Evolutionary Bilevel Optimization

Ankur Sinha
Pekka Malo
Aalto University School of Business
Helsinki, Finland

Kalyanmoy Deb
Indian Institute of Technology Kanpur, India
Aalto University School of Business, Helsinki, Finland
Michigan State University, East Lansing, USA
University of Skovde, Sweden

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An Optimization Problem

$$\mathrm{Min}_{(x,y)}\ F(x, y)$$

Such that

$G(x, y) \geq 0$	Inequality Constraint
$H(x, y) = 0$	Equality Constraint
$x_{min} \leq x \leq x_{max},\quad y_{min} \leq y \leq y_{max}$	Bound Constraint

The above is a single level of optimization task

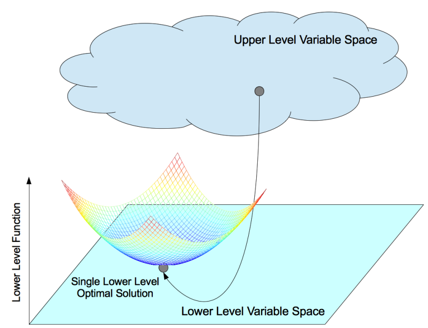
What is a Bilevel Problem?

$$\begin{aligned}
\mathrm{Min}_{(\mathbf{x}_u,\mathbf{x}_l)}\quad & F(\mathbf{x}_u,\mathbf{x}_l),\\
\mathrm{st}\quad & \mathbf{x}_l \in \mathrm{argmin}_{(\mathbf{x}_l)} \left\{ \begin{array}{c} f(\mathbf{x}_u,\mathbf{x}_l) \\ \mathbf{g}(\mathbf{x}_u,\mathbf{x}_l) \geq \mathbf{0}, \mathbf{h}(\mathbf{x}_u,\mathbf{x}_l) = \mathbf{0} \end{array} \right\},\\
& \mathbf{G}(\mathbf{x}_u,\mathbf{x}_l) \geq \mathbf{0}, \mathbf{H}(\mathbf{x}_u,\mathbf{x}_l) = \mathbf{0},\\
& (\mathbf{x}_u)_{min} \leq \mathbf{x}_u \leq (\mathbf{x}_u)_{max}, (\mathbf{x}_l)_{min} \leq \mathbf{x}_l \leq (\mathbf{x}_l)_{max}
\end{aligned}$$

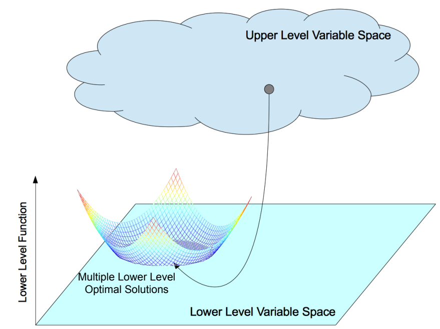
A mathematical program that contains an optimization problem in the constraints.

-Bracken and McGill (1973)

What is a Bilevel Problem? (cont.)



What is a Bilevel Problem? (cont.)



Origin of Bilevel Programming

- **An Extension of Mathematical Programming**
  - All optimization problems are special cases of bilevel programming

    –Bracken and McGill (1973)
- **Stackelberg Games**
  - Bilevel programs commonly appear in game theory when there is a leader and follower

    –Stackelberg (1952)

Extension of Mathematical Programming

- **Two Levels of Optimization**
  - Contains an upper and a lower level optimization tasks
  - Optimization problem within the constraints
  - Feasibility determined by solving the lower level problem
  - Can be generalized to multiple levels of optimization
- **Example**
  - Optimization problems having equilibrium satisfaction as a constraint
  - Optimization problems having stability satisfaction as a constraint
  - Optimization problems having differential equations as a constraint

Stackelberg Games

Two Player Game

- Leader makes the first move
- Follower reacts rationally and then makes its move
- Leader has the first mover's advantage
- Problem is non-symmetric

Example

- Hierarchical decision making
- Consider a leader dictating selling price and supply
- Customers respond rationally to the leader's decision
- Leader needs to anticipate this response to maximize profit

Properties of Bilevel Problems

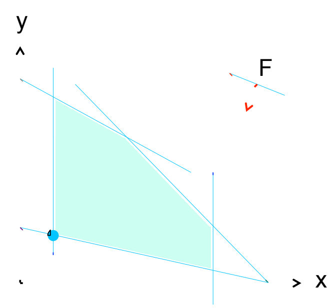
- Bilevel problems are typically non-convex, disconnected and strongly NP-hard
- Solving an optimization problem produces a one or more feasible solutions
- Multiple global solutions at lower level can induce additional challenges
- Two levels can be cooperating or conflicting

Single Level vs Bilevel

$$\text{Min}_{(x,y)}\ 3y + x$$

Such that

$$x + y \le 8$$
$$x + 4y \ge 8$$
$$x + 2y \le 13$$
$$1 \le x \le 6$$



A Single Level Linear Optimization Problem

Single Level vs Bilevel (cont.)

The single level linear optimization problem is now modified as follows:

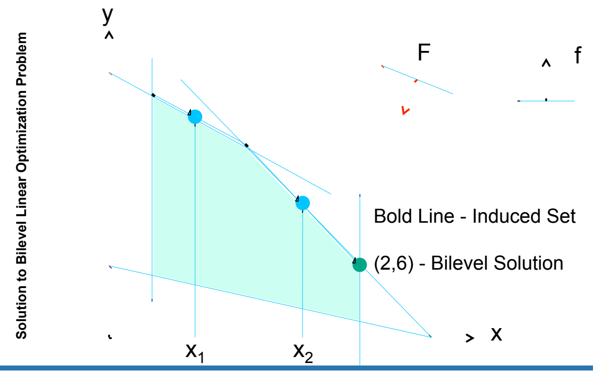
$$\text{Min}_{(x,y)}\ 3y + x$$

Such that

$$y \in \text{argmin}_{(y)} \left\{ \begin{array}{l} -y \\ \text{Such that} \\ \quad x + y \le 8,\ x + 4y \ge 8, \\ \quad x + 2y \le 13,\ 1 \le x \le 6 \end{array} \right\}$$

Bilevel Linear Optimization Problem

Single Level vs Bilevel (cont.)



Bilevel vs. Multi-objective Optimization

Bilevel linear optimization problem is now modified as follows:

$\text{Min}_{(x,y)}\ 3y + x$

$\text{Min}_{(x,y)}\ -y$

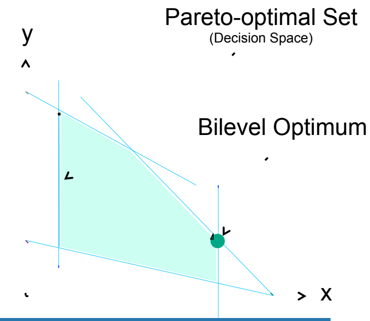
Such that

$x + y \leq 8$

$x + 4y \geq 8$

$x + 2y \leq 13$

$1 \leq x \leq 6$



Bilevel vs Multi-objective Opt. (cont.)

- Both problems involve two different objectives
- Multi-objective problems usually have multiple optimal solutions
- Bilevel problems usually have a single optimal solution
- A bilevel solution is not necessarily a Pareto-optimal solution
- It is not possible to directly use algorithms for multi-objective optimization for bilevel problems

Bilevel Multi-objective Problems

- Bilevel problems may involve optimization of multiple objectives at one or both of the levels
- Little work has been done in the direction of multi-objective bilevel problems (Eichfelder (2007), Deb and Sinha (2010))
- A general multi-objective bilevel problem may be formulated as follows:

$$
\begin{aligned}
\text{Min}_{(\mathbf{x}_u,\mathbf{x}_l)}\quad & \mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}),\ldots,F_M(\mathbf{x})),\\
\text{st}\quad & \mathbf{x}_l \in \text{argmin}_{(\mathbf{x}_l)} \left\{ \begin{array}{c} \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}),\ldots,f_m(\mathbf{x})) \\ \mathbf{g}(\mathbf{x}) \geq \mathbf{0}, \mathbf{h}(\mathbf{x}) = \mathbf{0} \end{array} \right\},\\
& \mathbf{G}(\mathbf{x}) \geq \mathbf{0}, \mathbf{H}(\mathbf{x}) = \mathbf{0},\\
& (\mathbf{x}_u)_{min} \leq \mathbf{x}_u \leq (\mathbf{x}_u)_{max}, (\mathbf{x}_l)_{min} \leq \mathbf{x}_l \leq (\mathbf{x}_l)_{max}
\end{aligned}
$$

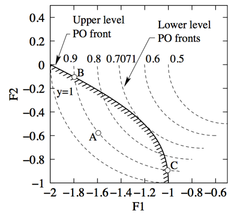
Bilevel Multi-objective: Example

$$\mathbf{x}_u = (x) \qquad \mathbf{x}_l = (y_1, y_2)$$

$$\begin{aligned}
\text{Minimize} \quad & \mathbf{F}(\mathbf{x}) = \begin{pmatrix} y_1 - x \\ y_2 \end{pmatrix}, \\
\text{Subject to} \quad & (y_1, y_2) \in \operatorname{argmin}_{(y_1,y_2)} \left\{ \mathbf{f}(\mathbf{x}) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \middle| g_1(\mathbf{x}) = x^2 - y_1^2 - y_2^2 \geq 0 \right\}, \\
& G_1(\mathbf{x}) = 1 + y_1 + y_2 \geq 0, \\
& -1 \leq y_1, y_2 \leq 1, \quad 0 \leq x \leq 1.
\end{aligned}$$

- Lower level Pareto front depends on x
- Upper level Pareto-optimal front lies on constraint G_1
  - Maximum two solutions from each x
  - Not all x in upper Pareto-optimal front
- Solutions possible even below the upper level Pareto-optimal front

- Dashed lines represent lower level Pareto fronts
- Solid line represents upper level Pareto front



Single vs Multi-objective Bilevel Problems

- Single-objective Bilevel Problems
  - One objective function at both the levels
  - Usually one target solution
- Multi-objective Bilevel Problems
  - Multiple objectives at one or both of the levels
  - Usually multiple solutions are targeted
  - Find and maintain many solutions at both levels
  - Decision making may lead to further complications
  - Not an easy matter

Applications

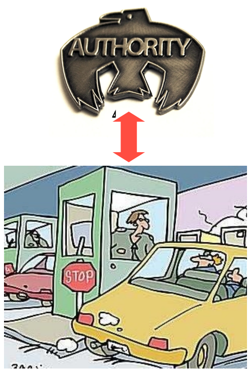
Toll Setting Problem

Authority's problem:

- Authority responsible for highway system wants to maximize its revenues earned from toll
- The authority has to solve the highway users optimization problem for all the possible tolls

Highway users' problem:

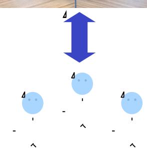
- For any toll chosen by the authority, highway users try to minimize their own travel costs
- A high toll will deter users to take the highway, lowering the revenues



Brotcorne et al. (2001)

Hierarchical Setting

- An owner of a company dictates the selling price and supply. He wants to maximize his profit.
- The buyers look at the product quality, pricing and various other options available to maximize their utility
- Mixed integer programs on similar lines have been formulated by Heliporn et al. (2010)



Max Profit (suppy, selling price, demand, other variables)

Such that
demand ∈ argmax { Utility | Lower Level Constraints}
Upper Level Constraints

Heilporn et al. (2010)

Stackelberg Duopoly Competition

Competition between a leader and a follower firm

Leader solves the following optimization problem to maximize its profit

$$
\begin{aligned}
\max_{q_l,q_f} \quad & \Pi_l = P(q_l,q_f)q_l - C(q_l) \\
\text{s.t.} \quad & q_f \in \arg\max_{q_f}\{\Pi_f = P(Q)q_f - C(q_f)\}, \\
& q_l + q_f \geq Q, \\
& q_l, q_f, Q \geq 0,
\end{aligned}
$$

where Q is the quantity demanded, $P(q_l, q_f)$ is the price of the goods sold, and $C(\cdot)$ is the cost of production of the respective firm. The variables in this model are the production levels of each firm q_l, q_f and demand Q.

If the leader and follower have similar functions, leader always makes a higher profit.
- First mover's advantage

Can be extended to multiple leaders and multiple followers

Frantsev et al. (2012)

Taxation Strategy

- Recently, there has been a controversy in Finland for gold mining in the Kuusamo region in Finland
- The region is a famous tourist resort endowed with immense natural beauty
- For any taxation strategy by the government, the mining company optimizes its own profits

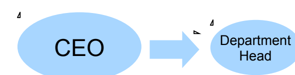
Leader: Government
Maximize revenue from taxes
Minimize Pollution

Follower: Mining Company
Maximize Profit

Sinha et al. (2013)

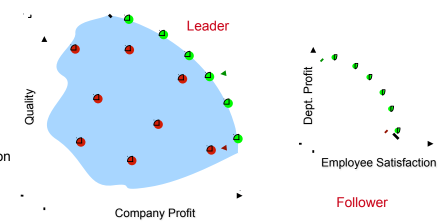
A Company Scenario

- Two levels of decision making
- Multiple objectives involved at both the levels



Leader: CEO of a Company
Objectives: Max Company Profit
Max Product Quality

Followers: Department Heads
Objectives: Max Department Profit
Max Employee Satisfaction



Zhang et al. (2007), Deb et al. (2009)

Other Applications

- Maximizing production in chemical industries where reactions occur under equilibrium
- Defense applications like, strategic offensive and defensive force structure design, strategic bomber force structure, and allocation of tactical aircraft to missions
- Other applications include, optimal shape design, optimal operating configuration and optimal facility location
- Wide applications in economics like Principal-Agency problems, Taxation policies etc.

Solution Methodologies

- KKT conditions of the lower level problem are used as constraints (Herskovits et al. 2000)
  - Lagrange multipliers increase the number of decision variables
  - Constrained search space
  - Applicable to differentiable problems only
- Another common approach: Nested optimization
  - For every $\mathbf{x}_u$, lower level problem is solved completely
  - Computationally very expensive
- Discretization of the lower level problem
  - The best solution obtained from discrete set for a given $\mathbf{x}_u$ is used as a feasible member at upper level

Solution Methodologies (cont.)

- Penalty based approaches
  - Special forms of penalty functions have been used
  - Lower level is usually required to be convex
  - Penalty function may require differentiability
- Branch and Bound techniques (Bard et al. 1982)
  - Used KKT conditions
  - Handled linear problems
  - Converted the problems into variable separable form
  - Utilized the branch and bound approach
- Taking an approximation of the lower level optimization problem such that its optimum is readily available
  - The optimal solutions from lower level might not be accurate

Solution Methodologies (cont.)

- Evolutionary algorithms have also been used for bilevel optimization
- Most of the methods are nested strategies
- Mathieu et al. (1994): LP for lower level and GA for upper level
- Yin (2000): Frank Wolfe Algorithm for lower level
- Oduguwa and Roy (2005): Proposed a co-evolutionary approach
- Wang et al. (2005):
  - Solved bilevel problems using a constrained handling scheme in EA
  - Method is computationally expensive, but successfully handles a number of test problems
- Li et al. (2006): Nested strategy using PSO
- EA researchers have also tried replacing the lower level problems using KKT (Wang et al. (2008), Li et al. (2007))

Bilevel Evolutionary Algorithm Based on Quadratic Approximations (BLEAQ)

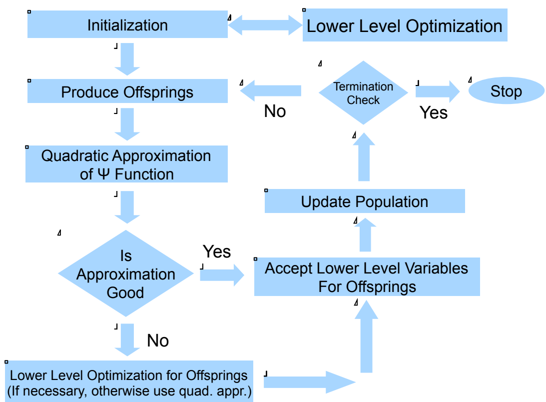
- Recently proposed by Sinha, Malo and Deb (2013) for solving bilevel optimization problems with various kinds of complexities
- Based on quadratic approximation of the inducible region
- The method is highly efficient when compared against nested approaches and other contemporary evolutionary techniques
- Capable of solving larger instances of bilevel programming problems
- Tested on a recently proposed SMD test-suit and other standard test problems from the literature

BLEAQ

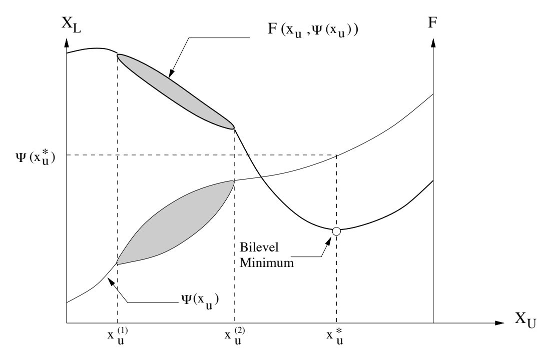
$$\begin{aligned}\text{Min}_{(\mathbf{x}_u,\mathbf{x}_l)} \quad & F(\mathbf{x}_u,\mathbf{x}_l),\\ \text{st} \quad & \mathbf{x}_l \in \text{argmin}_{(\mathbf{x}_l)} \left\{ \begin{array}{c} f(\mathbf{x}_u,\mathbf{x}_l) \\ \mathbf{g}(\mathbf{x}_u,\mathbf{x}_l) \geq \mathbf{0}, \mathbf{h}(\mathbf{x}_u,\mathbf{x}_l) = \mathbf{0} \end{array} \right\},\\ & \mathbf{G}(\mathbf{x}_u,\mathbf{x}_l) \geq \mathbf{0}, \mathbf{H}(\mathbf{x}_u,\mathbf{x}_l) = \mathbf{0},\\ & (\mathbf{x}_u)_{min} \leq \mathbf{x}_u \leq (\mathbf{x}_u)_{max}, (\mathbf{x}_l)_{min} \leq \mathbf{x}_l \leq (\mathbf{x}_l)_{max}\end{aligned}$$

$$\begin{aligned}\text{Min}_{(\mathbf{x}_u,\mathbf{x}_l)} \quad & F(\mathbf{x}_u,\mathbf{x}_l),\\ \text{st} \quad & \mathbf{x}_l \in \Psi(\mathbf{x}_u)\\ & \mathbf{G}(\mathbf{x}_u,\mathbf{x}_l) \geq \mathbf{0}, \mathbf{H}(\mathbf{x}_u,\mathbf{x}_l) = \mathbf{0},\\ & (\mathbf{x}_u)_{min} \leq \mathbf{x}_u \leq (\mathbf{x}_u)_{max}, (\mathbf{x}_l)_{min} \leq \mathbf{x}_l \leq (\mathbf{x}_l)_{max}\end{aligned}$$

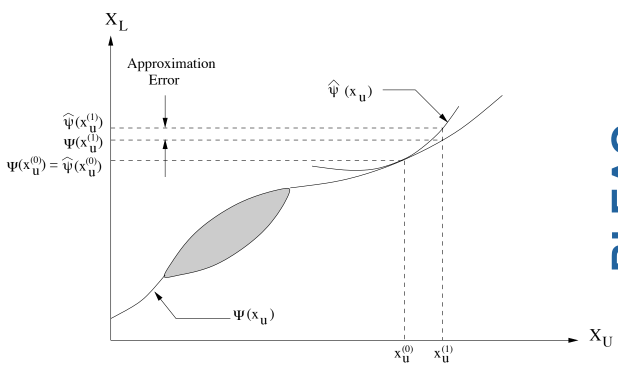
BLEAQ



BLEAQ

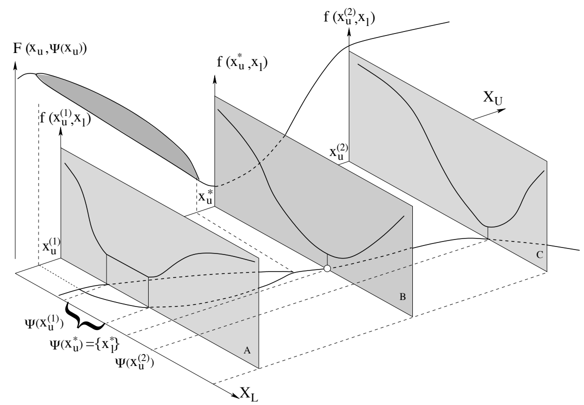


Upper level function with respect to x_u when x_l is optimal



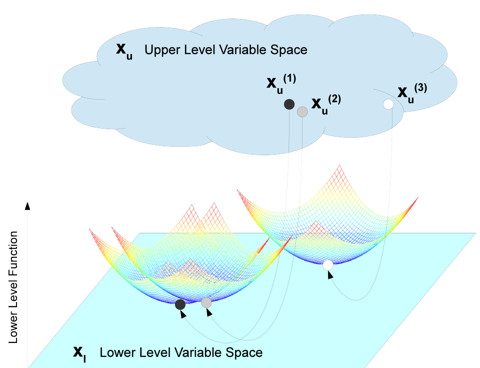
Approximation with localization around $x_u^{(0)}$ for $x_l^{(0)}$

BLEAQ



A graphical representation of a simple bilevel optimization problem

BLEAQ



Close upper level members are expected to have close lower level optimal solutions

BLEAQ

BLEAQ

Why Evolutionary Approach?

- Classical methods require various simplifying assumptions like continuity, differentiability and convexity
- Bilevel is more intriguing than a normal optimization problem as the necessary condition itself involves second order derivatives
- Most of the real world applications do not adhere to these assumptions
- Population based approach helps in approximation of the Ψ function
- Incorporation of a good member from one population to the population in the vicinity boosts convergence

Test Problem Construction

1. Controlled difficulty in convergence at upper and lower levels
2. Controlled difficulty caused by interaction of the two levels
3. Multiple global solutions at the lower level for any given set of upper level variables
4. Clear identification of relationship between lower level optimal solutions and upper level variables
5. Scalability to any number of decision variables at upper and lower levels
6. Constraints (preferably scalable) at upper and lower levels
7. Possibility to have conflict or cooperation at the two levels
8. The optimal solution of the bilevel optimization is known

SMD Test Problem Framework

The objectives and variables on both levels are decomposed as follows:

$$F(\mathbf{x}_u, \mathbf{x}_l) = F_1(\mathbf{x}_{u1}) + F_2(\mathbf{x}_{l1}) + F_3(\mathbf{x}_{u2}, \mathbf{x}_{l2})$$
$$f(\mathbf{x}_u, \mathbf{x}_l) = f_1(\mathbf{x}_{u1}, \mathbf{x}_{u2}) + f_2(\mathbf{x}_{l1}) + f_3(\mathbf{x}_{u2}, \mathbf{x}_{l2})$$

where

$$\mathbf{x}_u = (\mathbf{x}_{u1}, \mathbf{x}_{u2}) \quad \text{and} \quad \mathbf{x}_l = (\mathbf{x}_{l1}, \mathbf{x}_{l2})$$

Roles of Components

Panel A: Decomposition of decision variables

Upper-level variables		Lower-level variables	
Vector	Purpose	Vector	Purpose
$\mathbf{x}_{u1}$	Complexity on upper-level	$\mathbf{x}_{l1}$	Complexity on lower-level
$\mathbf{x}_{u2}$	Interaction with lower-level	$\mathbf{x}_{l2}$	Interaction with upper-level

Panel B: Decomposition of objective functions

Upper-level objective function		Lower-level objective function	
Component	Purpose	Component	Purpose
$F_1(\mathbf{x}_{u1})$	Difficulty in convergence	$f_1(\mathbf{x}_{u1}, \mathbf{x}_{u2})$	Functional dependence
$F_2(\mathbf{x}_{l1})$	Conflict / co-operation	$f_2(\mathbf{x}_{l1})$	Difficulty in convergence
$F_3(\mathbf{x}_{u2}, \mathbf{x}_{l2})$	Difficulty in interaction	$f_3(\mathbf{x}_{u2}, \mathbf{x}_{l2})$	Difficulty in interaction

Controlling Difficulty in Convergence

- Convergence difficulties can be induced via following routes
- Dedicated components: F_1 (upper) and f_2 (lower)
- Example:

$$F(\mathbf{x}_u, \mathbf{x}_l) = \boxed{F_1(\mathbf{x}_{u1})} + F_2(\mathbf{x}_{l1}) + F_3(\mathbf{x}_{u2}, \mathbf{x}_{l2})$$

Quadratic

$$f(\mathbf{x}_u, \mathbf{x}_l) = f_1(\mathbf{x}_{u1}, \mathbf{x}_{u2}) + \boxed{f_2(\mathbf{x}_{l1})} + f_3(\mathbf{x}_{u2}, \mathbf{x}_{l2})$$

Multi-modal

Controlling Difficulty in Interactions

- Interaction between variables x_{u2} and x_{l2} could be chosen as follows
- Dedicated components: F_3 and f_3
- Example:

$$F(\mathbf{x}_u, \mathbf{x}_l) = F_1(\mathbf{x}_{u1}) + F_2(\mathbf{x}_{l1}) + \boxed{F_3(\mathbf{x}_{u2}, \mathbf{x}_{l2})}$$

$$\sum_{i=1}^{r} (x_{u2}^i)^2 + \sum_{i=1}^{r} ((x_{u2}^i)^2 - \tan x_{l2}^i)^2$$

$$f(\mathbf{x}_u, \mathbf{x}_l) = f_1(\mathbf{x}_{u1}, \mathbf{x}_{u2}) + f_2(\mathbf{x}_{l1}) + \boxed{f_3(\mathbf{x}_{u2}, \mathbf{x}_{l2})}$$

$$\sum_{i=1}^{r} ((x_{u2}^i)^2 - \tan x_{l2}^i)^2$$

Difficulty due to Conflict/Co-operation

- Dedicated components: F_2 and f_2 or F_3 and f_3 may be used to induce conflict/cooperation

- Examples:
  - Co-operative interaction = improvement in lower-level improves upper-level (e.g. $F_2 = f_2$)
  - Conflicting interaction = improvement in lower-level worsens upper-level (e.g. $F_2 = -f_2$)
  - Mixed interaction = situation of both cooperation and conflict (e.g. $F_2 = f_2$ and $F_3 = \Sigma_i (x_{u2}^i)^2 - f_3$)

Controlled Multimodality

- Obtain multiple lower-level optima for every upper level decision
- Component used: f_2
- Example: Multimodality at lower-level

$$f_1(\mathbf{x}_{u1}, \mathbf{x}_{u2}) = (x_{u1}^1)^2 + (x_{u1}^1)^2 + (x_{u2}^1)^2 + (x_{u2}^2)^2$$
$$f_2(\mathbf{x}_{l1}) = \boxed{(x_{l1}^1 - x_{l1}^2)^2}$$
$$f_3(\mathbf{x}_{u2}, \mathbf{x}_{l2}) = (x_{u2}^1 - x_{l2}^1)^2 + (x_{u2}^2 - x_{l2}^2)^2$$

Induces multiple solutions: $x^1_{l1} = x^2_{l1}$

$$F_1(\mathbf{x}_{u1}) = (x_{u1}^1)^2 + (x_{u1}^1)^2$$
$$F_2(\mathbf{x}_{l1}) = \boxed{(x_{l1}^1)^2 + (x_{l1}^2)^2}$$
$$F_3(\mathbf{x}_{u2}, \mathbf{x}_{l2}) = (x_{u2}^1 - x_{l2}^2)^2 + (x_{u2}^2 - x_{l2}^2)^2$$

Gives best UL solution: $x^1_{l1} = x^2_{l1}=0$

Difficulty due to Constraints

Constraints are included at both the levels with one or more of the following properties

- Constraints exist, but are not active at the optimum
- A subset of constraints, or all the constraints are active at the optimum
- Upper level constraints are functions of only upper level variables, and lower level constraints are functions of only lower level variables
- Upper level constraints are functions of upper as well as lower level variables, and lower level constraints are also functions of upper as well as lower level variables
- Lower level constraints lead to multiple global solutions at the lower level
- Constraints are scalable at both levels

Test Problems: SMD1 – SMD3

- SMD1:
  - Interaction: co-operative
  - Lower level: convex (wrt lower level variables)
  - Upper level: convex (induced space)
- SMD 2:
  - Interaction: conflict
  - Lower level: convex (wrt lower level variables)
  - Upper level: convex (induced space)
- SMD 3:
  - Interaction: co-operative
  - Lower level: multimodality, Rastrigin's function
  - Upper level: convex (induced space)

Test Problems: SMD4 – SMD6

- SMD 4:
  - Interaction: conflict
  - Lower level: multimodality, Rastrigin's function
  - Upper level: convex (Induced Space)
- SMD 5:
  - Interaction: conflict
  - Lower level: multimodality, banana function
  - Upper level: convex (Induced Space)
- SMD 6:
  - Interaction: conflict
  - Lower level: infinitely many global solutions for any given x_u
  - Upper level: convex (Induced Space)

Test Problems: SMD7 – SMD9

- SMD 7:
  - Interaction: conflict
  - Lower level: convex (wrt lower level variables)
  - Upper level: multimodality
- SMD 8:
  - Interaction: conflict
  - Lower level: multimodality, banana function
  - Upper level: multimodality
- SMD 9:
  - Interaction: conflict
  - Lower level: non-scalable constraints
  - Upper level: non-scalable constraints

Test Problems: SMD10 – SMD12

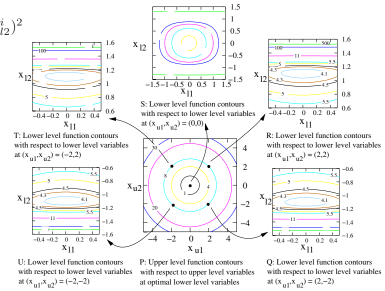
- SMD 10:
  - Interaction: conflict
  - Lower level: scalable constraints
  - Upper level: scalable constraints
- SMD 11:
  - Interaction: conflict
  - Lower level: non-scalable constraints, multiple global solutions
  - Upper level: scalable constraints
- SMD 12:
  - Interaction: conflict
  - Lower level: scalable constraints, multiple global solutions
  - Upper level: scalable constraints

Test Problem Construction

SMD1

$$F_1 = \sum_{i=1}^{p} (x_{u1}^i)^2$$
$$F_2 = \sum_{i=1}^{q} (x_{l1}^i)^2$$
$$F_3 = \sum_{i=1}^{r} (x_{u2}^i)^2 + \sum_{i=1}^{r} (x_{u2}^i - \tan x_{l2}^i)^2$$
$$f_1 = \sum_{i=1}^{p} (x_{u1}^i)^2$$
$$f_2 = \sum_{i=1}^{q} (x_{l1}^i)^2$$
$$f_3 = \sum_{i=1}^{r} (x_{u2}^i - \tan x_{l2}^i)^2$$

$$x_{u1}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, p\}$$
$$x_{u2}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, r\}$$
$$x_{l1}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, q\}$$
$$x_{l2}^i \in (\tfrac{-\pi}{2}, \tfrac{\pi}{2}), \ \forall \ i \in \{1, 2, \ldots, r\}$$



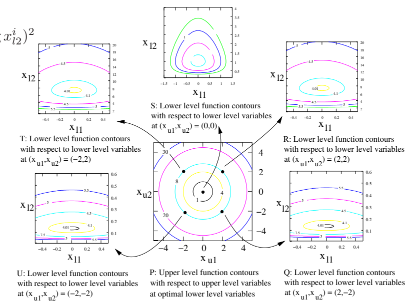
Upper and Lower Function Contours

Test Problem Construction

SMD2

$$F_1 = \sum_{i=1}^{p} (x_{u1}^i)^2$$
$$F_2 = -\sum_{i=1}^{q} (x_{l1}^i)^2$$
$$F_3 = \sum_{i=1}^{r} (x_{u2}^i)^2 - \sum_{i=1}^{r} (x_{u2}^i - \log x_{l2}^i)^2$$
$$f_1 = \sum_{i=1}^{p} (x_{u1}^i)^2$$
$$f_2 = \sum_{i=1}^{q} (x_{l1}^i)^2$$
$$f_3 = \sum_{i=1}^{r} (x_{u2}^i - \log x_{l2}^i)^2$$

$$x_{u1}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, p\}$$
$$x_{u2}^i \in [-5, 1], \ \forall \ i \in \{1, 2, \ldots, r\}$$
$$x_{l1}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, q\}$$
$$x_{l2}^i \in (0, e], \ \forall \ i \in \{1, 2, \ldots, r\}$$



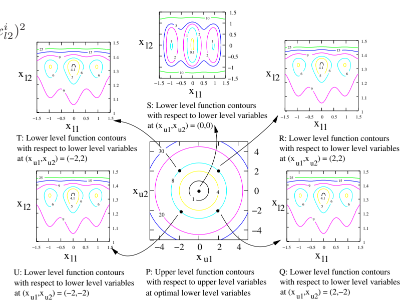
Upper and Lower Function Contours

Test Problem Construction

SMD3

$$F_1 = \sum_{i=1}^{p} (x_{u1}^i)^2$$
$$F_2 = \sum_{i=1}^{q} (x_{l1}^i)^2$$
$$F_3 = \sum_{i=1}^{r} (x_{u2}^i)^2 + \sum_{i=1}^{r} ((x_{u2}^i)^2 - \tan x_{l2}^i)^2$$
$$f_1 = \sum_{i=1}^{p} (x_{u1}^i)^2$$
$$f_2 = q + \sum_{i=1}^{q} \left((x_{l1}^i)^2 - \cos 2\pi x_{l1}^i \right)$$
$$f_3 = \sum_{i=1}^{r} ((x_{u2}^i)^2 - \tan x_{l2}^i)^2$$

$$x_{u1}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, p\}$$
$$x_{u2}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, r\}$$
$$x_{l1}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, q\}$$
$$x_{l2}^i \in (\tfrac{-\pi}{2}, \tfrac{\pi}{2}), \ \forall \ i \in \{1, 2, \ldots, r\}$$



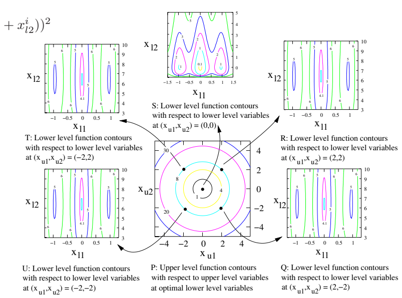
Upper and Lower Function Contours

Test Problem Construction

SMD4

$$F_1 = \sum_{i=1}^{p} (x_{u1}^i)^2$$
$$F_2 = -\sum_{i=1}^{q} (x_{l1}^i)^2$$
$$F_3 = \sum_{i=1}^{r} (x_{u2}^i)^2 - \sum_{i=1}^{r} (|x_{u2}^i| - \log(1 + x_{l2}^i))^2$$
$$f_1 = \sum_{i=1}^{p} (x_{u1}^i)^2$$
$$f_2 = q + \sum_{i=1}^{q} \left((x_{l1}^i)^2 - \cos 2\pi x_{l1}^i \right)$$
$$f_3 = \sum_{i=1}^{r} (|x_{u2}^i| - \log(1 + x_{l2}^i))^2$$

$$x_{u1}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, p\}$$
$$x_{u2}^i \in [-1, 1], \ \forall \ i \in \{1, 2, \ldots, r\}$$
$$x_{l1}^i \in [-5, 10], \ \forall \ i \in \{1, 2, \ldots, q\}$$
$$x_{l2}^i \in [0, e], \ \forall \ i \in \{1, 2, \ldots, r\}$$



Upper and Lower Function Contours

Results

- Following are the results for 10 variable instances of the SMD test problems using BLEAQ
- Comparison performed against nested evolutionary approach

Number of Runs: 21
Savings: Ratio of FE required by nested approach against BLEAQ

Pr. No.	Best Func. Evals.		Median Func. Evals.		Worst Func. Evals.	
	LL	UL	LL (Savings)	UL (Savings)	LL	UL
SDM1	99315	610	110716 (14.71)	740 (3.34)	170808	1490
SDM2	70032	376	91023 (16.49)	614 (3.65)	125851	1182
SDM3	110701	620	125546 (11.25)	900 (2.48)	137128	1094
SDM4	61326	410	81434 (13.59)	720 (2.27)	101438	1050
SDM5	102868	330	126371 (15.41)	632 (4.55)	168401	1050
SDM6	95687	734	118456 (14.12)	952 (3.25)	150124	1410

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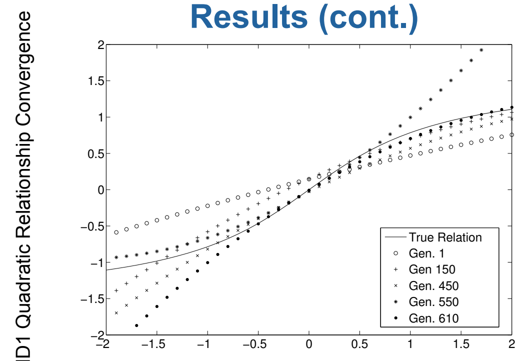
Results

Results (cont.)



Quadratic approximation at optima (0,0) improves with increasing generations

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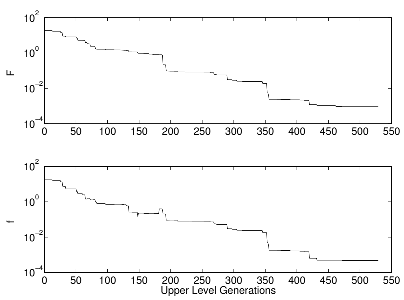
Results

Results (cont.)

Progress path for the elite member



Upper Level Function Value

Lower Level Function Value

Convergence Plot for SMD1

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Results

Results (cont.)

Median results for eight bilevel test problems collected from the literature.
Comparison against the evolutionary algorithm proposed by Wang et al. (2005)

	BLEAQ	WJL	Savings
TP1	15,432	85,499	5.54
TP2	15,632	256,227	16.39
TP3	4844	92,526	19.10
TP4	16,422	291,817	17.77
TP5	15,524	77,302	4.98
TP6	17,421	163,701	9.40
TP7	257,243	1,074,742	4.18
TP8	12,533	213,522	17.04

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EA's Niche

- Bilevel problems do exist in practice
- Classical approaches are not efficient
- Evolutionary algorithms so far show promise
  - Operator flexibility
  - Co-evolutionary approaches could be used
- Hard test problems suggested
- More open questions remain than answered
- Ideal platform to launch a coordinated research

Conclusions

- Bilevel problems are of interest both for theorists and practitioners
- Lack of efficient algorithms to handle such problems provide enormous scope for research
- The proposed evolutionary methodology is successfully able to handle non-linear bilevel problems
- More effort is required to reduce the number of lower level function evaluations
- Multiple objectives at one or both of the levels induce further challenges

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Contact any of the authors for further information:
• Ankur Sinha: ankur.sinha@aalto.fi
• Pekka Malo: pekka.malo@aalto.fi
• Kalyanmoy Deb: deb@iitk.ac.in

Thank You!