Particle Swarm Optimization Almost Surely Finds Local Optima

Manuel Schmitt Rolf Wanka
Department of Computer Science, University of Erlangen-Nuremberg, Germany
{manuel.schmitt, rolf.wanka}@cs.fau.de

ABSTRACT

Particle swarm optimization (PSO) is a popular nature-inspired meta-heuristic for solving continuous optimization problems. Although this technique is widely used, up to now only some partial aspects of the method have been formally investigated. In particular, while it is well-studied how to let the swarm converge to a single point in the search space, no general theoretical statements about this point or on the best position any particle has found have been known. For a very general class of objective functions, we provide for the first time results about the quality of the solution found. We show that a slightly adapted PSO almost surely finds a local optimum by investigating the newly defined *potential* of the swarm. The potential drops when the swarm approaches the point of convergence, but increases if the swarm remains close to a point that is not a local optimum, meaning that the swarm charges potential and continues its movement.

Categories and Subject Descriptors

I.2.8 [**Artificial Intelligence**]: Problem Solving, Control Methods, and Search—*heuristic methods*

General Terms

Algorithms, Performance, Theory

Keywords

Particle swarm optimization, quality of solution, fitness landscapes

1. INTRODUCTION

Background. Particle swarm optimization (PSO), introduced by Kennedy and Eberhart [10, 4], is a very popular meta-heuristic for solving continuous optimization problems. It is inspired by the social interaction of individuals living together in groups and supporting and cooperating with each other. Fields of very successful application are, among many others, Biomedical Image Processing [18], Geosciences [12], and Materials Science [14], to name just a few, where the continuous objective function on a multi-dimensional domain is not given in a closed form, but by a "black box." The popularity of the PSO framework in these scientific communities is due to the fact that it on the one hand can be realized and, if necessary, adapted to further needs easily, but on the other hand shows in experiments good performance results with respect to the quality of the obtained solution and the speed needed to obtain it. By adapting its parameters, users may in real-world applications easily and successfully control the swarm's behavior with respect to "exploration" ("searching where no one has searched before") and "exploitation" ("searching around a good position"). A thorough discussion of PSO can be found in [13].

To be precise, let an objective function $f : \mathbb{R}^D \to \mathbb{R}$ on a D-dimensional domain be given that (w. l. o. g.) has to be minimized. A population of *particles*, each consisting of a position (the candidate for a solution), a velocity and a local attractor, moves through the search space $\mathbb{R}^D$. The local attractor of a particle is the best position with respect to f this particle has encountered so far. The population in motion is the *swarm*. In contrast to evolutionary algorithms, the individuals of the swarm cooperate by sharing information about the search space via the global attractor, which is the best position any particle has found so far. The particles move in time-discrete iterations. The movement of a particle is governed by so-called movement equations that depend on both the particle's velocity and its two attractors and on some additional fixed parameters (for details, see Sec. 2).

Although this method is widely used in real-world applications, there has unfortunately not been any formal analysis explaining more than only partial aspects of the algorithm. A theoretical analysis of the particles' trajectories can be found in [2]. A discussion of runtime aspects is presented in [19]. For the case of a bounded high-dimensional search space, theoretical work about the initial behavior of the particle swarm is presented in [5, 6]. Some guidelines for the choice of the fixed parameters that control the impact of the current velocity and the attractors on the updated velocity of a particle can be found in [16, 8], where the authors prove that under certain conditions about the parameters the swarm provably converges. However, mathematical properties of the limit, i. e., the quality of the solution, are unknown. Van den Bergh/Engelbrecht [17] substantially modify the movement equations, enabling the particles to count the number of times they improve the global attrac-

GECCO'13, July 6–10, 2013, Amsterdam, The Netherlands.

tor and use that information. Empirical evidence for the capability of their method to find local optima on common benchmarks is given.

Closest to our work, Lehre/Witt [11] modify the movement equations by adding in *every* step a small random perturbation to the velocity. Their detailed analysis for one specific, simple, one-dimensional objective function shows that their modified PSO finds this function's unique optimum. But it is not clear how to apply this special analysis to higher dimensional search spaces and larger classes of functions.

An overview of known theoretical results about PSO can be found in [20].

New results. Up to now, every formal analysis providing results about the quality of the best search point obtained by the particle swarm has been restricted to a very special class of objective functions. In this paper, we provide the first *general* mathematical analysis of the quality of the global attractor when it is considered as a solution for objective functions from a very general class $\mathcal{F}$ of functions (see Def. 2 below) and therefore of the quality of the algorithm's return value. Informally, functions in $\mathcal{F}$ resemble (if restricted to 3-dimensional pictures) an (everted) island. On its almost arbitrary surface, the lowest point has to be found. The sea around the island is of no interest. We introduce the new approach of defining the *potential* Φ of the particle swarm that changes after every step. Φ covers two properties of the swarm: It tends to zero if the particles converge, but it increases if the whole swarm stays close to a search point that is no local minimum. In the latter case, the swarm charges potential and resumes its movement. As a consequence, we prove an emergent property of PSO for $\mathcal{F}$, namely that in the one-dimensional case the swarm almost surely (in the well-defined probabilistic sense) finds a local optimum. For the general D-dimensional case, we slightly modify the classical PSO and prove that this adapted swarm almost surely finds a local optimum. Necessity of such a modification is constituted in [15], where we report on significant experiments that show that even on the very simple sphere function the swarm does not necessarily converge towards the only local optimum at the center. Note that our analysis carries over to presumably all variants of PSO developed so far. One would expect such strong results to hold only for a very small class of objective functions. Indeed, we need some restrictions, but as it turns out the class $\mathcal{F}$ of objective functions for which our results hold is much more general than, e. g., the subset of the class of unimodal functions that is considered in [7] in the context of restricted $(1+1)$ evolutionary algorithms.

2. ANALYSIS OF THE CLASSICAL PSO ALGORITHM

First we present the model we are going to use for our analysis of the PSO algorithm. The model describes the positions of the particles, the velocities and the global and local attractors as real-valued stochastic processes. Furthermore, we define in Def. 5 the potential of the swarm which depends on the state of the particles and will be a measure for their movement. A swarm with high potential is more likely to reach search points far away from the current global attractor, while a swarm with potential approaching 0 is converging. (The basic mathematical tools from probability theory we need for our analysis can be found in, e. g., [3].)

Definition 1 (Classical PSO Process) *A* swarm $\mathcal{S}$ *of* N *particles moves through the* D*-dimensional search space* $\mathbb{R}^D$*. Let* $f : \mathbb{R}^D \to \mathbb{R}$ *be the objective function. For* $\mathcal{S}$*, we define the stochastic process* $(S_t)_{t\in\mathbb{N}_0} = ((X_t, V_t, L_t, G_t))_{t\in\mathbb{N}_0} = ((X_0, V_0, L_0, G_0), (X_1, V_1, L_1, G_1), \ldots)$*, consisting of*

- $X_t = (X_t^{n,d})_{1\le n\le N, 1\le d\le D}$ *(*d*-th coordinate of the* position *of particle* n *after step* t*),*
- $V_t = (V_t^{n,d})_{1\le n\le N, 1\le d\le D}$ *(*d*-th coordinate of the* velocity *of particle* n *after step* t*),*
- $L_t = (L_t^{n,d})_{1\le n\le N, 1\le d\le D}$ *(*d*-th coordinate of the* local attractor *of particle* n *after step* t*),*
- $G_t = (G_t^{n,d})_{1\le n\le N, 1\le d\le D}$*, (*$d$*-th coordinate of the* global attractor *before the* t*-th step of particle* n*).*

We will write X_t^n *for the vector* $(X_t^{n,1}, ..., X_t^{n,D})$ *(analogously,* V_t^n*,* L_t^n*,* G_t^n*) and* X_t^d *for the vector* $(X_t^{1,d}, ..., X_t^{N,d})$ *(analogously,* V_t^d*,* L_t^d*,* G_t^d*). Furthermore* $\tilde{G}_t^{n,d}$ *denotes the* d*-th coordinate of the global attractor* after *the* t*-th step of particle* n*, i. e.,* $\tilde{G}_t^{n,d} = G_t^{n+1,d}$ *if* $n < N$*, and* $\tilde{G}_t^{N,d} = G_{t+1}^{1,d}$*. With a given distribution for* (X_0, V_0) *and the values* $G_0^1 := \operatorname{argmin}_{1\le n\le N}\{f(X_0^n)\}$ *and* $L_0 := X_0$*,* $S_{t+1} = (X_{t+1}, V_{t+1}, L_{t+1}, G_{t+1})$ *is determined by the following recursive equations that are called the* movement equations*:*

- $G_t^{n+1} = \operatorname{argmin}\{f(L_t^n), f(G_t^n)\}$ *for* $t \ge 0$*,* $1 \le n \le N-1$*,*
- $G_{t+1}^1 = \operatorname{argmin}\{f(L_t^N), f(G_t^N)\}$ *for* $t \ge 0$*,*
- $V_{t+1}^{n,d} = \chi \cdot V_t^{n,d} + c_1 \cdot r_t^{n,d} \cdot (L_t^{n,d} - X_t^{n,d}) + c_2 \cdot s_t^{n,d} \cdot (G_{t+1}^{n,d} - X_t^{n,d})$ *for* $t \ge 0$*,*
- $X_{t+1}^{n,d} = X_t^{n,d} + V_{t+1}^{n,d}$ *for* $t \ge 0$*,*
- $L_{t+1}^n = \operatorname{argmin}\{f(X_{t+1}^n), f(L_t^n)\}$.

In case of a tie when applying argmin*, the new value prevails, i. e., whenever a particle finds a search point with value equal to the one of its local attractor, this point becomes the new local attractor. If additionally the value is equal to the one of the global attractor, this one is also updated. Here,* χ*,* c_1 *and* c_2 *are some positive constants called the* fixed parameters *of* $\mathcal{S}$*, and* $r_t^{n,d}$*,* $s_t^{n,d}$ *are uniformly distributed over* $[0, 1]$ *and all independent.*

If after the t-th step the process is stopped, the *solution* found by $\mathcal{S}$ so far is $\tilde{G}_t^N$.

Basically, this definition describes the common movement equations with two specifications: If a particle visits a point with the same objective value as its local attractor or the global attractor, then the attractor is updated to the new point. And the global attractor is updated after every step of a single particle, not only after every iteration of the whole swarm. Another common variant of PSO only updates the global attractor after every iteration of the whole swarm, however, due to our choice the information shared between the particles is as recent as possible.

PSO is designed to handle any objective function. But for the rest of this paper, we consider only objective functions from the set $\mathcal{F}$ defined below.

Definition 2 *Let $f : \mathbb{R}^D \to \mathbb{R}$ be a function. $f \in \mathcal{F}$ iff*

(i) there is a compact set $K \subset \mathbb{R}^D$ with positive Lebesgue measure, such that $P(X_0^n) \in K = 1$ for every n and $\left\{x \in \mathbb{R}^D \mid f(x) \leq \sup_K f\right\}$ (the island) is bounded;

(ii) f is continuous and has a continuous derivative.

Restriction (i) states that there is a compact set K such that for all $x \in K$, the set of all search points y as least as good as x, i. e., all y with $f(y) \leq f(x)$, is a bounded set. Since the particles are initialized inside K and since $f(G_t^n)$ is nonincreasing in t, (i) ensures that the possible area for the global attractor is limited if the positions of all particles are initialized inside of K (being on any point of the island is better than being in the sea). If for example $\lim_{|x|\to\infty} f(x) = \infty$ or if f has compact support and is negative on K, (i) is already satisfied. E. g., common benchmark functions like the sphere function $f(x) = \sum_{i=1}^{D} x_i^2$ or the Rosenbrock function $f(x) = \sum_{i=1}^{D-1}((1 - x_i)^2 + 100 \cdot (x_{i+1} - x_i^2)^2)$ are in $\mathcal{F}$. On functions that violate (i), the swarm might move forever because either they do not necessarily have a local optimum like $f(x) = x$ or they have an optimum, but improvements can be made arbitrary far away from it, like, e. g. in the case of the function $f(x) = x^2/(x^4 + 1)$, where $x = 0$ is the only local and the global optimum, but if the particles are far away from 0, they tend to further increase the distance because f converges to 0 as $|x|$ approaches ∞. Under such circumstances, convergence cannot be expected and it is necessary to restrict the function class in order to avoid this. However, (ii) might be the only true restriction.

From Def. 1 it follows that $(S_t)_{t\in\mathbb{N}}$ has the strong Markov property. Another interesting property of this stochastic process that follows immediately from the movement equations is the following:

Observation 1 *Let $\mathcal{S}$ be a swarm and $((X_t, V_t, L_t, G_t))_{t\in\mathbb{N}_0}$ its corresponding stochastic process. Let $\mathcal{L}^k$ denote the k-dimensional Lebesgue measure, $\mathcal{L}[Y]$ the distribution of a random variable Y and "$\ll$" (just in this observation) absolute continuity between two distributions. Assuming $\mathcal{L}[(X_0, V_0)] \ll \mathcal{L}^{2\cdot N\cdot D}$, it follows $\mathcal{L}[(X_t, V_t)] \ll \mathcal{L}^{2\cdot N\cdot D}$ for every $t \geq 0$. If $X_t^n \neq \tilde{G}_t^n$ for every n, then for every $t' > t$, $\mathcal{L}[(X_{t'}, V_{t'}) \mid S_t] \ll \mathcal{L}^{2\cdot N\cdot D}$ almost surely*[1].

If the positions and velocities of the initial population are distributed in some natural way, e. g., uniformly at random over K, Obs. 1 states that the swarm has similar restrictions as a process consisting only of variables that are sampled u. a. r. in the sense that events with probability 0 in the latter case also have probability 0 in the first case. If the function contains no plateaus, we cannot expect the swarm to ever hit a local optimum since the probability to hit that particular point by just sampling is zero and so is the probability for hitting it with the swarm. Another consequence is that no point in $\mathbb{R}^D$ is visited more than once because that would not happen under uniform sampling since $\mathbb{R}^D$ is not enumerable. This implies that the well-studied equilibrium when every particle is at the global attractor and has velocity 0 is a state that the process may converge to but that will never be reached. Therefore we give a definition of the kind of convergent behavior that can be expected.

[1]Let $(\Omega, \mathcal{A}, P)$ be a probability space. An event $E \in \mathcal{A}$ happens almost surely iff $P(E) = 1$.

Definition 3 (Convergence) *Swarm $\mathcal{S}$* converges *if there almost surely is a point z such that the following two conditions hold:*

1. $\lim_{t\to\infty} V_t = 0$ *(the movement of the particles tends to zero),*
2. $\lim_{t\to\infty} X_t^n = z$ *for each $n \in \{1, \ldots, N\}$ (every particle moves towards z).*

A consequence of the above conditions is that for every $n \in \{1, \ldots, N\}$ $\lim_{t\to\infty} G_t^n = z$ almost surely and $\lim_{t\to\infty} L_t^n = z$ almost surely. Although the convergence analysis in the literature ([8]) usually makes the assumption that at least the global attractor is constant forever, a prerequisite that because of Observation 1 we cannot assume, the generalization of the convergence proof from [8], showing that their results still hold under the weaker assumption of only the convergence of the attractors, is straight-forward.

We want to prove more, namely, that under the stated assumptions about f the swarm is able to find a local minimum. Here, the notion of the potential of a swarm comes into play. Roughly speaking, as long as the swarm has potential high enough to overcome the distance to at least one local minimum, the probability to find it within a few steps is still positive. A problem occurs when the value of the potential is too low for the swarm to overcome the distance to the next optimum by only a small number of steps. In other words, if f is monotonically decreasing in some direction and on an area that is large in comparison to the potential of the swarm, the particles must be able to "run down the hill," i. e., they must be able to surpass every distance as long as f decreases. The following definition formally describes the situation of a swarm while it is "running down the hill" and will lead to the definition of the potential.

Definition 4 (Running Particle Swarm) *Let $d_0 \leq D$ be an arbitrary dimension. We call $\mathcal{S}$ positively running in direction d_0, if the following two properties hold: First, $G_t^{n,d_0} = \max_{1\leq i\leq N}\{X_{t'}^{i,d_0}\}$ for $t' = t$ if $i < n$ and $t' = t - 1$ otherwise. Second, $L_t^n = X_t^n$ for every n. In other words, while the swarm is running, each particle updates its local attractor at every step and the global attractor is always the local attractor with greatest value in the d_0-th dimension. If the global attractor is always the position with smallest d_0-value instead, we call the swarm negatively running in direction d_0.*

Note that while a particle swarm is positively running, every particle has positive velocity in direction d_0, in terms: $V_t^{n,d_0} > 0$. If the swarm is negatively running in direction d_0, then $V_t^{n,d_0} < 0$. Intuitively one may think of running as the behavior a swarm shows when it moves through an area that is monotone in one dimension and changes in any other dimension are insignificant.

For our analysis, we now define the potential of $\mathcal{S}$.

Definition 5 (Potential) *For $a > 0$, the potential of swarm $\mathcal{S}$ in dimension d right before the t-th step of particle n is $\Phi_t^{n,d}$ and its potential in dimension d after the t-th step of*

particle n is $\tilde{\Phi}_t^{n,d}$ with

$$\Phi_t^{n,d} := \sqrt{\sum_{n'=1}^{N} \left(a \cdot |V_{t-1}^{n',d}| + |G_t^{n,d} - X_{t-1}^{n',d}|\right)},$$

$$\tilde{\Phi}_t^{n,d} := \sqrt{\sum_{n'=1}^{N} \left(a \cdot |V_t^{n',d}| + |\tilde{G}_t^{n,d} - X_t^{n',d}|\right)}.$$

Basically, the potential Φ is an extension of the physical interpretation of the particle swarm model. If the particles move faster and get farther away from their global attractor, the potential increases. If the swarm converges, the potential tends towards 0. However, for technical reasons explained later, we need the additional parameter a and the square root. Note that general tendencies towards 0 or ∞ are invariant under different choices of a.

Example 1 *Consider a 1-dimensional particle swarm and the objective function $f(x) = -x$. Assume that the velocities of the particles are all positive. Then the swarm is positively running in direction 1 from 0 until ∞. It is obvious that the position with the greatest x-value leads to the smallest value of $f(x)$ and therefore becomes the global attractor. It remains to prove that the velocity of every particle stays positive. Given the old velocity $V_t^{n,1}$, the new velocity $V_{t+1}^{n,1}$ is a positive linear combination of the three components $V_t^{n,1}$, $G_t^{n,1} - X_t^{n,1}$ and $L_t^{n,1} - X_t^{n,1}$. The value for $V_t^{n,1}$ is positive by assumption, $G_t^{n,1} - X_t^{n,1}$ and $L_t^{n,1} - X_t^{n,1}$ are non-negative since $G_t^{n,1} \geq L_t^{n,1} \geq X_t^{n,1}$. Therefore, the velocity stays positive and the swarm will stay positively running forever. In that situation, a good behavior would be increasing (or at least non-decreasing) Φ.*

Informally speaking, if a swarm $\mathcal{S}$ has a too little potential Φ left to make it to the next local minimum, it is necessary that Φ increases after $\mathcal{S}$ has become running, and so Φ enables the swarm to overcome every distance. The following lemma is the central technical observation of our work and makes a statement about how to choose the parameters to make sure that a running swarm has an increasing potential.

Lemma 1 (Running to Infinity Lemma) *For certain parameters N, χ, c_1, c_2 and the swarm $\mathcal{S}$ positively (negatively, resp.) running in direction d_0, $V_t^{n,d_0} + X_t^{n,d_0}$ ($-V_t^{n,d_0} - X_t^{n,d_0}$, resp.) tends to ∞ for every n almost surely. In particular, the swarm leaves every bounded set $B \subset \mathbb{R}^D$ almost surely.*

Proof. W. l. o. g., assume that the swarm is positively running. The main idea of this proof is to show that on expectation the potential significantly increases after every iteration of the particle swarm. More precisely, we want to show that $\mathrm{E}[1/\Phi_t^{n,d_0}] \xrightarrow{t\to\infty} 0$, which is equivalent to $1/\Phi_t^{n,d_0} \xrightarrow{t\to\infty} 0$ almost surely and therefore $\Phi_t^{n,d_0} \xrightarrow{t\to\infty} \infty$ almost surely. Our strategy to prove the convergence of $\mathrm{E}[1/\Phi_t^{n,d_0}]$ is to prove that for $\Phi_t^{n,d_0} \neq 0$ the bound $\mathrm{E}[\Phi_t^{n,d_0}/\Phi_{t+1}^{n,d_0} \mid S_t] \leq q$ holds for some fixed $q < 1$ almost surely (due to Lemma 1, the case $\Phi_t^{n,d_0} = 0$ for some t has probability 0). We bound $\mathrm{E}[\Phi_t^{n,d_0}/\Phi_{t+1}^{n,d_0} \mid S_t]$ for our concrete choice of the potential from Def. 5. Although there are better choices for the definition of a potential leading to larger areas of parameters that match our requirements, for our existence proof this one is sufficient.

We need to determine the values for N, χ, c_1 and c_2, for which the potential fulfills

$$\mathrm{E}\left[\Phi_t^{n,d_0}/\Phi_{t+1}^{n,d_0} \mid S_t\right] \leq q \tag{1}$$

for a $q < 1$. In other words, during one iteration of all particles, we want the reciprocal of the potential to decrease on expectation by at least a factor of q. Since the contributions of the different particles to the whole potential might differ arbitrarily much, we can not expect that every step of a single particle decreases the potential of the swarm by a constant factor. More reasonable and sufficient for our purpose is to prove that the expectation during each step of a single particle is bounded from above by 1 and that there is at least one particle (e. g., the one with the largest contribution to the potential) with an expected decrease of the term bounded from above by q. Therefore we now focus only on the movement of one single particle n. If we can verify $\mathrm{E}[\Phi_t^{n,d_0}/\tilde{\Phi}_t^{n,d_0}] \leq 1$ for every n and $\mathrm{E}[\Phi_t^{n,d_0}/\tilde{\Phi}_t^{n,d_0}] \leq q$ for at least one n, that would imply (1). We show the calculation only for $n = 1$ since the situation is symmetric and we want to avoid too much mess with the indices.

$$\mathrm{E}\left[\frac{\Phi_{t+1}^{1,d_0}}{\tilde{\Phi}_{t+1}^{1,d_0}} \mid S_t\right] = \mathrm{E}\left[\sqrt{\frac{a \cdot V_t^{1,d_0} + N \cdot G_{t+1}^{1,d_0} - X_t^{1,d_0} + R}{a \cdot V_{t+1}^{1,d_0} + N \cdot \tilde{G}_{t+1}^{1,d_0} - X_{t+1}^{1,d_0} + R}} \mid S_t\right]$$

where we substituted R for $\sum_{n=2}^{N} a \cdot V_t^{n,d_0} - X_t^{n,d_0}$, describing the state of the other particles that did not move.

Now there are two cases. In the first case, the position of particle 1 is the global attractor, i. e., $G_{t+1}^{1,d_0} = X_t^{1,d_0}$. Then its move is deterministic and its new position will be the new global attractor. In this case, we obtain:

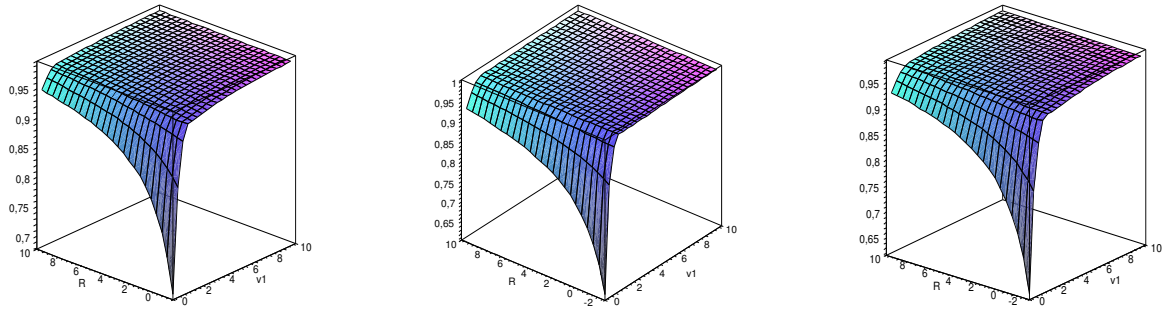
$$\mathrm{E}\left[\frac{\Phi_{t+1}^{1,d_0}}{\tilde{\Phi}_{t+1}^{1,d_0}} \mid S_t\right] = \mathrm{E}\left[\sqrt{\frac{a \cdot V_t^{1,d_0} + (N-1) \cdot G_{t+1}^{1,d_0} + R}{a \cdot V_{t+1}^{1,d_0} + (N-1) \cdot \tilde{G}_{t+1}^{1,d_0} + R}} \mid S_t\right]$$

$$= \sqrt{\frac{a \cdot V_t^{1,d_0} + (N-1) \cdot G_{t+1}^{1,d_0} + R}{a \cdot \chi \cdot V_t^{1,d_0} + (N-1) \cdot \left(G_{t+1}^{1,d_0} + \chi \cdot V_t^{1,d_0}\right) + R}}$$

$$= \sqrt{\frac{a \cdot V_t^{1,d_0} + (N-1) \cdot G_{t+1}^{1,d_0} + R}{(a + N - 1) \cdot \chi \cdot V_t^{1,d_0} + (N-1) \cdot G_{t+1}^{1,d_0} + R}},$$

which is less than 1 for every possible V_t^{1,d_0} and every possible R if and only if $a < (a + N - 1) \cdot \chi$. Given that, we can furthermore find the desired bound of a $q < 1$, if $(N-1) \cdot \tilde{G}_{t+1}^{1,d_0} + R < \mathrm{const} \cdot V_t^{1,d_0}$ for some constant. That is the case, if, e. g., particle 1 has the largest value of $a \cdot V + (G - X)$ among all particles of the swarm and therefore makes the largest contribution to the potential.

The second case when $G_{t+1}^{1,d_0} > X_t^{1,d_0}$ is more difficult. Since the potential is invariant under translation, we can assume $X_t^{1,d_0} = 0$. Furthermore, w. l. o. g., we set $G_{t+1}^{1,d_0} = 1$. Otherwise we could scale $\hat{\Phi}_t^1$ and $\tilde{L}_t^1$ by the factor $1/G_{t+1}^{1,d_0}$ and replace $V_t^{1,d_0}/G_{t+1}^{1,d_0}$ by V_t^{1,d_0} and $R/G_{t+1}^{1,d_0}$ by R which does not change the result since we need to bound the term for every positive V_{t-1}^{1,d_0} and every $R > -(N-1) \cdot G_{t+1}^{1,d_0}$. We



(a) $\chi = 0.729$, $c_2 = 1.49$, $N = 2$, $a = 2.69$ [2]

(b) $\chi = 0.729$, $c_2 = 1.3 \cdot \chi$, $N = 3$, $a = 5.38$ [1]

(c) $\chi = 0.6$, $c_2 = 1.7$, $N = 3$, $a = 2.98$ [16]

Figure 1: The integral $I(\chi, c_2, N, a)$ for different parameter choices

obtain:

$$\begin{aligned}
&\mathrm{E}\left[\frac{\Phi_{t+1}^{1,d_0}}{\tilde{\Phi}_{t+1}^{1,d_0}} \mid S_t\right] \\
&= \mathrm{E}\left[\sqrt{\frac{a \cdot V_t^{1,d_0} + N \cdot G_{t+1}^{1,d_0} - X_t^{1,d_0} + R}{a \cdot V_{t+1}^{1,d_0} + N \cdot \tilde{G}_{t+1}^{1,d_0} - X_{t+1}^{1,d_0} + R}} \mid S_t\right] \\
&= \mathrm{E}\left[\sqrt{\frac{a \cdot V_t^{1,d_0} + N + R}{a \cdot V_{t+1}^{1,d_0} + N \cdot \tilde{G}_{t+1}^{1,d_0} - X_{t+1}^{1,d_0} + R}} \mid S_t\right]
\end{aligned}$$

Now we insert the movement equations from Def. 1

$$\begin{aligned}
&= \mathrm{E}\left[\left(\frac{a \cdot V_t^{1,d_0} + N + R}{(a-1) \cdot \left(\chi \cdot V_t^{1,d_0} + c_2 \cdot s_t^{1,d_0}\right) + \right.\right. \\
&\qquad \left.\left. + N \cdot \max\left\{1, \chi \cdot V_t^{1,d_0} + c_2 \cdot s_t^{1,d_0}\right\} + R}\right)^{1/2} \mid S_t\right] \\
&= \int_0^1 \left(\frac{a \cdot V_t^{1,d_0} + N + R}{(a-1) \cdot \left(\chi \cdot V_t^{1,d_0} + c_2 \cdot s\right) + \right. \\
&\qquad \left. + N \cdot \max\left\{1, \chi \cdot V_t^{1,d_0} + c_2 \cdot s\right\} + R}\right)^{1/2} ds \\
&=: I(\chi, c_2, N, a)
\end{aligned}$$

Here we make use of the square root in our potential! The integral $I(\chi, c_2, N, a)$ is a function containing a case distinction (because of the max) and square root terms. So, the desired bound can be shown by tedious, but straight-forward calculation, after eliminating the square roots, a polynomial remains. If we had left out the square root in the definition of the potential, the result of the integral would have contained logarithmic terms that can not be handled that easily. We plotted the integral (see Fig. 1) for three different choices of the parameters, so one can easily verify that for the typical parameter sets found in the literature and a good choice of a, only two or three particles are sufficient to keep the value always less than 1 and that under the assumption that particle 1 makes the largest contribution to the potential, one can find an upper bound $\hat{q} < 1$. That finishes the proof. □

In Fig. 2, one can see the borderlines between choices for c_2 and χ that satisfy the conditions of Lemma 1 and those that do not. The parameters that satisfy both Lemma 1 and convergence requirements will be referred to as good parameters. Note that in [11], the authors have shown the existence of "bad" parameters that allow stagnation on arbitrary search points with positive probability. They have proven that for $N = 1$, all choices of parameters are bad, so a swarm with good parameters needs at least two particles. From here on we assume the parameters to be good.

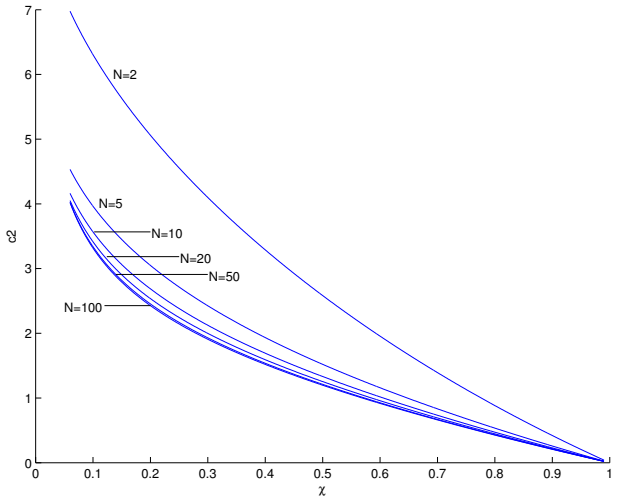


Figure 2: Borders between the too low values for c_2 and χ and the ones large enough to satisfy the requirements in Lemma 1 for some swarm sizes N.

Lemma 1 says that, given the parameters are good, a swarm that moves into the right direction can overcome every distance and increase its potential, no matter how small it was in the beginning. In other words: The equilibrium when all attractors and particles are on the same point and every velocity is zero is not stable because arbitrary small changes of an attractor, a position or a velocity can be sufficient to lead the swarm far away from this equilibrium, as long as there is a direction with decreasing value of the fitness function.

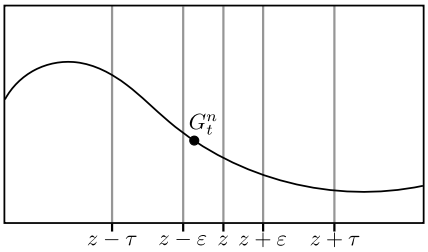


Figure 3: Fitness function f monotonic on $B_\tau(z)$, global attractor in ε-neighborhood of z

Theorem 1 *If $D = 1$, then every accumulation point of $(G_t^n)_{n=1,\dots,N;t\in\mathbb{N}}$ is a local minimum of f almost surely.*

Proof. Assume for contradiction, that there is an accumulation point $z \in \mathbb{R}$ such that (w. l. o. g.) f is monotonically decreasing on $B_\tau(z) = (z-\tau, z+\tau)$ for some $\tau > 0$. Since z is an accumulation point, for every $\varepsilon > 0$ G_t^n is inside the neighborhood $B_\varepsilon(z) = (z-\varepsilon, z+\varepsilon)$ of z infinitely often. Fig. 3 gives a visualization of the described situation. For no n and no t, G_t^n enters $[z, z+\tau]$ because otherwise it could never again get closer to z since $f(G_t^n)$ is decreasing with t. Let for some n_0 and some t_0 be $G_{t_0}^{n_0} \in B_\varepsilon(z)$. Now we consider two cases: The first case is that there is at least one particle n (maybe n_0 itself) that has position outside of $B_\tau(z)$, i. e., if the swarm has large potential, this will happen with positive probability. The set $M := \{x \in \mathbb{R}^D \mid f(x-\varepsilon) > f(x) > f(z)\}$, which is the set of all search points that improve the global attractor without abandoning z as an accumulation point, can be made arbitrary small by choosing ε small. So we can assume that with positive probability $M \setminus B_\tau(z)$ will not be hit by particle n_0 within the next iteration. The only situation violating this assumption is when a particle has position and local attractor close to $G_t^{n_0}$ and a velocity greater than τ and pointing towards a point in M, because the influence of the random terms in the movement equation for its next move will be small unless another particle updates the global attractor before. However, for this situation to occur a very specific value of the velocity is necessary, so it will with positive and actually comparatively high probability not happen.

Now we will outline a sequence of iterations that leads particle n to $[z, z+\tau]$. In the worst case, particle n has a large velocity compared to the distances to the local and global attractor. However, after the first constant number of iterations this is no longer the case because this huge velocity will lead the particle far away from the attractors, so it points away from $G_t^{n_0}$ and can therefore be consumed. How many iterations this takes depends on the parameters of the swarm. When a state is reached where the velocity of particle n is much smaller than the distance to the attractors, the global attractor in particular, it is easy to see that the probability for hitting $[z, z+\tau]$ within the next constant number of iterations is positive. I. e., assuming $c_2 > 1$, one decreases the velocity between 0 and $\tau/(3\cdot\chi)$. Then, with $r_t^n \le \tau/(3\cdot c_1\cdot(L_t^n - X_t^n))$ and $s_t^n \in [1/c_2 + (z - G_{t+1}^n)/(c_2\cdot(G_{t+1}^n - X_t^n)), 1/c_2 + \tau/(3\cdot c_2\cdot(G_{t+1}^n - X_t^n))]$, the next step leads to $[z, z+\tau]$. If $c_2 < 1$, the argument is similar, one needs just more iterations. Note that the probability for the described sequence is constant and therefore the probability for this sequence to happen never within the infinite number of times when the global attractor is within an ε from z is 0.

It remains to cover the case when no particle has position outside of $B_\tau(z)$, so assume the potential is low enough such that all particles will stay inside $B_\tau(z)$ forever, which means that the local attractors are inside $B_\tau(z)$ as well. That is the point were Lemma 1 becomes useful. Since f is monotonic on $B_\tau(z)$, the local and the global attractor are always greater or equal to the current position of the particle. Therefore the velocities will with probability 1 all become positive after a finite number of iterations and stay positive. It follows that each particle will exceed its local attractor almost surely after a finite number of iterations. Let t_1 be the time when the last particle has exceeded its local attractor. Then the swarm is positively running from t_1 to t_2 with t_2 being the time when the first particle surpasses a local minimum and therefore leaves or has already left $B_\tau(z)$. With Lemma 1, this will happen after a finite number of iterations almost surely. So, z is no accumulation point of G_t^n. □

Corollary 1 *If $D = 1$, then $f(G_t^n)$ converges towards the value of a local minimum. Particularly, if no two local minima have the same value, then G_t^n converges towards a local minimum. If the swarm converges towards a point $z \in \mathbb{R}$, then z is a local minimum.*

Proof. The first statement follows directly from Thm. 1. From Def. 2 it follows that the sequence of the global attractors over the time is bounded and therefore has at least one accumulation point. If there is more than one accumulation point, then f has the same value on each of them because f is continuous. Due to Thm. 1 every accumulation point is a local minimum, so if there are no two local minima with the same value, there is only one accumulation point that therefore is the limit of G_t^n. That proves the second statement. The third statement again is a direct consequence of Thm. 1 because convergence of the swarm implies convergence of G_t^n. □

3. MODIFIED PSO ALGORITHM

Now the question arises how much of the results from the 1-dimensional PSO can be transferred to the general case. Unfortunately, the stated result is not true in a D-dimensional situation with $D > 1$. The main problem is the following: Assume that the whole swarm is close to a point that allows improvements neither in positive not in negative changes of the first coordinate. Furthermore let the swarm have high potential in the first and low potential in any other dimension. Then an improvement of the global attractor is still possible and will indeed happen infinitely often, but it is very unlikely and between two updates are many iterations without an update. The reason is that any improvement in some of the dimensions $2, ..., D$ is voided by the much larger worsening in dimension 1. In the meantime, the swarm tends to converge and therefore continuously looses potential and never gets running. A small and simple modification of the PSO algorithm avoids that problem by enabling the swarm to rebalance the potentials in the different dimensions:

Definition 6 (Modified PSO) *For some arbitrary small but fixed $\delta > 0$, we define the modified PSO via the same equations as the classic PSO in Def. 1, only modifying the*

third equation to

$$V_{t+1}^{n,d} = \begin{cases} (2 \cdot r_t^{n,d} - 1) \cdot \delta, \\ \quad \textit{if } \forall\, d' \in \{1, ..., D\} : |V_t^{n,d'}| + |G_{t+1}^{n,d'} - X_t^{n,d'}| < \delta, \\ \chi \cdot V_t^{n,d} + c_1 \cdot r_t^{n,d} \cdot (L_t^{n,d} - X_t^{n,d}) \\ \qquad + c_2 \cdot s_t^{n,d} \cdot (G_{t+1}^{n,d} - X_t^{n,d}), \\ \quad \textit{otherwise.} \end{cases}$$

Whenever the first case applies, we call the step forced.

In words: As soon as for one particle the sum of the velocity and the distance between the position and the global attractor are below the bound of δ in every single dimension, the updated velocity of this particle is drawn u. a. r. from the interval $[-\delta, \delta]$. Note the similarity between this condition and the definition of the potential. Indeed, we could have used the condition $\Phi_{t+1}^{n,d} < \delta$ (with some fixed a) instead, but we decided to keep the modification as simple and independent from the terms occurring in the analysis as possible. Now the potential can no longer converge to 0 while staying unbalanced because if it decreases below a certain bound, we randomly assign a value to the velocity which on expectation has an absolute value of $\delta/2$.

This modified PSO is similar to the Noisy PSO proposed by Lehre and Witt in [11] where they generally add a random perturbation drawn u. a. r. from $[-\delta/2, \delta/2]$ for some small δ and prove that their swarm is able to find a local optimum. However, their analysis is restricted to one specific 1-dimensional fitness function.

In case of our modified PSO, we consider the change from the classic PSO as comparatively simple. The main difference to previous approaches (e. g., [17]) is that the PSO uses the modification not as its engine. Rather, we will see that the number of forced steps is small and if the swarm is not already within an δ-neighborhood of a local optimum, after some forced steps the potential increases and the swarm switches back to classical steps, a behavior we also observed experimentally [15].

Note that we sacrifice the convergence of the swarm in order to increase the quality of the solution, since the potential cannot approach 0 anymore. Instead, we can only expect the global attractor to converge.

Theorem 2 *Using the modified PSO algorithm, every accumulation point of G_t^n is a local minimum of f almost surely.*

Proof. Assume, for contradiction, that there is some accumulation point z of G_t^n that is no local minimum. Then, in any neighborhood of z and therefore in particular in $B_\delta(z)$, there is a point $x_0 \in B_\delta(z)$ with $f(x_0) < f(z)$. Since f is continuous, x_0 has some neighborhood $B_\tau(x_0)$, such that $f(x) < f(z)$ for every $x \in B_\tau(x_0)$. Fig. 4 gives an overview over the situation.

The set $B_\tau(x_0)$ plays the role of the interval $(z, z+\tau)$ from the proof of Thm. 1. Now we investigate what happens when G_t^n enters $B_\varepsilon(z)$. This will for each $\varepsilon > 0$ happen infinitely often because z is an accumulation point. Like in the first case of the proof of Thm. 1, one can explicitely construct a sequence of iterations, leading a particle into $B_\tau(x_0)$ if the potential is sufficiently high. However, due to the modification of the movement equation, the potential cannot converge to 0 anymore, so the second case from the proof of 1 is unnecessary here. □

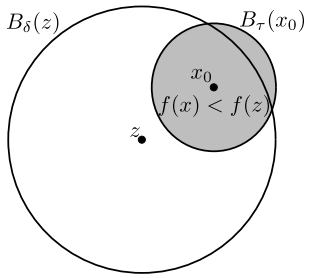


Figure 4: Every point x_0 with $f(x_0) < f(z)$ has a neighborhood $B_\tau(x_0)$, such that $f(x) < f(z)$ for $x \in B_\tau(x_0)$

This result is not surprising because in the modified PSO random perturbations occur when the swarm tends to converge and it is easy to see that small random perturbations can optimize any continuous function (but with a very poor runtime). Note that the proof of Thm. 2 does neither make use of f having a continuous derivative nor of Lemma 1. To supplement this result, we need to prove a statement about how often the modification actually applies. It is obvious that for δ chosen too large, every step of the particles could be forced. The case of δ being small with respect to the structure of the function is the interesting one. On the other hand, if the distance of a particle and a local optimum is smaller than δ, presumably all upcoming steps will be forced because there is no room for further improvements. But one can show that, given the swarm is sufficiently far away from the closest local optimum, the forced steps only balance the potentials between the different dimensions and enable the swarm to become running. In particular, consider the following situation: Let for some dimension d_0 and some $c \gg 1$ be $\frac{\partial f}{\partial d_0} < 0$ on a $(c \cdot \delta)$-neighborhood of the current global attractor and let the swarm have low potential, i. e., every particle has in every dimension potential of order δ. Instead of only being driven by the random perturbation, we would like the swarm to become running in direction d_0, increasing the potential in that direction, so the velocity updates can be done according to the classical case again.

Theorem 3 *In the situation described above, the probability for the swarm to become running within a constant number of iterations is positive and independent of δ.*

Proof. We will explicitely describe a possible sequence of iterations enabling the swarm to become running. First, the particles decrease their distance to the global attractor in every single dimension to at most $\delta \cdot \epsilon/2$ with $\varepsilon \ll 1$ and a velocity of absolute value less than $\delta \cdot (1 - \varepsilon/2)$, such that the local attractor is updated for all particles except the one whose local attractor is equal to the global attractor. If the current global attractor G_t^n is no local maximum, this can be done because every local attractor has a function value worse than the global attractor and since f is continuous, so the function values of f approach $f(G_t^n)$ when x approaches G_t^n. The case of G_t^n being a local maximum has probability 0. Then the next step of each particle is forced. In the next iteration, the velocity of every particle gets smaller than $\delta \cdot \varepsilon/2$ in each dimension except d_0. In dimension d_0, one particle obtains velocity greater than $\delta \cdot (1+\varepsilon)/2$, such that it gets to a search point that is in dimension d_0 more than $\delta/2$ and in any other dimension at most $\varepsilon \cdot \delta$ away from the previous global attractor. For ε sufficiently small, this particle

will update the global attractor since f has a positive partial derivative in dimension d_0. Every other particle obtains in d_0 a velocity less than $-\delta\cdot(1+\varepsilon)/2$, making sure that its new position and the new global attractor after that step differ by more than δ. So the next step will not be forced and the potentials have order $\sqrt{\delta}$ in dimension d_0 and only $\sqrt{\delta\cdot\epsilon}$ in every other dimension. So for ε sufficiently small with respect to the function f, the swarm will become running and therefore the steps will actually become unforced. □

The behavior of the modified PSO is the same as of the classic PSO, except that due to the modification the particles can overcome "corners," i. e., if the global attractor stagnates because the potential of wrong dimensions is too high and the potential of dimensions where the function value can be decreased is too low, the modification helps to balance the potentials of the different dimensions. The "blind" algorithm that just randomly checks a point around the previous best solution with range δ would of course find a local minimum but the running time would tend towards ∞ if δ approaches 0.

4. CONCLUSION

This paper focuses on the capability of a particle swarm to find a local minimum. The PSO algorithm is analyzed under this aspect and it is pointed out why the swarm might not always find a local minimum, namely the swarm gets stuck if the differences of the potentials between the dimensions are too large. A suggestion to modify the algorithm by randomly assigning a small velocity when the potential of a particle falls below a certain bound is suggested. It is proven that this modification together with some new parameter selection guidelines enables the swarm to find a local minimum for a large class of objective functions. Additionally, it is shown that the modification does not take over the swarm, it just corrects the direction before the classic movement equations are applied again.

Acknowledgement

We would like to thank Berthold Vöcking for useful advice on an earlier version of the paper, and the anonymous referees for their valuable remarks.

5. REFERENCES

[1] A. Carlisle and G. Dozier. An off-the-shelf PSO. In *Proc. Particle Swarm Optimization Workshop*, pages 1–6, 2001.

[2] M. Clerc and J. Kennedy. The particle swarm – explosion, stability, and convergence in a multidimensional complex space. *IEEE Transactions on Evolutionary Computation*, 6:58–73, 2002.

[3] R. Durrett. *Probability: Theory and Examples*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2010.

[4] R. C. Eberhart and J. Kennedy. A new optimizer using particle swarm theory. In *Proc. 6th International Symposium on Micro Machine and Human Science*, pages 39–43, 1995.

[5] S. Helwig. *Particle Swarms for Constrained Optimization*. PhD thesis, Department of Computer Science, University of Erlangen-Nuremberg, 2010. urn:nbn:de:bvb:29-opus-19334.

[6] S. Helwig and R. Wanka. Theoretical analysis of initial particle swarm behavior. In *Proc. 10th Int. Conf. on Parallel Problem Solving from Nature (PPSN)*, pages 889–898, 2008.

[7] J. Jägersküpper. Analysis of a simple evolutionary algorithm for minimization in Euclidean spaces. *Theoretical Comput. Sci.*, 379:329–347, 2007.

[8] M. Jiang, Y. P. Luo, and S. Y. Yang. Particle swarm optimization – stochastic trajectory analysis and parameter selection. In F. T. S. Chan and M. K. Tiwari, editors, *Swarm Intelligence – Focus on Ant and Particle Swarm Optimization*, pages 179–198. 2007. Corrected version of [9].

[9] M. Jiang, Y. P. Luo, and S. Y. Yang. Stochastic convergence analysis and parameter selection of the standard particle swarm optimization algorithm. *Inf. Process. Lett.*, 102:8–16, 2007. Corrected by [8].

[10] J. Kennedy and R. C. Eberhart. Particle swarm optimization. In *Proc. IEEE International Conference on Neural Networks*, volume 4, pages 1942–1948, 1995.

[11] P. K. Lehre and C. Witt. Finite first hitting time versus stochastic convergence in particle swarm optimisation, 2011. arXiv:1105.5540.

[12] J. E. Onwunalu and L. J. Durlofsky. Application of a particle swarm optimization algorithm for determining optimum well location and type. *Computational Geosciences*, 14:183–198, 2010.

[13] B. K. Panigrahi, Y. Shi, and M.-H. Lim, editors. *Handbook of Swarm Intelligence — Concepts, Principles and Applications*. Springer, 2011.

[14] K. Ramanathan, V. M. Periasamy, M. Pushpavanam, and U. Natarajan. Particle swarm optimisation of hardness in nickel diamond electro composites. *Archives of Computational Materials Science and Surface Engineering*, 1:232–236, 2009.

[15] M. Schmitt and R. Wanka. Particles prefer walking along the axes: Experimental insights into the behavior of a particle swarm. In *Proc. Genetic and Evolutionary Computation Conference (GECCO)*, 2013. Full version arXiv:1303.6145.

[16] I. C. Trelea. The particle swarm optimization algorithm: Convergence analysis and parameter selection. *Inf. Process. Lett.*, 85:317–325, 2003.

[17] F. van den Bergh and A. P. Engelbrecht. A new locally convergent particle swarm optimiser. In *Proc. IEEE Int. Conf. on Systems, Man and Cybernetics (SMC)*, volume 3, pages 94–99, 2002.

[18] M. P. Wachowiak, R. Smolíková, Y. Zheng, J. M. Zurada, and A. S. Elmaghraby. An approach to multimodal biomedical image registration utilizing particle swarm optimization. *IEEE Transactions on Evolutionary Computation*, 8:289–301, 2004.

[19] C. Witt. Why standard particle swarm optimisers elude a theoretical runtime analysis. In *Proc. 10th ACM SIGEVO Workshop on Foundations of Genetic Algorithms (FOGA)*, pages 13–20, 2009.

[20] C. Witt. Theory of particle swarm optimization. In A. Auger and B. Doerr, editors, *Theory of Randomized Search Heuristics*, chapter 7, pages 197–223. World Scientific Publishing, 2011.