B_2 1…1 in row 1 and learn answer A_2
  - Guess B_1 A_1 B_2 A_2 1…1 in row 3 and ignore answer
  - …
- General:
  - Do an honest guess as on the previous slide.
  - Use the next guess to store guess+answer+what you learned before.
- Needs 3+ rows: current guess + old storage → new storage

"A_1": Suitably encoded with $O(\log n)$ of letters

Proof Idea (3): Two Rows Only

- Difficulty:
  - To enter a new guess, one of the two rows must be emptied
  - You must store and guess in the same row
    - Problem: Storage influences CodeMaker's answers!
  - All control information must also be stored in this one row
    - what is the block I'm just optimizing?
    - what am I currently doing (guessing, storing, finding the unique solution, finding the last few letters in a different way…)
- Solution:
  - technical.
  - read the paper at STACS'12 or `arxiv.org/abs/1110.3619`.

Summary: Memory-BBC of OneMax

- Result: The complexity of Mastermind remains at $\Theta(n / \log n)$ guesses even if we allow only two rows.
  - Key proof argument: Clever guesses inspired by random guesses
- Open problems / future work:
  - Our proof works for any constant number of colors – what happens for larger numbers of colors?
  - constant factors: "what's hidden in the Θ(…)"
    - does a memory restriction lose us a constant factor?

Finally: A New Guessing Game

- So far: BBC is strongly related to guessing games
  - In particular: BBC(OneMax) ≈ Mastermind
  - Therefore: Use game theoretic arguments to solve BBC problems
- Now [next few slides]: Use BBC problems to derive a fun game ☺
  - LeadingOnes Game

LeadingOnes Test Functions

- Classic test function:
  - LeadingOnes: $\{0,1\}^n \rightarrow \{0,\ldots,n\}$; $x \mapsto \max\{i \in \{0,\ldots,n\} \mid x_1 = \ldots = x_i = 1\}$
    - "how many bits counted from the left are one"
  - Unique optimum (1,…,1)
  - "Harder than OneMax": Each non-optimal solution has only one superior Hamming neighbor
- LeadingOnes function *class* LO_n:
  - Let σ be a permutation of $\{1,\ldots,n\}$
  - Let $z \in \{0,1\}^n$ ("target string")
  - $f_{z\sigma} : \{0,1\}^n \rightarrow \{0,\ldots,n\}$; $x \mapsto \max\{i \in \{0,\ldots,n\} \mid x_{\sigma(1)} = z_{\sigma(1)}, \ldots, x_{\sigma(i)} = z_{\sigma(i)}\}$
    - "how many bits, counted in the order of σ, are as in z
  - same fitness landscape as LeadingOnes

The LeadingOnes Game

- Transfer the BBC(LO_n) problem into a game:
- CodeMaker: Picks a secret code z and a secret permutation σ
- Round:
  - CodeBreaker guesses a bit-string $x \in \{0,1\}^n$
  - CodeMaker's answer: $f_{z\sigma}(x)$ = "how many code letters in the order of σ are correct?"
- How many rounds until CodeBreaker guesses the secret code z?
- Practice: Fun to play with n=5 or n=6 [and that's the message of this slide]
- Theory: next few slides, fun as well, but doesn't help you play the actual game

Black-Box Complexity of LeadingOnes

- Reminder: LO_n consists of all functions
  - $f_{z\sigma} : \{0,1\}^n \rightarrow \{0,\ldots,n\}$; $x \mapsto \max\{i \in \{0,\ldots,n\} \mid x_{\sigma(1)} = z_{\sigma(1)}, \ldots, x_{\sigma(i)} = z_{\sigma(i)}\}$
- Black-box complexity of LO_n, lower bound
  - $\Omega(n)$, because you need $\Theta(n)$ fitness evaluations even if σ = id
- Black-box complexity of LO_n, upper bounds
  - $O(n^2)$, run-time of RLS, (1+1) EA, …
  - $O(n \log(n))$: determine "the next bit" with $\log(n)$ queries by simulating binary search (more details next slide)
    - Would be a natural lower bound:
      - "next bit"-position is a number in $\{1,\ldots,n\}$, coding length $\log(n)$
      - a typical query teaches you a constant amount of information
  - DW (EA'11): $O(n \log(n) / \log\log(n))$ is enough…

The BinarySearch Trick

- Assume that you have a solution x with $f_{z\sigma}(x) = k$ and you know which k bit-positions are responsible for this. Denote by I the remaining bit-positions.
- `While |I| > 1 do`
  - `Choose J ⊆ I with |J| ≈ |I|/2`
  - `Obtain y from x by flipping the bits in J`
  - `If` $f_{z\sigma}(y)$ `>` $f_{z\sigma}(x)$ `then I := J`
  - `else I := I \ J`
- Determines "the next bit" with at most $\log_2(n)$ fitness queries
  - $n \log_2(n)$ queries suffice to optimize LO_n
- How can we do better?

Proving $O(n \log(n) / \log\log(n))$: Outline

- Assume that you have a solution x with $f_{z\sigma}(x) = k$ and you know which k bit-positions are responsible for this. Denote by I the remaining bit-positions. Let $L := \log(n)^{1/2}$
- Step 1: Use $L^2 = \log(n)$ iterations to find a y with $f_{z\sigma}(y) = k + L$
  - Flip the bits in I with probability $1/L$, accept if improvement
  - Note: We don't learn which L bit-positions lead to the improvement!!!
- Step 2: Use $\log(n)^{3/2} / \log\log(n)$ queries to determine the L bit-positions
  - In y, flip the I-bits with probability $1/L$. Do so $\log(n)^{3/2} / \log\log(n)$ times.
  - Look at all outcomes with fitness $k+j$ and find out bit number $k+j+1$.
  - With high probability, the $\log(n)^{3/2} / \log\log(n)$ samples suffice to learn all L bit-positions
- Step 1+2: $\log(n)^{3/2} / \log\log(n)$ fitness evaluations to gain $\log(n)^{1/2}$ bits…

Final Summary ☺

- Black-box complexity: Expected number of fitness evaluations the best black-box algorithm needs to query the optimum of the hardest instance.
  - $\min_A \max_I T(A,I)$
  - Note: lower bound on the performance of any EA, ACO, …
- Strongly related to guessing games
  - BBC(OneMax) ≈ Mastermind
  - BBC(LeadingOnes) ≈ what you should play in the next tutorial ☺
- Techniques:
  - Information theory: BBC ≥ log(|SearchSpace|) / log(|fitness_values|)
  - Random guesses: Often ≤ log(|SearchSpace|) / log(|typical_answers|)
  - Clever guesses: Be creative!