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TIGHT BOUNDS FOR AMERICAN OPTIONS VIA MULTILEVEL MONTE CARLO

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ABSTRACT

This paper is an overview of recent results by Belomestny and Schoenmakers 2011 and Belomestny, Ladkau, and Schoenmakers 2012, on dual and primal Monte Carlo evaluation of American style derivatives using multilevel principles. It presents a novel and generic approach to reduce the complexity of nested simulations problems arising in Monte Carlo pricing of American options. The approach genuinely uses the multilevel idea where each level corresponds to a given number of inner simulations. A thorough complexity analysis of the respective nested dual algorithm and nested policy improvement algorithm shows that a significant complexity reduction can be achieved by using the multilevel versions of the algorithms.

1 INTRODUCTION

Pricing high-dimensional American options in an efficient way has been a challenge for decades. For low or moderate dimensions, deterministic (PDE) based methods may be applicable, but for higher dimensions Monte Carlo based methods are practically the only way out. Besides the dimension independent convergence rates, Monte Carlo methods are also popular because of their generic applicability. In the late nineties several regression methods for constructing "good" exercise policies yielding price lower bounds were introduced in the literature (see Carriere 1996, Longstaff and Schwartz 2001, and Tsitsiklis and Van Roy 2001, for a detailed description see also Glasserman 2004). Among many other approaches we mention that Broadie and Glasserman 2004 developed a stochastic mesh method, Bally and Pages 2003 introduced quantization methods, and Kolodko and Schoenmakers 2006 considered a class of policy iterations. In Bender, Kolodko, and Schoenmakers 2008 it is demonstrated that the latter approach can be combined effectively with the Longstaff-Schwartz approach.

The methods mentioned above commonly provide a (generally suboptimal) exercise policy, hence a lower price for an American product. They are therefore called primal methods. As a next breakthrough in the Monte Carlo simulation of American options, a dual approach was developed by Rogers 2002 and independently by Haugh and Kogan 2004, related to earlier ideas in Davis and Karatzas 1994. Due to the dual formulation one considers "good" martingales rather than "good" stopping times. In fact, based on a "good" martingale the price of an American derivative can be bounded from above by a "look-back" option due to the difference of the cash-flow and this martingale. Probably one of the most popular numerical methods for computing dual upper bounds is the method of Andersen and Broadie 2004. However, this method has a drawback, namely a high computational complexity due to the need for nested Monte Carlo

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simulations. As a remedy, Belomestny, Bender, and Schoenmakers 2009 developed a non-nested simulation algorithm using regression for approximating the integrand of a suitable martingale representation.

In this paper we review two new multilevel Monte Carlo simulation approaches where the multilevel concept is applied to the number of inner Monte Carlo simulations, rather than the discretization step size as in Giles 2008. While the first approach relies on the dual method leading to a multilevel version of the Andersen and Broadie 2004 algorithm, the second one leads to a multilevel version of the policy iteration approach presented in Kolodko and Schoenmakers 2006. Regarding the latter part only standard (Howard) policy iteration is considered, but, with no doubt the method may be applied successfully to the more refined policy iteration procedure in Kolodko and Schoenmakers 2006 as well. Finally we note that in this review all theorems are stated without proofs. For proofs and further details we refer to Belomestny and Schoenmakers 2011 and Belomestny, Ladkau, and Schoenmakers 2012, respectively.

2 PRIMAL AND DUAL VALUATION OF AMERICAN OPTIONS

Let $(Z_j)_{j\geq 0}$ be a nonnegative adapted process on a filtered probability space $(\Omega, \mathbb{F}=(\mathscr{F}_j)_{j\geq 0}, \mathbb{P})$ representing the discounted payoff of an American option, so that the holder of the option receives Z_j if the option is exercised at time $j \in \{0,\ldots,T\}$ with $T \in \mathbb{N}_+$. The pricing of American options can be formulated as a primal-dual problem. Let Y_j denote the time j solution to this problem. The primal representation corresponds to the following optimal stopping problems

$$Y_j^* := \max_{\tau \in \mathscr{T}[j,\ldots,T]} \mathbb{E}_{\mathscr{F}_j}[Z_\tau], \quad j=0,\ldots,T,$$

where $\mathscr{T}[j,\ldots,T]$ is the set of $\mathbb{F}$-stopping times taking values in $\{j,\ldots,T\}$. The process $\left(Y_j^*\right)_{j\geq 0}$ is called the Snell envelope. It is well known that Y^* is a supermartingale satisfying the Bellman principle

$$Y_j^* = \max\left(Z_j, \mathbb{E}_{\mathscr{F}_j}[Y_{j+1}^*]\right), \quad 0 \leq j < T, \quad Y_T^* = Z_T.$$

An exercise policy is a family of stopping times $(\tau_j)_{j=0,\ldots,T}$ such that $\tau_j \in \mathscr{T}[j,\ldots,T]$.

During the nineties the primal approach was the only method available. Some years later a quite different "dual" approach was discovered by Rogers 2002 and Haugh and Kogan 2004. The next theorem summarizes their results.

Theorem 1 Let $\mathscr{M}$ denote the space of adapted martingales, then we have the following dual representation for the value process Y_j^*

$$\begin{aligned} Y_j^* &= \inf_{\pi \in \mathscr{M}} \mathbb{E}_{\mathscr{F}_j}\left[\max_{s\in\{j,\ldots,T\}}(Z_s - \pi_s + \pi_j)\right] \\ &= \max_{s\in\{j,\ldots,T\}}(Z_s - \pi_s^* + \pi_j^*) \quad \text{a.s.}, \end{aligned}$$

where

$$Y_j^* = Y_0^* + \pi_j^* - A_j^* \tag{1}$$

is the (unique) Doob decomposition of the supermartingale Y_j^*. That is, π^* is a martingale and A^* is an increasing process with $\pi_0 = A_0 = 0$ such that (1) holds.

3 UPPER AND LOWER BOUNDS FOR BERMUDAN OPTIONS VIA NESTED MONTE CARLO

Assume that the cash-flow Z_j is of the form (with a slight abuse of notation) $Z_j = Z_j(X_j)$ for some underlying (possibly high-dimensional) Markovian process X. As a consequence, the Snell envelope then has the form $Y_j^* = Y_j^*(X_j)$, $j=0,\ldots,T$, as well. Furthermore, it is assumed that we are given a stopping family (τ_j) that is *consistent,* i.e.

$$\tau_j > j \Rightarrow \tau_j = \tau_{j+1}, \quad j=0,\ldots,T-1,$$

and that (τ_j) depends on ω only through the path $X_\cdot$ in the following way. For each j the event $\{\tau_j = j\}$ is measurable w.r.t. $\sigma\{X_j\}$, and τ_j is measurable w.r.t. $\sigma\{X_k,\, j \le k \le T\}$, i.e.,

$$\tau_j(\omega) = h_j(X_j(\omega), ..., X_T(\omega)) \tag{2}$$

for some Borel measurable function h_j. A typical example of such a stopping family is

$$\tau_j = \inf\{k : j \le k \le T, Z_k(X_k) > f_k(X_k)\} \wedge T,$$

for a set of real valued functions $f_k(x)$, with $j = 0, ..., T$, and $\wedge$ denoting the minimum operator. The stopping policy defines a lower bound for Y^* via

$$Y_j = \mathbb{E}_{\mathscr{F}_j}[Z_{\tau_j}], \quad j = 0, \ldots, T.$$

Consider now a new family $(\widehat{\tau}_j)_{j=0,...,T}$ defined by

$$\widehat{\tau}_j := \inf\left\{k : j \le k < T,\ Z_k \ge \mathbb{E}_{\mathscr{F}_k}[Z_{\tau_{k+1}}]\right\} \wedge T. \tag{3}$$

The basic idea behind (3) goes back to Howard 1960 in fact. For more general versions of policy iteration and their analysis, see Kolodko and Schoenmakers 2006. Next, we introduce the $(\mathscr{F}_j)$-martingale

$$\pi_j = \sum_{k=1}^{j} \left(\mathbb{E}_{\mathscr{F}_k}[Z_{\tau_k}] - \mathbb{E}_{\mathscr{F}_{k-1}}[Z_{\tau_k}]\right), \quad j = 0, \ldots, T, \tag{4}$$

and then consider,

$$\widetilde{Y}_j := \mathbb{E}_{\mathscr{F}_j}\left[\max_{k=j,...,T}(Z_k - \pi_k + \pi_j)\right],$$

along with

$$\widehat{Y}_j := \mathbb{E}_{\mathscr{F}_j}[Z_{\widehat{\tau}_j}], \quad j = 0, \ldots, T.$$

The following theorem states that $\widehat{Y}$ is an improvement of Y and that the Snell envelope process Y_j^* lies between $\widehat{Y}_j$ and $\widetilde{Y}_j$ with probability 1.

Theorem 2 It holds

$$Y_j \le \widehat{Y}_j \le Y_j^* \le \widetilde{Y}_j, \quad j = 0, \ldots, T \quad a.s.$$

The main issue in the Monte Carlo construction of $\widehat{Y}$ and $\widetilde{Y}$ is the estimation of the conditional expectations in (3) and (4). A canonical approach is the use of sub-simulations. In this respect we consider an enlarged probability space $(\Omega, \mathbb{F}', \mathbb{P})$, where $\mathbb{F}' = (\mathscr{F}'_j)_{j=0,...,T}$ and $\mathscr{F}_j \subset \mathscr{F}'_j$ for each j. By assumption, $\mathscr{F}'_j$ specified as

$$\mathscr{F}'_j = \mathscr{F}_j \vee \sigma\left\{X_\cdot^{i,X_i},\ i \le j\right\} \text{ with } \mathscr{F}_j = \sigma\left\{X_i,\ i \le j\right\},$$

where for a generic $(\omega, \omega_{in}) \in \Omega$, $X_\cdot^{i,X_i} := X_k^{i,X_i(\omega)}(\omega_{in})$, $k \ge i$ denotes a sub trajectory starting at time i in the state $X_i(\omega) = X_i^{i,X_i(\omega)}$ of the outer trajectory $X(\omega)$. In particular, the random variables $X_\cdot^{i,X_i}$ and $X_\cdot^{i',X_{i'}}$ are by assumption independent, conditionally $\{X_i, X_{i'}\}$, for $i \ne i'$. On the enlarged space we consider $\mathscr{F}'_j$ measurable estimations $\mathscr{C}_{j,M}$ of $C_j = \mathbb{E}_{\mathscr{F}_j}\left[Z_{\tau_{j+1}}\right]$ as being standard Monte Carlo estimates based on M sub simulations. More precisely

$$\mathscr{C}_{j,M} = \frac{1}{M}\sum_{m=1}^{M} Z_{\tau_{j+1}^{(m)}}(X_{\tau_{j+1}^{(m)}}^{j,X_j})$$

where

$$\tau_{j+1}^{(m)} = h_{j+1}(X_{j+1}^{j,X_j,(m)}, ..., X_T^{j,X_j,(m)}), \quad 0 \le j < T$$

are evaluated on M sub trajectories all starting at time j in X_j. Obviously, $\mathscr{C}_{j,M}$ is an unbiased estimator for C_j with respect to $\mathbb{E}_{\mathscr{F}_j}[\cdot]$. We thus end up with simulation based versions of (3) and (4) respectively,

$$\widehat{\tau}_{j,M} := \inf\{k : j \le k < T,\ Z_k > \mathscr{C}_{k,M}\} \wedge T, \quad j = 0, ..., T,$$

$$\begin{aligned}\pi_{j,M} \quad &:= \quad \sum_{k=1}^{j} (Z_k - \mathscr{C}_{k-1,M})\, 1_{\{\tau_k = k\}} \\ &\quad + \sum_{k=1}^{j} (\mathscr{C}_{k,M} - \mathscr{C}_{k-1,M})\, 1_{\{\tau_k > k\}}.\end{aligned}$$

Denote

$$\widehat{Y}_{j,M} := \mathbb{E}_{\mathscr{F}_j}[Z_{\widehat{\tau}_{j,M}}], \quad j = 0, \ldots, T$$

and

$$\widetilde{Y}_{j,M} := \mathbb{E}_{\mathscr{F}_j}\left[\max_{k=j,\ldots,T}(Z_k - \pi_{k,M} + \pi_{j,M})\right].$$

Concerning the properties of $\widehat{Y}_{j,M}$ and $\widetilde{Y}_{j,M}$ one can prove the following results under mild regularity conditions.

Theorem 3 Let us assume that there exist constants $B_0 > 0$, $B_1 > 0$ and $\alpha > 0$, such that for any $\delta > 0$ and $j = 0, \ldots, T-1$,

$$\mathbb{P}(|\mathbb{E}_{\mathscr{F}_j}[Z_{\tau_{j+1}}] - Z_j| \le \delta) \le B_0 \delta^{\alpha}, \quad \left|Z_j\right| < B_1.$$

It then holds,

$$|\widehat{Y}_0 - \widehat{Y}_{0,M}| \le M^{-\frac{1+\alpha}{2}} B,$$

with some constant B depending only on α, B_0 and B_1. Moreover

$$\mathbb{E}[(Z_{\widehat{\tau}_{0,M}} - Z_{\widehat{\tau}_0})^2] \le C M^{-\alpha/2}.$$

for some $C > 0$.

Theorem 4 Introduce for $\mathscr{Z} := \max_{j=0,\ldots,T}(Z_j - \pi_j)$, the random set

$$\mathscr{Q} = \left\{j : Z_j - \pi_j = \mathscr{Z}\right\},$$

and the $\mathscr{F}_T$ measurable random variable

$$\Lambda := \min_{j \notin \mathscr{Q}} (\mathscr{Z} - Z_j + \pi_j),$$

with $\Lambda := +\infty$ if $\mathscr{Q} = \{0, ..., T\}$. Obviously $\Lambda > 0$ a.s. Further suppose that

$$\mathbb{E}[\Lambda^{-\xi}] < \infty \text{ for some } 0 < \xi \le 1, \quad \text{and} \quad \#\mathscr{Q} = 1.$$

It then holds,

$$\left|\widetilde{Y}_0 - \widetilde{Y}_{0,M}\right| \le C M^{-\frac{\xi+1}{2}}$$

for some constant C.

Example 1 Let us assume that Λ has a density g that is continuous and finite in a right neighborhood of zero. We then have

$$\mathbb{E}\frac{1}{\Lambda^{\xi}} = \int_0^{\infty} z^{-\xi} g(z) dz < \infty \text{ for any } 0 \le \xi < 1.$$

4 POLICY IMPROVED LOWER BOUND BY MULTILEVEL MONTE CARLO

For a fixed natural number L and a set of natural numbers $\mathbf{m} := (m_0, ..., m_L)$ satisfying $1 \leq m_0 < ... < m_L$, we consider in the spirit of Giles 2008 the telescoping sum

$$\widehat{Y}_{m_L} = \widehat{Y}_{m_0} + \sum_{l=1}^{L} \left(\widehat{Y}_{m_l} - \widehat{Y}_{m_{l-1}} \right),$$

where $\widehat{Y}_m := \widehat{Y}_{0,m}$. Next we take a set of natural numbers $\mathbf{n} := (n_0, \ldots, n_L)$ satisfying $n_0 > ... > n_L \geq 1$, and simulate an initial set of cash-flows

$$\left\{ Z^{(j)}_{\widehat{\tau}_{m_0}}, \quad j = 1, ..., n_0 \right\},$$

due to an initial set of trajectories $\{X^{0,x,(j)}_{\cdot}, \, j = 1, ..., n_0\}$, where

$$Z^{(j)}_{\widehat{\tau}_{m_0}} := Z_{\widehat{\tau}^{(j)}_{0,m_0}} \left(X^{0,x,(j)}_{\widehat{\tau}^{(j)}_{0,m_0}} \right).$$

Next we simulate *independently* for each level $l = 1, ..., L$, a set of pairs

$$\left\{ (Z^{(j)}_{\widehat{\tau}_{m_l}}, Z^{(j)}_{\widehat{\tau}_{m_{l-1}}}), \quad j = 1, \ldots, n_l \right\}$$

due to a set of trajectories $X^{0,x,(j)}_{\cdot}$, $j = 1, ..., n_l$, to obtain the multilevel estimator

$$\widehat{\mathscr{Y}}_{\mathbf{n},\mathbf{m}} := \frac{1}{n_0} \sum_{j=1}^{n_0} Z^{(j)}_{\widehat{\tau}_{m_0}} + \sum_{l=1}^{L} \frac{1}{n_l} \sum_{j=1}^{n_l} \left(Z^{(j)}_{\widehat{\tau}_{m_l}} - Z^{(j)}_{\widehat{\tau}_{m_{l-1}}} \right) \tag{5}$$

as an approximation to $\widehat{Y}$. Henceforth we always take $\mathbf{m}$ to be a geometric sequence $m_l = m_0 \kappa^l$, for some $m_0, \ \kappa \in \mathbb{N}, \ \kappa \geq 2$.

5 DUAL UPPER BOUND BY MULTILEVEL MONTE CARLO

With the notations of the previous section we define

$$\widetilde{Y}_{m_L} = \widetilde{Y}_{m_0} + \sum_{l=1}^{L} [\widetilde{Y}_{m_l} - \widetilde{Y}_{m_{l-1}}],$$

where $\widetilde{Y}_m := \widetilde{Y}_{0,m}$. Given a sequence $\mathbf{n} = (n_0, \ldots, n_L)$ with $n_0 > \ldots > n_L \geq 1$, we then simulate for $l = 0$ an initial set of trajectories

$$\left\{ (Z^{(i)}_j, \pi^{(i)}_{j,m_0}), \quad i = 1, ..., n_0, \quad j = 0, \ldots, T, \right\}$$

of the two-dimensional vector process (Z_j, π_{j,m_0}), and then for each level $l = 1, \ldots, L$, *independently,* a set of trajectories

$$\left\{ (Z^{(i)}_j, \pi^{(i)}_{j,m_{l-1}}, \pi^{(i)}_{j,m_l}), \quad i = 1, ..., n_l, \quad j = 0, \ldots, T \right\}$$

of the vector process $(Z_j, \pi_{j,m_{l-1}}, \pi_{j,m_l})$. Based on this simulation we consider the following multilevel estimator:

$$\widetilde{\mathscr{Y}}_{\mathbf{n},\mathbf{m}} := \frac{1}{n_0} \sum_{i=1}^{n_0} \mathscr{Z}^{(i)}_{m_0} + \sum_{l=1}^{L} \frac{1}{n_l} \sum_{i=1}^{n_l} [\mathscr{Z}^{(i)}_{m_l} - \mathscr{Z}^{(i)}_{m_{l-1}}] \tag{6}$$

with $\mathscr{Z}^{(i)}_{m_l} := \max_{j=0,\ldots,T} \left(Z^{(i)}_j - \pi^{(i)}_{j,m_l} \right)$, $i = 1, \ldots, n_l$, $l = 0, \ldots, L$.

Complexity analysis

Let us now compute the numerical complexity of the multilevel estimators (5) and (6). To this end we consider a "generic" multi-level estimator for a target quantity X of the form:

$$X_{\mathbf{n},\mathbf{m}} := \frac{1}{n_0}\sum_{i=1}^{n_0} \mathscr{X}_{m_0}^{(i)} + \sum_{l=1}^{L} \frac{1}{n_l}\sum_{i=1}^{n_l} [\mathscr{X}_{m_l}^{(i)} - \mathscr{X}_{m_{l-1}}^{(i)}], \tag{7}$$

where for any fixed l the random variables $\mathscr{X}_{m_l}^{(i)}, i = 1,\ldots,n_l$, are i.i.d. and the pairs $(\mathscr{X}_{m_l}^{(i)}, \mathscr{X}_{m_{l-1}}^{(i)})$, $l = 1,\ldots,L$, are independent. Assume that there are some positive constants γ, β, μ_∞, σ_∞ and $\mathscr{V}_\infty$ such that $\mathrm{Var}[\mathscr{X}_m] \le \sigma_\infty^2$,

$$|X - \mathbb{E}[\mathscr{X}_m]| \quad \le \quad \mu_\infty m^{-\gamma}, \quad m \in \mathbb{N} \tag{8}$$

and

$$\mathbb{E}[\mathscr{X}_{m_l} - \mathscr{X}_{m_{l-1}}]^2 \quad \le \quad \mathscr{V}_\infty m_l^{-\beta}, \quad l = 1,\ldots,L. \tag{9}$$

These assumptions immediately imply

$$|\mathbb{E}[X_{\mathbf{n},\mathbf{m}}] - X| \le \mu_\infty m_L^{-\gamma}$$

and

$$\mathrm{Var}[X_{\mathbf{n},\mathbf{m}}] \le \frac{\sigma_\infty^2}{n_0} + \sum_{l=1}^{L} \frac{\mathscr{V}_\infty}{n_l m_l^\beta}.$$

Note that in the case of the multilevel policy iteration algorithm we have (under assumptions of Theorem 3) $\gamma = (1+\alpha)/2$ and $\beta = \alpha/2$, where typically $\alpha \ge 1$. For the multilevel dual algorithm it obviously holds (under premises of Theorem 4) $\gamma = (\xi+1)/2$ and $\beta = 1$.

Theorem 5 Let us assume that $0 < \beta < 1$, $\gamma \ge \frac{1}{2}$ and $m_l = m_0\kappa^l$ for some fixed κ and $m_0 \in \mathbb{N}$. Fix some $0 < \varepsilon < 1$. Let $L = L(\varepsilon)$ be the integer part of

$$\frac{1}{\gamma \ln \kappa} \ln\left[\frac{\sqrt{2}\mu_\infty}{m_0^\gamma \varepsilon}\right],$$

and

$$\begin{aligned} n_l &= n_0 \kappa^{-l(1+\beta)/2} \quad \text{with} \\ n_0 &= n_0(\varepsilon) = \frac{2\sigma_\infty^2}{\varepsilon^2} + \frac{2\mathscr{V}_\infty}{\varepsilon^2 m_0^\beta} \frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1} \kappa^{(1-\beta)/2}. \end{aligned}$$

Then the number of numerical operations needed to achieve the accuracy ε, i.e., to get $\sqrt{\mathbb{E}[(X - X_{\mathbf{n},\mathbf{m}})^2]} < \varepsilon$ is given, up to a constant, by

$$\begin{aligned} \mathscr{C}_{\mathrm{ML}}^{\mathbf{n},\mathbf{m}}(\varepsilon) &= n_0 m_0 + \sum_{l=1}^{L} n_l (m_l + m_{l-1}) \\ &= \frac{2\mathscr{V}_\infty \kappa^{1-\beta} m_0^{1-\beta} \left(1 + \kappa^{-1}\right)}{\varepsilon^2} \\ &\times \left(\frac{(\sqrt{2}\mu_\infty / m_0^\gamma \varepsilon)^{(1-\beta)/2\gamma} - 1}{\kappa^{(1-\beta)/2} - 1} + \frac{\sigma_\infty^2 m_0^\beta}{\mathscr{V}_\infty \kappa^{(1-\beta)/2}} \right) \\ &\times \left(\frac{(\sqrt{2}\mu_\infty / m_0^\gamma \varepsilon)^{(1-\beta)/2\gamma} - 1}{\kappa^{(1-\beta)/2} - 1} + \frac{\kappa^{-(1-\beta)/2}}{(1 + \kappa^{-1})} \right) \\ &= O(\varepsilon^{-2-\frac{1-\beta}{\gamma}}), \quad \varepsilon \searrow 0. \end{aligned}$$

Corollary 6 By letting $\beta \nearrow 1$ we derive from Theorem 5 that under the choice

$$n_l = n_0 \kappa^{-l} \quad \text{with}$$
$$n_0 = n_0(\varepsilon) = \frac{2\sigma_\infty^2}{\varepsilon^2} + \frac{2\mathscr{V}_\infty}{\varepsilon^2 m_0} L,$$

for $\beta = 1$ it holds

$$\mathscr{C}_{\mathrm{ML}}^{\mathbf{n},\mathbf{m}}(\varepsilon) = O(\varepsilon^{-2} \ln^2 \varepsilon), \quad \varepsilon \searrow 0.$$

(cf. the situation in Belomestny and Schoenmakers 2011).

Let $\mathscr{C}_{\mathrm{stan}}^{N,M}(\varepsilon)$ stand for the numerical complexity of the standard Monte Carlo estimate of X based on M inner and N outer paths.

Corollary 7 It holds for $\beta < 1$,

$$\frac{\mathscr{C}_{\mathrm{ML}}^{\mathbf{n},\mathbf{m}}(\varepsilon)}{\mathscr{C}_{\mathrm{stan}}^{N,M}(\varepsilon)} = O(\varepsilon^{\beta/\gamma}), \quad \varepsilon \searrow 0$$

and for $\beta = 1$,

$$\frac{\mathscr{C}_{\mathrm{ML}}^{\mathbf{n},\mathbf{m}}(\varepsilon)}{\mathscr{C}_{\mathrm{stan}}^{N,M}(\varepsilon)} = O(\varepsilon^{1/\gamma} \ln^2 \varepsilon), \quad \varepsilon \searrow 0.$$

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