Theory of Swarm Intelligence

Dirk Sudholt

University of Sheffield, UK

Tutorial at GECCO 2012

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Swarm Intelligence

Collective behavior of a "swarm" of agents.

Examples from Nature

- dome construction by termites
- communication of bees
- ant trails
- foraging behavior of fish schools and bird flocks
- swarm robotics

Plenty of inspiration for optimization.

ACO and PSO

Ant colony optimization (ACO)

- inspired by foraging behavior of ants
- artificial ants construct solutions using pheromones
- pheromones indicate attractiveness of solution component

Particle swarm optimization (PSO)

- mimics search of bird flocks and fish schools
- particles "fly" through search space
- each particle is attracted by own best position and best position of neighbors

Theory

What "theory" can mean

- convergence analysis
- analysis of simplified models of algorithms
- empirical studies on test functions
- runtime analysis / computational complexity analysis
- ...

Example Question

How long does it take on average until algorithm A finds a target solution on problem P?

Notion of time: number of iterations, number of function evaluations

Content

What this tutorial is about

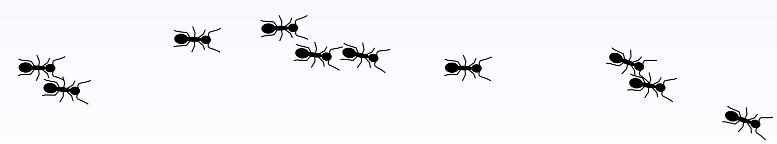
- runtime analysis
- simple variants of swarm intelligence algorithms
- insight into their working principles
- impact of parameters and design choices on performance
- what distinguishes ACO/PSO from evolutionary algorithms?
- performance guarantees for combinatorial optimization
- methods and proof ideas

What this tutorial is not about

- convergence results
- analysis of models of algorithms
- no intend to be exhaustive

Overview

Ant Colony Optimization (ACO)



Main idea: artificial ants communicate via pheromones.

Scheme of ACO

Repeat:

- construct ant solutions guided by pheromones
- update pheromones by reinforcing good solutions

Pseudo-Boolean Optimization

Goal: maximize $f \colon \{0,1\}^n \to \mathbb{R}$.

Illustrative test functions

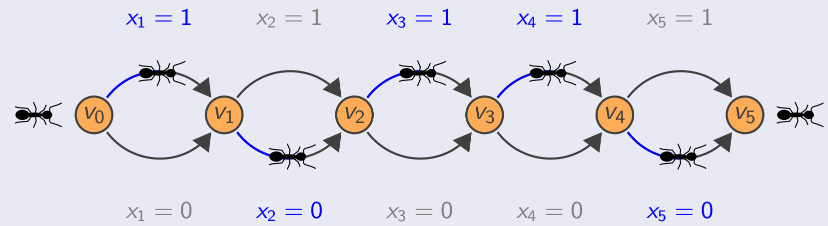
$$\textsc{OneMax}(x) = \sum_{i=1}^{n} x_i$$

$$\textsc{LeadingOnes}(x) = \sum_{i=1}^{n} \prod_{j=1}^{i} x_j$$

$$\textsc{Needle}(x) = \prod_{i=1}^{n} x_i$$

ACO in Pseudo-Boolean Optimization

Solution Construction



Probability of choosing an edge equals pheromone on the edge.

Initial pheromones: $\tau(x_i = 0) = \tau(x_i = 1) = 1/2$.

Note: no linkage between bits. No heuristic information used.

Pheromones $\tau(x_i = 1)$ suffice to describe all pheromones.

ACO in Pseudo-Boolean Optimization (2)

Pheromone update: reinforce some good solution x.

Strength of update determined by evaporation factor $0 \leq \rho \leq 1$:

$$\tau'(x_i = 1) = \begin{cases} (1-\rho) \cdot \tau(x_i = 1) & \text{if } x_i = 0 \\ (1-\rho) \cdot \tau(x_i = 1) + \rho & \text{if } x_i = 1 \end{cases}$$

Pheromone borders as in MAX-MIN Ant System (Stützle and Hoos, 2000):

$$\tau_{\min} \leq \tau' \leq 1 - \tau_{\min}$$

Default choice: $\tau_{\min} := 1/n$ (cf. standard mutation in EAs).

One Ant?

Most ACO algorithms analyzed: one ant per iteration.

One ant at a time, many ants over time.

Steady-state GA

- Probabilistic model: Population
- New solutions: selection + variation
- Environmental selection

Ant Colony Optimization

- Probabilistic model: Pheromones
- New solutions: construction graph
- Selection for reinforcement

Evolutionary Algorithms vs. ACO

MMAS* (Gutjahr and Sebastiani, 2008)

Start with uniform random solution x^* and repeat:

- Construct x.
- Replace x^* by x if $f(x) > f(x^*)$.
- Update pheromones w. r. t. x^* (best-so-far update).

Note: best-so-far solution x^* is constantly reinforced.

(1+1) EA

Start with uniform random solution x^* and repeat:

- Create x by flipping each bit in x^* independently with probability $1/n$.
- Replace x^* by x if $f(x) \geq f(x^*)$.

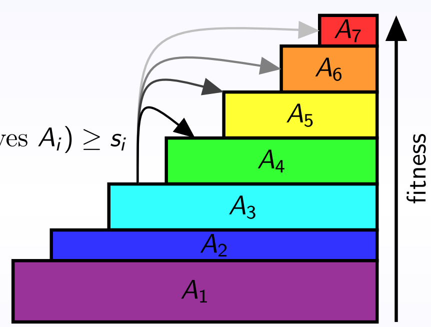
(1+1) EA: Probability of setting bit to 1 is in $\{1/n, 1-1/n\}$.

MMAS*: Probability of setting bit to 1 is in $[1/n, 1-1/n]$ (unless $\rho \approx 1$).

Overview

Fitness-level Method for the (1+1) EA

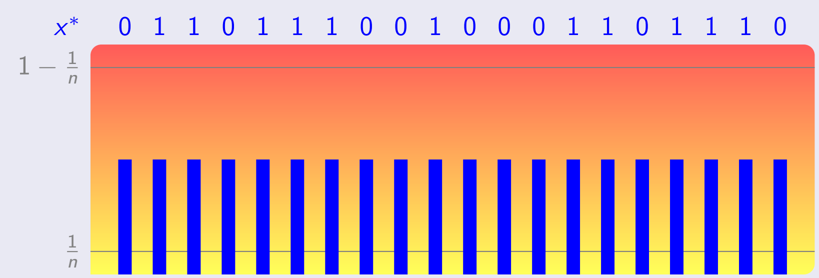
$\Pr((1+1)\text{ EA leaves } A_i) \geq s_i$



Expected optimization time of (1+1) EA at most $\sum_{i=1}^{m-1} \frac{1}{s_i}$.

MMAS*

Pheromones on 1-edges



After $(\ln n)/\rho$ reinforcements of x^* MMAS* temporarily behaves like (1+1) EA.

Fitness-Level Method with A_i = search points with i-th fitness value

(1+1) EA: $\leq \sum_{i=1}^{m-1} \frac{1}{s_i}$ MMAS*: $\leq \sum_{i=1}^{m-1} \frac{1}{s_i} + m \cdot \frac{\ln n}{\rho}$

Upper bounds: time for finding improvements + time for pheromone adaptation.

MMAS*

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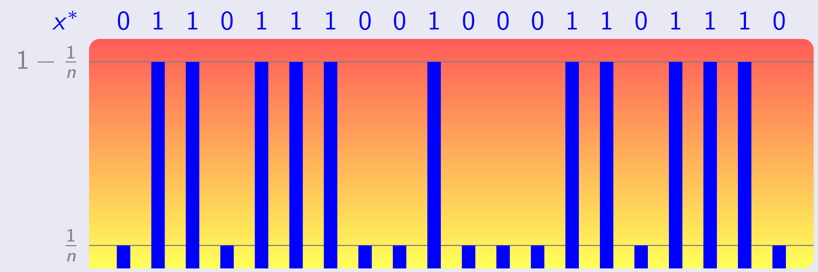

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Bounds with Fitness Levels

OneMax:

$$s_i \geq (n-i) \cdot \frac{1}{n} \cdot \left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{n-i}{en}$$

Theorem

$$\text{(1+1) EA:} \quad en \sum_{i=0}^{n-1} \frac{1}{n-i} = O(n \log n)$$

$$\text{MMAS*:} \quad en \sum_{i=0}^{n-1} \frac{1}{n-i} + n \cdot \frac{\ln n}{\rho} = O((n \log n)/\rho)$$

Bounds with Fitness Levels (2)

LeadingOnes

$$s_i \geq \frac{1}{n} \cdot \left(1 - \frac{1}{n}\right)^{n-1} \geq \frac{1}{en}$$

Theorem

$$\text{(1+1) EA:} \quad en^2 \qquad \text{MMAS*:} \quad en^2 + n \cdot \frac{\ln n}{\rho} = O(n^2 + (n \log n)/\rho)$$

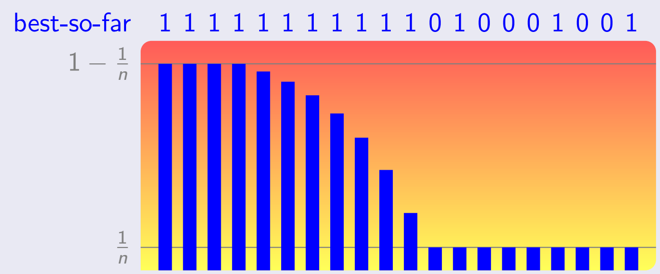
Unimodal functions with d function values:

Theorem

$$\text{(1+1) EA:} \quad end \qquad \text{MMAS*:} \quad end + d \cdot \frac{\ln n}{\rho} = O(nd + (d \log n)/\rho)$$

Layering of Pheromones for LeadingOnes

(Lower bounds on) pheromones on LeadingOnes



Improved results for

- LeadingOnes (Neumann, Sudholt, and Witt, 2009)
- shortest paths (Sudholt and Thyssen, 2012)
- Binary PSO on OneMax (Sudholt and Witt, 2010)

Overview

Strict Selection

Most ACO algorithms replace x^* only if $f(x) > f(x^*)$.

Drawback

Cannot explore plateaus.

Theorem (Neumann, Sudholt, Witt, 2009)

If $\rho \geq 1/poly(n)$ the expected time of MMAS* *on* NEEDLE *is* $\Omega(2^{-n} \cdot n^n) = \Omega((n/2)^n)$.

Define variant MMAS of MMAS* replacing x^* if $f(x) \geq f(x^*)$.
Pheromones on each bit perform a random walk.

Theorem (Neumann, Sudholt, Witt, 2009 and Sudholt, 2011)

The expected time of MMAS *on* NEEDLE *is* $O(n^2/\rho^2 \cdot \log n \cdot 2^n)$.

Mixing time estimates (Sudholt, 2011)

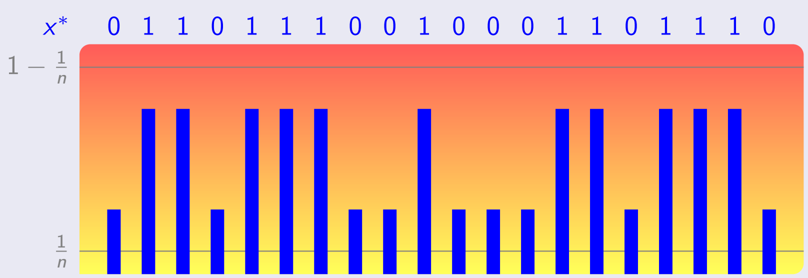
MMAS "forgets" initial pheromones on bits that have been irrelevant for the last $\Omega(n^2/\rho^2)$ steps.

MMAS and Fitness Levels

How does MMAS cope with plateaus on fitness levels?

Switching between equally fit solutions can prevent freezing.

Pheromones on 1-edges



Fitness-level method breaks down!

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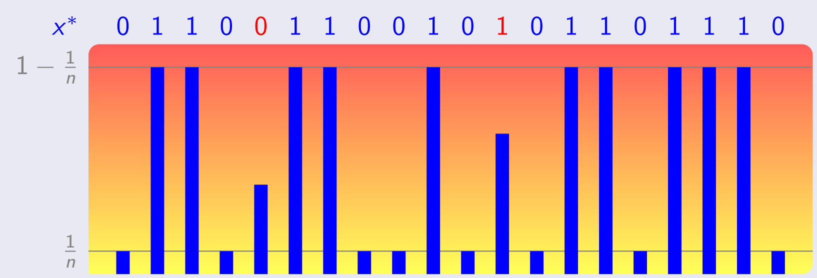


Fitness-level method breaks down!

Is this Behavior Detrimental?

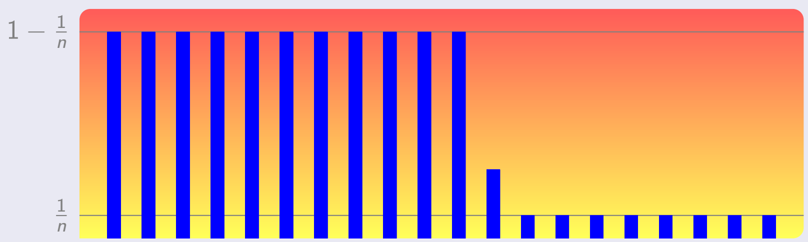
Probably not.

Theorem (Kötzing, Neumann, Sudholt, and Wagner, 2011)

$O(n \log n + n/\rho)$ on OneMax *for both* MMAS* *and* MMAS.

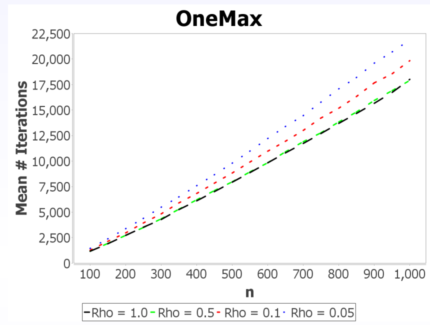
Assuming the sum of pheromones is fixed. Worst possible pheromone distribution for finding improvements on OneMax (Gleser, 1975):

Pheromones on 1-edges

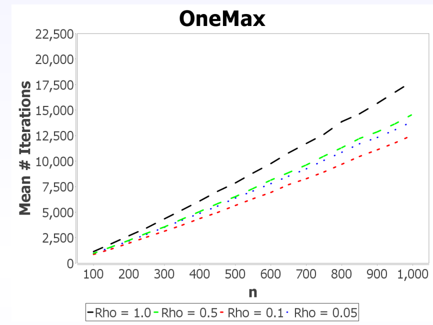


Worst case: all pheromones (but one) at borders.

Experiments (Kötzing et al., 2011)



MMAS*



MMAS

- MMAS better than MMAS*
- MMAS with $\rho < 1$ better than (1+1) EA (=MMAS at $\rho = 1$)!
- does not hold for MMAS*

Open Problem

Prove that MMAS with proper ρ is faster than MMAS* and (1+1) EA.

Overview

Iteration-Best Update

λ-MMAS$_{\text{ib}}$

Repeat:
- construct λ ant solutions
- update pheromones by reinforcing the best of these solutions

Advantages:
- can escape from local optima
- inherently parallel
- simpler ants

Iteration-Best vs. Comma Strategies

Jägersküpper and Storch, 2007

(1,λ) EA: $\lambda \geq c \log n$ necessary, even for OneMax.

If $\lambda \leq c' \log n$ then (1,λ) EA needs exponential time.

Reason: (1,λ) EA moves away from optimum if close and λ too small.

Behavior too chaotic to allow for hill climbing!

Iteration-Best on OneMax

Slow pheromone adaptation effectively eliminates chaotic behavior.

Theorem (Neumann, Sudholt, and Witt, 2010)

If $\rho \leq 1/(cn^{1/2} \log n))$ for a sufficiently large constant $c > 0$ and $\rho \geq 1/poly(n)$ then 2-MMAS$_{ib}$ optimizes OneMax in expected time $O(\sqrt{n}/\rho)$.
For $\rho = 1/(cn^{1/2} \log n)$ the time bound is $O(n \log n)$.

Two ants are enough!

Proof idea: as long as all pheromones are at least 1/3, the sum of pheromones grows steadily.

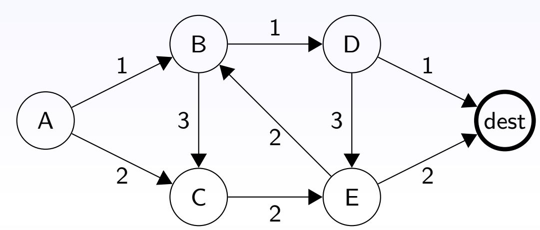
Large ρ or small λ: pheromones come crashing down to $1/n$.

Theorem

Choosing $\lambda/\rho \leq (\ln n)/244$, the optimization time of λ-MMAS$_{ib}$ on every function with a unique optimum is $2^{\Omega(n^{\varepsilon})}$ for some constant $\varepsilon > 0$ w. o. p.

Overview

ACO System for Single-Destination Shortest Path Problem



MMAS$_{SDSP}$

For each vertex u the ant

- memorizes and keeps track of its best-so-far path
- constructs a simple path from u to the destination
- updates pheromones on edges $(u, \cdot)$ (local pheromone update)

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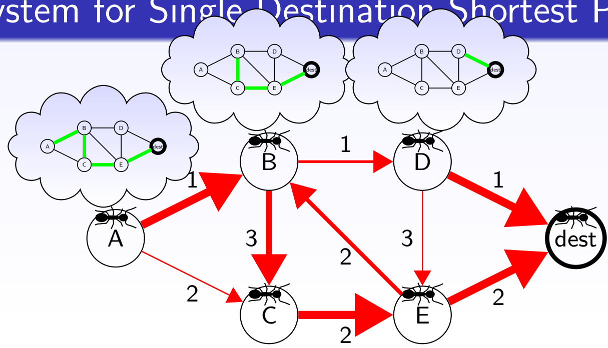

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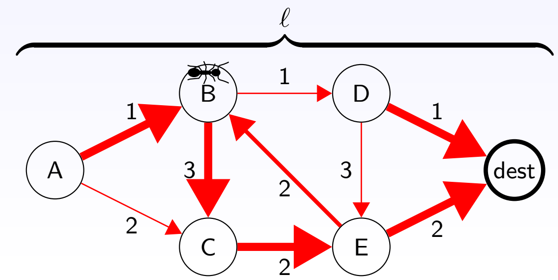
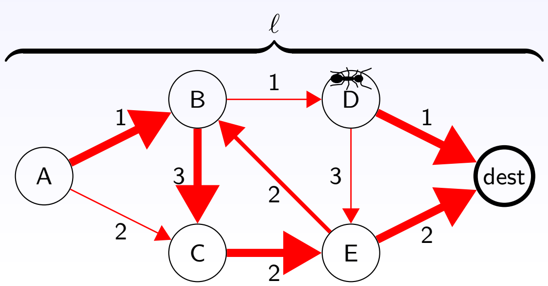
Shortest Paths Propagate Through the Graph



Let $\tau_{\min} := 1/(\Delta\ell)$. Consider vertex u such that all ants on its shortest paths have found shortest paths and adapted their pheromones.

Shortest Paths Propagate Through the Graph



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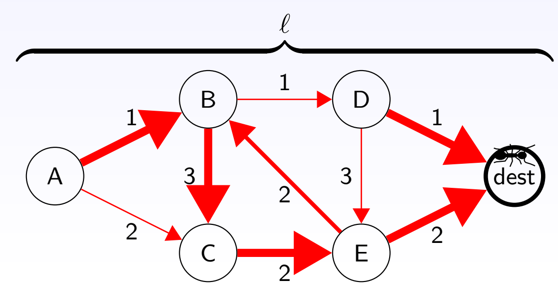

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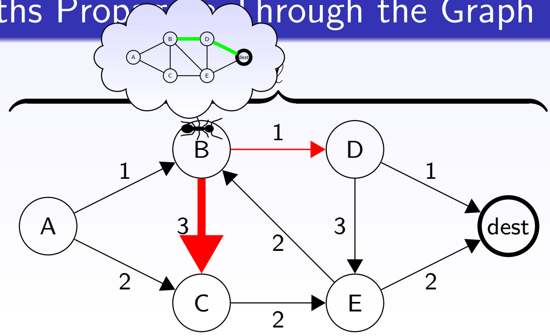

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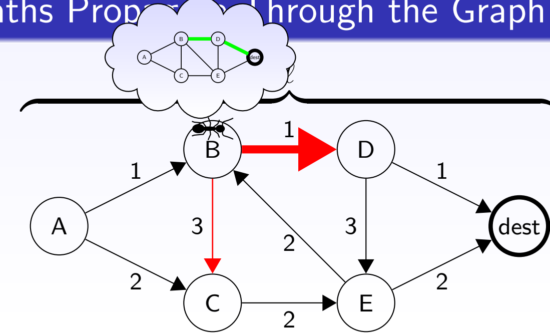

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Expected time until ant at u has done the same $\leq 2e/\tau_{\min} + \ln(1/\tau_{\min})/\rho$.

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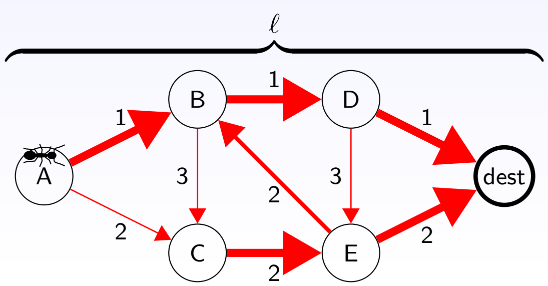

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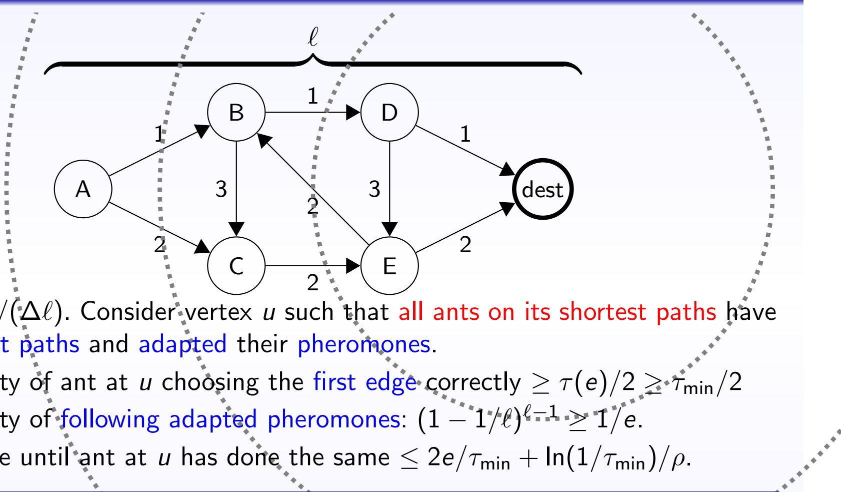
Upper bounds for MMAS$_{\text{SDSP}}$ (Sudholt and Thyssen, 2012)

- Consider all vertices sequentially: $O(n\Delta\ell + n\ln(\Delta\ell)/\rho)$.

Shortest Paths Propagate Through the Graph



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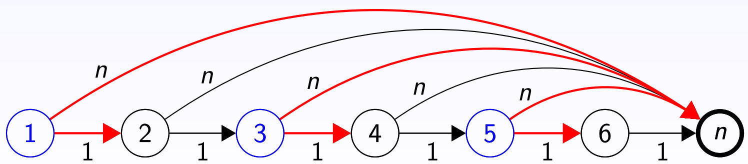
Upper bounds for MMAS$_{\text{SDSP}}$ (Sudholt and Thyssen, 2012)

- Consider all vertices sequentially: $O(n\Delta\ell + n\ln(\Delta\ell)/\rho)$.
- Slice graph into "layers" and use pheromone layering: $O(\Delta\ell^2 + \ell/\rho)$.

A Worst-Case Graph



Expected time $O(\Delta\ell^2 + \ell/\rho)$ and $\Omega\left(\Delta\ell^2 + \frac{\ell}{\rho\log(1/\rho)}\right)$

- #wrong vertices decreases on average by $O(\rho\log(1/\rho))$.
- expected time for decrease of $\Omega(\ell) \Rightarrow \Omega\left(\frac{\ell}{\rho\log(1/\rho)}\right)$.

After pheromone adaptation still $\Omega(\ell)$ wrong vertices left

- #wrong vertices decreases on average by $O(\tau_{\min})$
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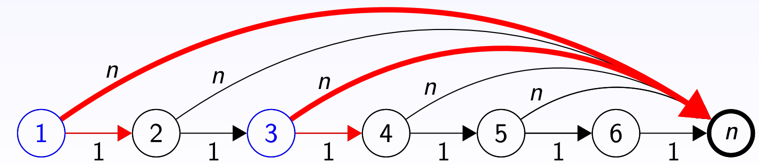

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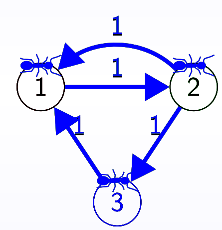
Overview

All-Pairs Shortest Path Problem

Use distinct pheromone function $\tau_v \colon E \to \mathbb{R}_0^+$ for each destination v:



A Simple Interaction Mechanism

Path construction with interaction

For each ant traveling from u to v
- with prob. $1/2$
  - use τ_v to travel from u to v
- with prob. $1/2$
  - choose an intermediate destination $w \in V$ uniformly at random
  - uses τ_w to travel from u to w
  - uses τ_v to travel from w to v

Speed-up by Interaction

Theorem

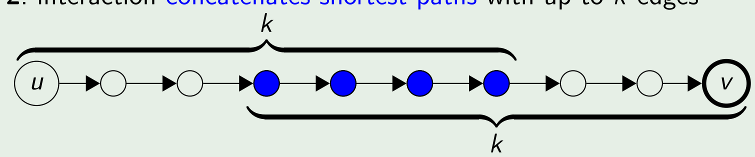
If $\tau_{\min} = 1/(\Delta\ell)$ and $\rho \leq 1/(23\Delta \log n)$ the number of iterations using interaction w. h. p. is $O(n \log n + \log(\ell) \log(\Delta\ell)/\rho)$.

Possible improvement: $O(n^3) \to O(n \log^3 n)$

Proof Sketch

Phase 1: find all shortest paths with one edge
slow evaporation $\longrightarrow$ near-uniform search

Phase 2: interaction concatenates shortest paths with up to k edges



$\longrightarrow$ find shortest paths with up to $3/2 \cdot k$ edges.

Note: slow adaptation helps!

Overview

Stochastic Shortest Paths

Directed acyclic graph $G = (V, E, w)$ with non-negative weights

For a path $p = (e_1, \ldots, e_\ell)$

$w(p) := \sum_{i=1}^{\ell} w(e_i)$ is the real length of p.

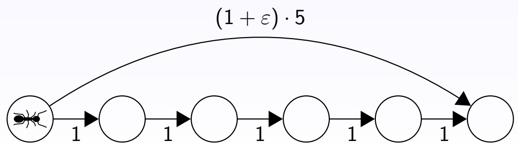
$\tilde{w}(p) := \sum_{i=1}^{\ell} (1 + \eta(e_i)) \cdot w(e_i)$ is the noisy length of p.

Goal

Find or approximate real shortest paths despite noise.

Ants Become Risk-Seeking

Every edge has independent noise $\sim \Gamma(k, \theta)$.



Algorithm: $\text{MMAS}_{\text{SDSP}}$, no re-evaluation of best-so-far paths.

Ant tends to store path with high variance as best-so-far path.

Theorem (Sudholt and Thyssen, 2012)

There is a graph where with probability $1 - \exp(-\Omega(\sqrt{n}/\log n))$ $\text{MMAS}_{\text{SDSP}}$ does not find a $(1 + k\theta/3)$-approximation for all vertices within e^{cn} iterations.

Re-evaluate best-so-far paths: Doerr, Hota, and Kötzing (this GECCO).

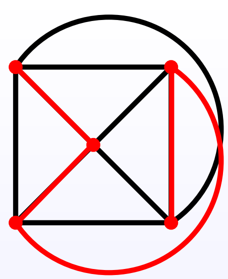
Overview

Broder's Algorithm

Problem: Minimum Spanning Trees

Consider the input graph itself as construction graph.

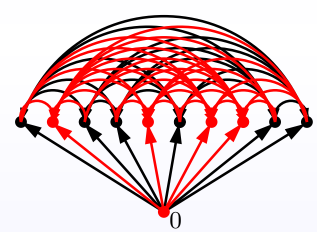
Spanning tree can be chosen uniformly at random using random walk algorithms (e. g. Broder, 1989).



Reward chosen edges ⇒ next solution will be similar to constructed one
But: local improvements are possible

Component-based Construction Graph

- Vertices correspond to edges of the input graph
- Construction graph $C(G) = (N, A)$ satisfies $N = \{0, \ldots, m\}$ (start vertex 0) and $A = \{(i,j) \mid 0 \le i \le m, 1 \le j \le m, i \ne j\}$.



For a given path $v_1, \ldots, v_k$ select the next edge from its neighborhood $N(v_1, \ldots, v_k) := (E \setminus \{v_1, \ldots, v_k\}) \setminus \{e \in E \mid (V, \{v_1, \ldots, v_k, e\})$ contains a cycle$\}$ (problem-specific aspect of ACO). Reward: all edges, that point to visited vertices (neglect order of chosen edges)

Algorithm

1-ANT: (following Neumann/Witt, 2010)

- two pheromone values
- value h: if edge has been rewarded
- value ℓ: otherwise
- heuristic information η, $\eta(e) = \frac{1}{w(e)}$ (used before for TSP)
- Let v_k the current vertex and N_{v_k} be its neighborhood.
- Prob(to choose neighbor y of v_k) $= \frac{[\tau_{(v_k,y)}]^\alpha \cdot [\eta_{(v_k,y)}]^\beta}{\sum_{y \in N(v_k)} [\tau_{(v_k,y)}]^\alpha \cdot [\eta_{(v_k,y)}]^\beta}$ with $\alpha, \beta \geq 0$.
- Consider special cases where either $\beta = 0$ or $\alpha = 0$.

Results for Pheromone Updates

Case $\alpha = 1$, $\beta = 0$: proportional influence of pheromone values

Theorem (Broder-based construction graph)

Choosing $h/\ell = n^3$, the expected time until the 1-ANT *with the Broder-based construction graph has found an MST is $O(n^6(\log n + \log w_{max}))$.*

Theorem (Component-based construction graph)

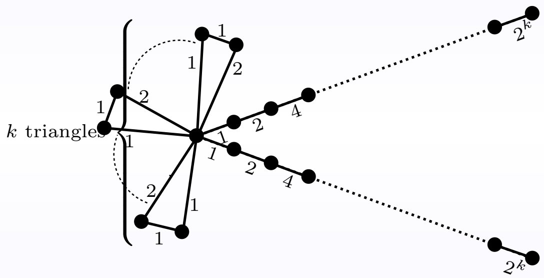
Choosing $h/\ell = (m - n + 1) \log n$, the expected time until the 1-ANT *with the component-based construction graph has found an MST is $O(mn(\log n + \log w_{max}))$.*

Better than (1+1) EA!

Broder Construction Graph: Heuristic Information

Example graph G^* with $n = 4k + 1$ vertices.

- k triangles of weight profile $(1, 1, 2)$
- two paths of length k with exponentially increasing weights.



Theorem (Broder-based construction graph)

Let $\alpha = 0$ and β be arbitrary, then the probability that the 1-ANT *using the Broder construction procedure does not find an MST in polynomial time is $1 - 2^{-\Omega(n)}$.*

Component-based Construction Graph/Heuristic Information

Theorem (Component-based construction graph)

Choosing $\alpha = 0$ and $\beta \geq 6w_{\max} \log n$, the expected time of the 1-ANT *with the component-based construction graph to find an MST is constant.*

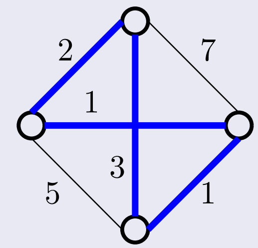
Proof Idea

- Choose edges as Kruskal's algorithm.
- Calculation shows: probability of choosing a lightest edge is at least $1 - 1/n$.
- $n - 1$ steps $\Longrightarrow$ probability for an MST is $\Omega(1)$.

Overview

Traveling Salesman Problem

Traveling Salesman Problem (TSP)



- Input: weighted complete graph $G = (V, E, w)$ with $w : E \to \mathbb{R}$.
- Goal: Find Hamiltonian cycle of minimum weight.

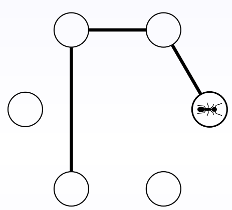
MMAS for TSP (Kötzing, Neumann, Röglin, Witt 2010)

Best-so-far pheromone update with $\tau_{\text{min}} := 1/n^2$ and $\tau_{\text{max}} := 1 - 1/n$.

Initialization: same pheromone on all edges.

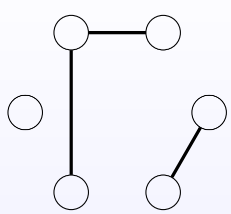
"Ordered" tour construction

Append a feasible edge chosen with probability proportional to pheromones.



"Arbitrary" tour construction

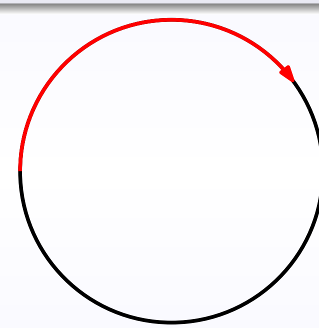
Add an edge chosen with probability proportional to pheromones as long as no cycle is closed or a vertex get degree at least 3.



Locality

Lemma

MMAS* with saturated pheromones exchanges $\Omega(\log(n))$ edges in expectation.



Length of unseen part roughly halves each time.

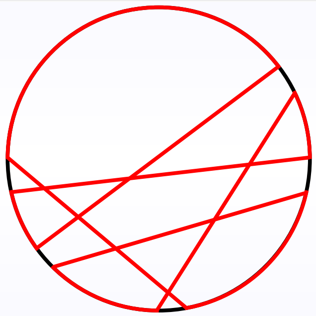
Lemma

For any constant k: MMAS^*_{Arb} with saturated pheromones creates exactly k new edges with probability $\Theta(1)$.

Locality

Lemma

MMAS* with saturated pheromones exchanges $\Omega(\log(n))$ edges in expectation.



Length of unseen part roughly halves each time.

Lemma

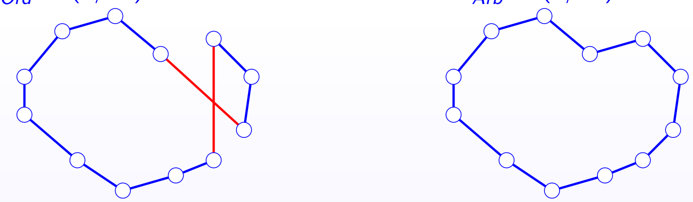
For any constant k: MMAS$^*_{Arb}$ with saturated pheromones creates exactly k new edges with probability $\Theta(1)$.

ACO Simulating 2-OPT

Zhou (2009): ACO can simulate 2-OPT.

Probability of particular 2-Opt step (for constant ρ):

MMAS$^*_{Ord}$: $\Theta(1/n^3)$ MMAS$^*_{Arb}$: $\Theta(1/n^2)$



Average Case Analysis

Assume that n points placed independently, uniformly at random in the unit hypercube $[0,1]^d$.

Theorem [Englert, Röglin, Vöcking 2007]

2-Opt finds after $O(n^{4+1/3} \cdot \log n)$ iterations with probability $1 - o(1)$ a solution with approximation ratio $O(1)$.

Theorem

For $\rho = 1$, MMAS$^*_{Arb}$ finds after $O(n^{6+2/3})$ iterations with probability $1 - o(1)$ a solution with approximation ratio $O(1)$.

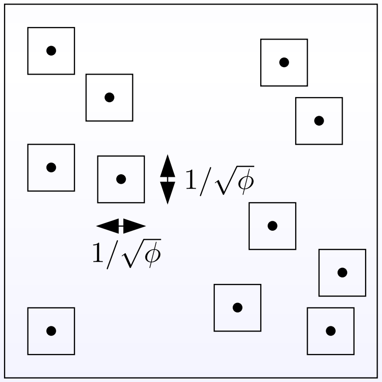
Theorem

For $\rho = 1$, MMAS$^*_{Ord}$ finds after $O(n^{7+2/3})$ iterations with probability $1 - o(1)$ a solution with approximation ratio $O(1)$.

Smoothed Analysis

Smoothed Analysis

Each point $i \in \{1, \ldots, n\}$ is chosen independently according to a probability density $f_i : [0,1]^d \to [0, \phi]$.



2-Opt:
$O(\sqrt[d]{\phi})$-approximation in $O(n^{4+1/3} \cdot \log(n\phi) \cdot \phi^{8/3})$ steps

MMAS$^*_{Ord}$: $O(\sqrt[d]{\phi})$-approximation in $O(n^{7+2/3} \cdot \phi^3)$ steps

MMAS$^*_{Arb}$: $O(\sqrt[d]{\phi})$-approximation in $O(n^{6+2/3} \cdot \phi^3)$ steps

ACO: Summary and Open Questions

(Stochastic) Shortest Paths

Natural and interesting test-bed for the robustness of ACO algorithms.

- global pheromone updates?
- other strategies to deal with noise
- where does slow pheromone adaptation help?
- average-case analyses with heuristic information

Strength of ACO

Problem-specific construction procedures can make ACO more powerful.

- how to find a fruitful combination of metaheuristic search and problem-specific components?

Main Challenge in Analysis of ACO

Understand dynamics of pheromones within borders.

- results for MST and TSP with more natural pheromone models

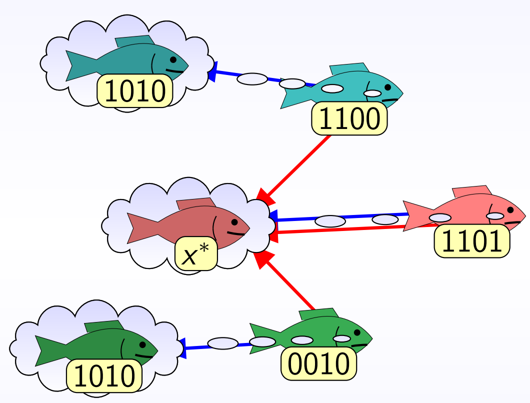
Overview

Particle Swarm Optimization

Particle Swarm Optimization

- Bio-inspired optimization principle developed by Kennedy and Eberhart (1995).
- Mostly applied in continuous spaces.
- Swarm of particles, each moving with its own velocity.
- Velocity is updated according to
  - own best position and
  - position of the best individual in its neighborhood (here: swarm).

Particle Swarm Optimization

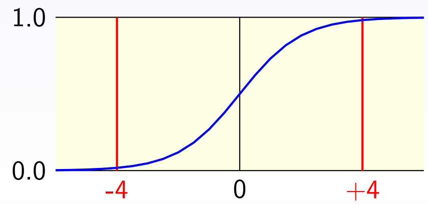


Binary PSO (Kennedy und Eberhart, 1997)

Creating New Positions

Probabilistic construction using velocity v and sigmoid function $s(v)$:

$$\text{Prob}(x_j = 1) = s(v_j) = \frac{1}{1+e^{-v_j}}$$



Restrict velocities to $v_j \in [-v_{\max}, +v_{\max}]$.

- Common practice: $v_{\max} = 4$ (good for $n \in [50, 500]$)
- Sudholt and Witt, 2010: $v_{\max} := \ln(n-1)$ (good across all n):

$$\frac{1}{n} \leq \text{Prob}(x_j = 1) \leq 1 - \frac{1}{n}.$$

Updating Velocities

Update current velocity vector according to

- cognitive component → towards own best: $x^{*(i)} - x^{(i)}$ and
- social component → towards global best: $x^* - x^{(i)}$.

Learning rates c_1, c_2 affect weights for the two components.

Random scalars $r_1 \in U[0, c_1]$, $r_2 \in U[0, c_2]$ chosen anew in each generation:

$$v^{(i)} = v^{(i)} + r_1(x^{*(i)} - x^{(i)}) + r_2(x^* - x^{(i)})$$

Understanding Velocities

Assume bit i is 1 in global best and own best. Create x.

- **ACO**: reinforce bit value 1 in probabilistic model if $x_i = 1$
- **PSO**: reinforce bit value 1 in probabilistic model if $x_i = 0$

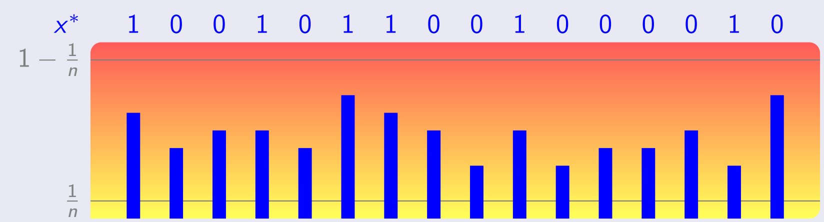
Probability to increase v_i is

$$1 - s(v_i) = s(-v_i) = \frac{1}{1+e^{v_i}}.$$

⇒ at least 1/2 for $v_i < 0$, but decreases rapidly with growing v_i.

Velocity Freezing

Particle with best-so-far solution: own best = global best

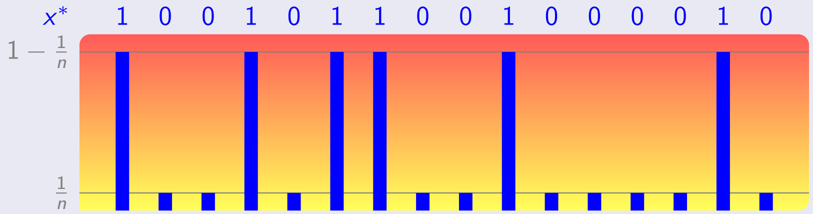


Lemma

Expected freezing time to $v_{\max}$ or $-v_{\max}$ is $O(n)$ for single bits and $O(n \log n)$ for n or μn bits if $\mu = poly(n)$.

Velocity Freezing

Particle with best-so-far solution: own best = global best



Lemma

Expected freezing time to $v_{\max}$ or $-v_{\max}$ is $O(n)$ for single bits and $O(n \log n)$ for n or μn bits if $\mu = poly(n)$.

Fitness-Level Method for Binary PSO

Upper bound for the (1+1) EA

$$\sum_{i=0}^{m-1} \frac{1}{s_i}$$

Upper bound for #generations of Binary PSO

$$\sum_{i=0}^{m-1} \frac{1}{s_i} + O(m \cdot n \log n)$$

Upper bound for #generations of "social" Binary PSO, i. e., $c_1 := 0$

$$O\left(\frac{1}{\mu} \sum_{i=0}^{m-1} \frac{1}{s_i} + m \cdot n \log n\right)$$

1-PSO vs. (1+1) EA on OneMax

More detailed analysis: average adaptation time of $384 \ln n$ is sufficient.

Theorem (Sudholt and Witt, 2010)

The expected optimization time of the 1-PSO *on* OneMax *is $O(n \log n)$.*

Proof uses layering argument and amortized analysis.

Experiments: 1-PSO 15% slower than (1+1) EA on OneMax.

Overview

Continuous PSO

Search space: (bounded subspace of) $\mathbb{R}^n$.

Objective function: $f\colon \mathbb{R}^n \to \mathbb{R}$.

Particles represent positions $x^{(i)}$ in this space.

Particles fly at certain velocity: $x^{(i)} := x^{(i)} + v^{(i)}$.

Velocity update with inertia weight ω:

$$v^{(i)} = \omega v^{(i)} + r_1(x^{*(i)} - x^{(i)}) + r_2(x^* - x^{(i)})$$

Convergence of PSO

Swarm can collapse to points or other low-dimensional subspaces.

Convergence results for standard PSO, $\omega < 1$ (Jiang, Luo, and Yang, 2007)

PSO converges ... somewhere.

Extensions of standard PSO

- Bare-bones PSO (Kennedy, 2003)
- PSO with mutation (several variants)
- PSO using gradient information (several variants)
- Guaranteed Convergence PSO (GCPSO) (van den Bergh and Engelbrecht, 2002)

Guaranteed Convergence PSO

Van den Bergh and Engelbrecht, 2002:

- Make a cube mutation of a particle's position by adding $p \in U[-\ell, \ell]^n$.
- Adapt "step size" ℓ in the course of the run by doubling or halving it, depending on the number of successes.

Possible step size adaptation (Witt, 2009)

After an observation phase consisting of n steps has elapsed, double ℓ if the total number of successes was at least $n/5$ in the phase and halve it otherwise. Then start a new phase.

$\longrightarrow$ 1/5-rule known from evolution strategies!

GCPSO with 1 Particle

GCPSO with one particle is basically a (1+1) ES with cube mutation.

Can be analyzed like classical (1+1) ES (Jägersküpper, 2007)

$$\textsc{Sphere}(x) := ||x|| = x_1^2 + x_2^2 + \cdots + x_n^2$$

Theorem (Witt, 2009)

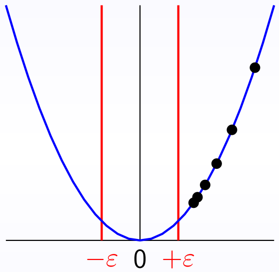
Consider the GCPSO$_1$ on $\textsc{Sphere}$. *If* $\ell = \Theta(||x^*||/n)$ *for the initial solution* x^**, the runtime until the distance to the optimum is no more than* $\varepsilon||x^*||$ *is* $O(n\log(1/\varepsilon))$ *with probability at least* $1 - 2^{-\Omega(n)}$ *provided that* $2^{-n^{O(1)}} \le \varepsilon \le 1$.

Same result as for (1+1) ES using Gaussian mutations in Jägersküpper, 2007.

Stagnation of Standard PSO

Lehre and Witt, MIC 2011

Standard PSO with one/two particles stagnates even on one-dimensional Sphere!



Expected first hitting time of ε-ball around optimum is infinite.

Noisy PSO (Lehre and Witt, 2011)

Adding noise $U[-\delta/2, \delta/2]$ for $\delta \leq \varepsilon$ yields finite expected first hitting time on (half-)Sphere.

PSO: Summary and Open Questions

Summary

- analysis of Binary PSO and its probabilistic model
- initial result on runtime of GCPSO
- results on expected first hitting time of ε-ball for Standard PSO & Noisy PSO

Neighborhood topologies

- ring topology, etc. instead of global best of swarm
- where does a restricted topology help?

Swarm dynamics

- analyze combined impact of cognitive and social components
- more results on swarms in continuous spaces

Overview

Conclusions

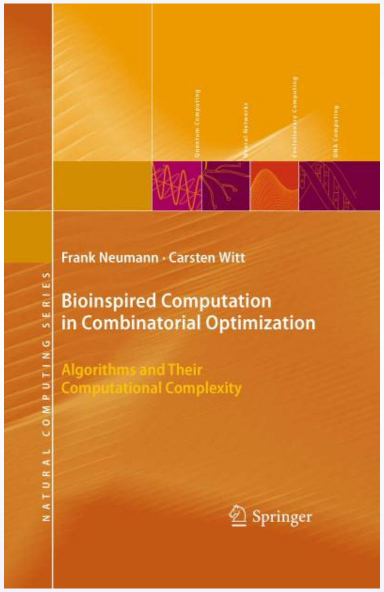
Summary

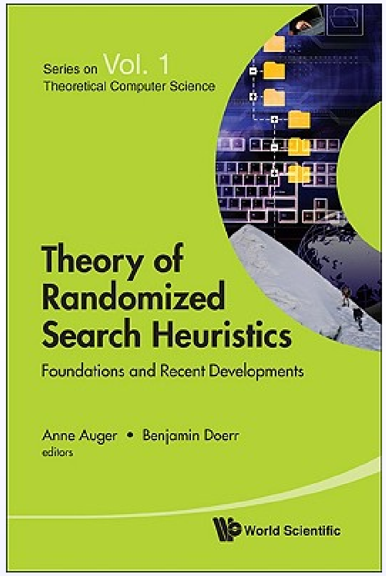
- Insight into probabilistic models underlying ACO and PSO
- How design choices and parameters affect (bounds on) running times
- How simple ACO algorithms optimize unimodal functions and plateaus
- Results for ACO in combinatorial optimization
- First analyses of basic PSO algorithms in discrete and continuous spaces

Future Work

- A unified theory of randomized search heuristics?
- More results on multimodal problems
- When and how diversity and slow adaptation help

Further Reading





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Thank you!

Questions?