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Online Objective Reduction for Many-Objective Optimization Problems

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***Abstract*—For many-objective optimization problems, i.e. the number of objectives is greater than three, the performance of most of the existing Evolutionary Multi-objective Optimization algorithms will deteriorate to a certain degree. It is therefore desirable to reduce many objectives to fewer essential objectives, if applicable. Currently, most of the existing objective reduction methods are based on objective selection, whose computational process is, however, laborious. In this paper, we will propose an online objective reduction method based on objective extraction for the many-objective optimization problems. It formulates the essential objective as a linear combination of the original objectives with the combination weights determined based on the correlations of each pair of the essential objectives. Subsequently, we will integrate it into NSGA-II. Numerical studies have show the efficacy of the proposed approach.**

I. Introduction

The Evolutionary Multiobjective Optimization (EMO) algorithms have successfully used in a variety of real-world applications. However, most of the well-known EMO algorithms, such as NSGA-II [1], [2], MOEA/D [3], [4], cannot work well for the many-objective optimization problem [5], i.e. the number of objectives is greater than three, because of the poor scalability of the most existing EMO algorithms, difficulty in visualization [6], [7], and high computational cost. Undoubtedly, many-objective optimization problems are more challenging compared to the 2-objective or 3-objective problems.

Recently, a number of efforts, e.g. modifying dominance relations [8], [9], [10], indicator based ranking [11], and substitute distance assignments [12], [13], have been made to deal with the many-objective optimization problems. Nevertheless, it is still desirable to first investigate whether the many-objective problem is really a many-objective one. More often than not, the problem at hand may have many objectives, but they may be reducible to fewer essential objectives, which can therefore be solved by an existing optimization algorithm.

In recent years, as summarized in [7], [14], a number of dimensionality reduction (also called *objective reduction* interchangeably) schemes for many-objective problems have been presented. The early seminal work discussing the issue of redundancy in objectives was presented by Gal and Leberling [15]. More extensive studies in this domain have been made in the last decade. For example, Saxena and Deb [7] have proposed a *Correlation-Based* Reduction method, in which a set of non-dominant solutions for dimensionality analysis is obtained by running NSGA-II for a large number of generations. Thereafter, the correlation matrix R is computed using the objective values of the final population. The eigenvalues and corresponding eigenvectors are then analyzed in order to reduce the objectives. Also, a *Dominance Structure-Based* Reduction method is presented in [6], [16]. They have investigated how adding and omitting an objective affects the problem characteristics. Formal definitions of conflict and redundancy of objective sets are discussed. It aims at finding a subset of objectives such that either the entire or most of the dominance relation is preserved. However, its time complexity is quite high, which limits its applications from the practical viewpoint. Recently, Jaimes et al. [17], [18] have developed a dimensionality reduction scheme based on *Feature Selection*. In their approach, the objective set is first divided into homogeneous neighborhoods based on a correlation matrix of a set of non-dominant solutions obtained by an EMO algorithm. The conflict between the objectives can be utilized as a distance metric. That is, the more conflict between the objectives, the more distance they are in the objective "conflict" space. Thereafter, the most compact neighborhood is chosen, and in which all the objectives except the center one are dropped as they are the least conflicting.

The methods stated above all try to find an objective subset from the original objectives set such that the entire or most of dominance relation is preserved. It is a binary hard decision on selecting the objectives. The numerical results have shown that these algorithms can successfully identify the redundant objectives. Nevertheless, these approaches are quite time-consuming and inapplicable to online objective reduction. In this paper, we will propose an online objective reduction method for many-objective optimization problems. It formulates the essential objective as a linear combination of the original objectives with the combination weights determined based on the correlations of each pair of the essential objectives. It is a continuous optimization problem and can be solved by an

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analysis method. Then, we integrate the proposed objective reduction method into an EMO algorithm to deal with many-objective optimization problems. Experimental results have shown the efficacy of the proposed method.

The remainder of this paper is organized as follows. Section II briefly introduces the many-objective optimization problem and the existing objective reduction methods. Section III gives the detailed description of the proposed objective reduction method, as well as its characteristics. We shall integrate the proposed objective reduction method into the NSGA-II algorithm and conduct the experiments in Section IV and Section V, respectively. Finally, we draw a conclusion in Section VI.

II. Correlation-based Objective Reduction

Without loss of generality, the multi-objective optimization problem can be formulated as follows:

$$\begin{aligned} \min \ & F(x) = (f_1(x), f_2(x), \dots, f_M(x))^T \\ s.t. \ & x \in \Omega, \end{aligned} \tag{1}$$

where $\Omega \subset R^n$ is the decision space and n is the dimensionality of the decision variable x. $F : \Omega \rightarrow R^M$ consists of M real-value objective functions. When $M \geq 4$, the problem (1) is regarded as a many-objective optimization problem.

Let $u = (u_1, \dots, u_M)^T$ and $v = (v_1, \dots, v_M)^T \in R^M$ be two solutions, u is said to dominate v ($u \prec v$) if and only if $u_i \leq v_i$ for all $i = 1, \dots, M$ and $u \neq v$. x^* is called Pareto optimal solution if there is no solution $x \in \Omega$ such that $F(x) \prec F(x^*)$. The set of all Pareto optimal solutions in Ω is denoted as $E(f, D)$. Also, the set of Pareto optimal solutions in the objective space is called as Pareto Front (PF).

Typically, most of the existing objective reduction methods select an objective subset from the original objective set such that the dominance relation with respect to (w.r.t.) the non-dominant set obtained by an EMO algorithm is preserved as much as possible. The mathematical description of the objective reduction in many-objective optimization problem can be formulated as follows: Let $\mathcal{X} = \{x_1, x_2, \dots, x_N\}$ be N non-dominant solutions obtained by an EMO algorithm. Then, an m-size ($m \leq M$) objective subset can be defined as

$$\begin{aligned} F'(x) &= (f_{k_1}(x), \dots, f_{k_m}(x))^T \\ &= \mathcal{I}^T (f_1(x), \dots, f_M(x))^T \end{aligned} \tag{2}$$

where $k_1, k_2, \dots, k_m \in \{1, 2, \dots, M\}$ and $\mathcal{I}_{Mm} = [e_{k_1}, e_{k_2}, \dots, e_{k_m}]$ is an index matrix with

$$e_{k_i} = (\underbrace{0, \dots, 0}_{k_i - 1}, 1, \underbrace{0, \dots, 0}_{M - k_i})^T.$$

We have $f_{k_i} = e_{k_i}^T (f_1, f_2, \dots, f_M)^T$. The objective reduction, or more accurately, objective selection can be regarded as finding the index matrix $\mathcal{I}$ such that the non-dominant relation w.r.t. $\mathcal{X}$ is preserved as much as possible. Since it is a discrete combinatorial optimization problem, the computation cost is quite high, which make it inapplicable to online objective reduction. Under the circumstances, It is desired to relax the discrete index matrix into contiguous weighted matrix to express the essential objective as a linear combination of the original objectives and solve it by an analysis method. Along this line, we will propose an online objective reduction method in the next section.

III. The Proposed Objective Reduction Method

A. A Novel Correlation-based Objective Reduction Model

In general, a negative correlation between each pair of objectives means that one objective increases while the other one decreases and vice versa [19]. Therefore, we claim that the more negative the correlation between two objectives, the more conflict between them. That is, we can use the correlation between two objectives to measure the degree of conflict between them. Subsequently, we propose an objective reduction (also called *objective extraction*) method based on correlation to deal with many-objective optimization problems. Specifically, the model of the objective reduction method proposed in this paper can be formulated as follows:

$$\begin{aligned} \min \ & L(W) = \sum_{i=1}^{m} \sum_{j=1, j \neq i}^{m} \left[\rho(g_i, g_j) + \lambda W_i^T W_j \right] \\ s.t. \ & \mathbf{0} \prec W_i, \quad i = 1, \cdots, m, \\ & \|W_i\|_2 = 1, \quad i = 1, \cdots, m. \end{aligned} \tag{3}$$

where $W = [W_1, \dots, W_m] \in R_+^{Mm}$ is the weighted matrix. The ith reduced objective $g_i(x) = W_i^T (f_1, \dots, f_M)^T$ expresses as a linear combination of the original objectives. $\rho(g_i, g_j)$ is the correlation between objectives g_i and g_j. The term $W_i^T W_j$ is the inner product of the vectors of W_i and W_j. This term makes the solutions of the model (3) are sparse. That is, most of the elements of W_i equal to zero, $i = 1, \dots, m$. $\lambda = 1$ is a constant parameter, which is used to control the level of the sparsity. The greater value of λ is, the sparser the solution of the model is. $\mathbf{0} \prec W_i$ means $W_i, i = 1, \dots, m$ is a positive non-zero vector. Then, the reduced objective optimization problem is:

$$\begin{aligned} & G(x) = W^T (f_1(x), \dots, f_M(x))^T \\ & s.t. \ x \in \Omega. \end{aligned} \tag{4}$$

In this problem, the variable W is a contiguous variable. According to the definition of correlation, problem (3) can be converted into the following optimization problem:

$$\begin{aligned} \min L(W) = & \sum_{i=1}^{m} \sum_{j=1, j \neq i}^{m} \left[\frac{W_i^T R W_j}{\|W_i\|_R \|W_j\|_R} + \lambda W_i^T W_j \right] \\ s.t. \ & \mathbf{0} \prec W_i, \quad i = 1, \cdots, m, \\ & \|W_i\|_2 = 1, \quad i = 1, \cdots, m. \end{aligned} \tag{5}$$

where R is the correlation matrix of $F(x)$ with respect to $\mathcal{X} = \{x_1, x_2, \dots, x_N\}$ and $\|W_i\|_R = \sqrt{W_i^T R W_i}$.

To solve problem (5), We optimize W_i alternately for $i = 1, \dots, m$. That is, supposing W_js with $j = 1, \dots, i-1, i+1, \dots, m$ are fixed, the ith optimization problem w.r.t. W_i, which is deduced from the problem (5), is given as follows:

$$\min l_i(W_i) = \sum_{j=1,j\neq i}^{m}\left[\frac{W_i^T R W_j}{\|W_i\|_R \|W_j\|_R} + \lambda W_i^T W_j\right] \\ s.t.\ \mathbf{0} \prec W_i, \\ \|W_i\|_2 = 1. \tag{6}$$

As this problem is a constraint optimization problem, the gradient projection method [20] can therefore be used to solve it in this paper.

The gradient projection method is based on projecting the search direction into the subspace tangent to the active constraints. The gradient of Eq.(6) is given as follows:

$$\nabla l_i = \sum_{j=1,j\neq i}^{m}\left[\frac{RW_j}{\|W_i\|_R\|W_j\|_R} - \frac{W_i^T R W_j R W_i}{\|W_i\|_R^3 \|W_j\|_R} + \lambda W_j\right]. \tag{7}$$

We choose an initial $\mathbf{0} \prec W_i$ and repeat

$$W_i^k = P(W_i^{k-1} - \alpha \nabla l_i) \tag{8}$$

where $P(x) = \frac{x_+}{\|x_+\|_2}$, x_+ is componentwise max of 0 and x, $\|x_+\|_2$ is the 2-norm of vector x_+, and α is the step length, which can be calculated by a linear search method. Since this is a constraint optimization problem, α is finite. According to the constraint $\mathbf{0} \prec W_i$, we have

$$\alpha_{max} < \max_j \left\{ \frac{W_{i,j}}{\nabla l_{i,j}} \Big| \nabla l_{i,j} > 0 \right\}, \tag{9}$$

where $W_{i,j}$, $\nabla l_{i,j}$ are the jth element of W_i and ∇l_i, respectively. The pseudocode of computing the weighted matrix is given in Algorithm 1.

Algorithm 1: Compute the Weighted Matrix W

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input :
- A stopping criterion;
- The correlation matrix R;
- The initial weighted matrix W.

output: The weighted matrix W.

while *the stopping criteria is not met* **do**
- **for** $i \leftarrow 1$ **to** m **do**
  - Compute the gradient w.r.t. W_i by Eq. (7);
  - Perform the linear search on α to obtain the optimal step length α by solving problem:
  $$\alpha_{best} \leftarrow arg \min_{0\leq\alpha<\alpha_{max}} l_i(P(W_i - \alpha\nabla l_i));$$
  - $W_i \leftarrow P(W_i - \alpha_{best}\nabla l_i)$.
- **end**

end

B. Characteristics of the Proposed Objective Reduction Method

The characteristics of the proposed objective reduction method are two-fold:

1) Preservation of the Dominance Relation: The reduced problem $G(x)$ defined in Eq. (4) can preserve the dominance relation. Subsequently, we have the following proposition:

Proposition 1: Let $\mathbf{0} \prec W_i$ with $i = 1, \ldots, m$. For any $u, v \in \Omega$, if $F(u) \prec F(v)$, we have $G(u) \prec G(v)$. That is, $E(G, \Omega) \subset E(F, \Omega)$.

This proposition claims that for any non-dominant solution of the reduced problem $G(x)$, it is also a non-dominant solution of the original problem $F(x)$, but not vice versa. It means that we can obtain a subset of the non-dominant solutions of the original problem. Moreover, it also gives the reason to limit $\mathbf{0} \prec W_i$.

2) Preservation of the Non-dominant Relation as Much as Possible: The proposed objective reduction method can preserve the non-dominant as much as possible. Subsequently, we have the following proposition:

Proposition 2: Let $\mathcal{X} = \{x_1, x_2, \ldots, x_N\}$ be N non-dominant solutions w.r.t. the original F. For any $k_1, k_2, \ldots, k_m \in \{1, 2, \ldots, M\}$, we have

$$\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m} \rho(g_i, g_j) \leq \sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m} \rho(f_{k_i}, f_{k_j}).$$

Proof: It is easy to verify that

$$L(\mathcal{I}) = \sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\left[\rho(f_{k_i}, f_{k_j}) + \lambda\langle e_{k_i}, e_{k_j}\rangle\right]$$

Noting that $e_{k_i}^T e_{k_j} = 0$ as $i \neq j$, we have

$$L(\mathcal{I}) = \sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m} \rho(f_{k_i}, f_{k_j}).$$

Moreover, based on the model in Eq. (3), we have

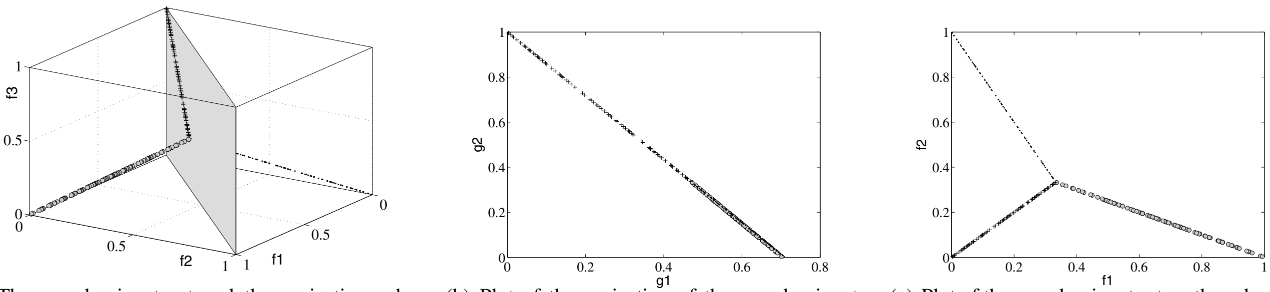
$$\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m} \rho(g_i, g_j) \leq \sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\left[\rho(g_i, g_j) + \lambda W_i^T W_j\right] = L(W).$$

Since W is the optimal solution of problem (3), we have $L(W) \leq L(\mathcal{I})$. This implies that

$$\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \rho(g_i, g_j) \leq \sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \rho(f_{k_i}, f_{k_j}).$$

■

From this proposition, we can know that, as $\mathcal{I}$ is the optimal solution of problem (3), the proposed objective reduction method is equivalent to the objective selection. It means that the most of existing objective selection is a special case of the proposed objective reduction method. This proposition also implies that one can use fewer objectives to preserve the non-dominant relation than the objective reduction method based on objective selection. This claim can be further justified by a test problem: Its PF is three line segments that join the center of the plane $f_1 + f_2 + f_3 = 1$ with the points $(1, 0, 0)^T, (0, 1, 0)^T$ and $(0, 0, 1)^T$, as illustrated in Fig. 1(a). We randomly generated 300 points on the PF as the non-dominant solution. The optimal value of W obtained via



(a) The non-dominant set and the projection subspace spanned with W_1 and W_2

(b) Plot of the projection of the non-dominant set on the subspace spanned with W_1 and W_2

(c) Plot of the non-dominant set on the subspace of f_1 versus f_2

Fig. 1. Illustration of the proposed objective reduction method on a synthetic test problem.

Eq. (5) was: $W_1 = (0.7041, 0.7101, 0)^T$ and $W_2 = (0, 0, 1)^T$. The distribution of the reduced objectives, i.e. the projection of the original objective values on the subspace spanned with W_1 and W_2, is shown in Fig. 1(b). Moreover, the distribution of the projection of the original objective on the subspace $\{f_1, f_2\}$, $\{f_1, f_3\}$ and $\{f_2, f_3\}$ is the same. Also, we plot the original objectives of the non-dominant solution on subspace $\{f_1, f_2\}$ in Fig. 1(c). From Fig. 1(b) it can be seen that the non-dominant relation of the most solutions is preserved. That is, we can find most of the non-dominant solutions in this subspace. However, the non-dominant relation cannot be preserved in the subspace $\{f_1, f_2\}$. This means that this problem cannot be reduced. Therefore, we can claim that the proposed objective reduction method can preserve the non-dominant relation with fewer objectives.

IV. Integration of the Proposed Objective Reduction Method into an EMO Algorithm

This section will integrate the objective reduction method into an EMO algorithm to deal with the many-objective optimization problems. We obtain a set of non-dominated solutions for an M-objective problem, and initialize the weight matrix W. Then, we compute the weighted matrix W by Algorithm (1) on the current non-dominated solutions and run the EMO algorithm corresponding to the new problem $G(x)$ described in Eq. (4). Algorithm 2 gives the details of the EMO algorithm with the proposed online objective reduction method.

V. Simulation Results

In order to demonstrate the performance of the proposed objective reduction method, we conducted NSGA-II with the proposed objective reduction method, denoted as NSGA-II-OE, on one test instance P1 presented in [21]. This test instance can be compacted into the following form:

$$\text{P1}: f_i(x) = \alpha_i(x_I) + \beta_i(y_{m+1:n}), \text{ for } i = 1, \ldots, M$$

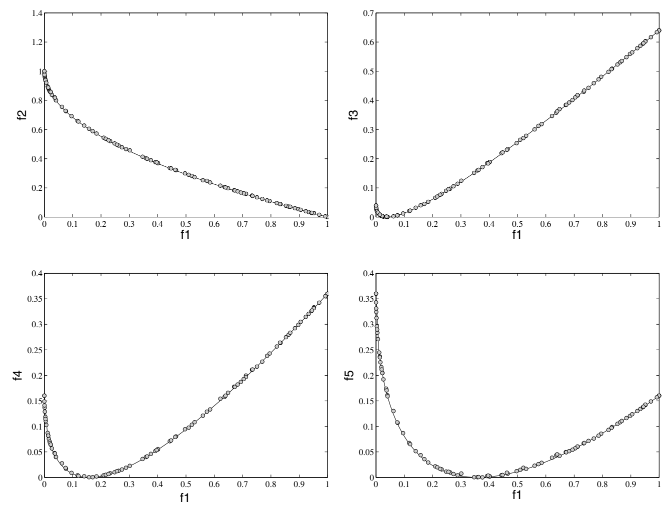


Fig. 2. The final population obtained by NSGA-II-OE in the objective space of f_1 versus f_2, f_1 versus f_3, f_1 versus f_4, f_1 versus f_5 on P1 with 5-objective.

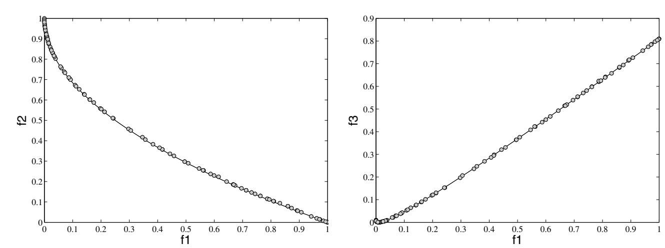


Fig. 3. The final population obtained by NSGA-II-OE in the objective space of f_1 versus f_2, f_1 versus f_3 on P1 with 10-objective.

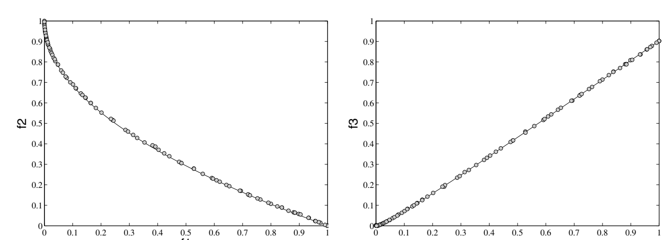
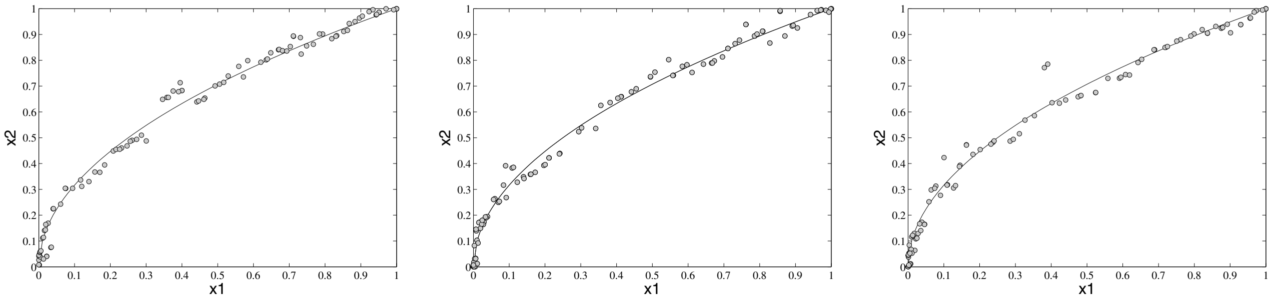


Fig. 4. The final population obtained by NSGA-II-OE in the objective space of f_1 versus f_2, f_1 versus f_3 on P1 with 20-objective.



(a) 5-objective (b) 10-objective (c) 20-objective

Fig. 5. The final population found by NSGA-II-OE in the subspace of x_1 versus x_2 on P1 with 5-objective, 10-objective and 20-objective.

Algorithm 2: Pseudocode of the EMO Algorithm with the Objective Reduction Method

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input :
- The maximum number of the generations: G_{max};
- Number of reductions during the search: O;
- The size of the population: N;
- Genetic operators and their associated parameters.

output: All the non-dominated solutions in P_t.
$\diamond$ Initialize a random population P_t in the decision space.
$\diamond$ Run the EMO algorithm k generations for original objectives until all the solutions in the population are non-dominant solutions.
$\diamond$ Compute the weighted matrix W on current non-dominant solutions.
$\diamond$ $G_{pre} \leftarrow \lceil (G_{max} - k)/O \rceil$.
for $o \leftarrow 1$ **to** O **do**
  for $g \leftarrow 1$ **to** G **do**
    $\diamond$ Run the EMO algorithm on the new optimization problem $F^{'}(x) = W^T F(x)$ and obtain the population P_t;
  end
  $\diamond$ Compute the correlation matrix R of $F(x)$ w.r.t. P_t;
  $\diamond$ Update the weighted matrix W as described in Algorithm 1;
end

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where $x \in [0,1]^n$ and $n = 10$. Furthermore, α functions are defined as follows:

$$\alpha_1(x_I) = x_1$$
$$\alpha_2(x_I) = 1 - \sqrt{x_1}$$
$$\alpha_k(x_I) = \left(\frac{k-2}{M} - \sqrt{x_1}\right)^2 \quad \forall\ k = 3, 4, \ldots, M$$

and β functions are defined as

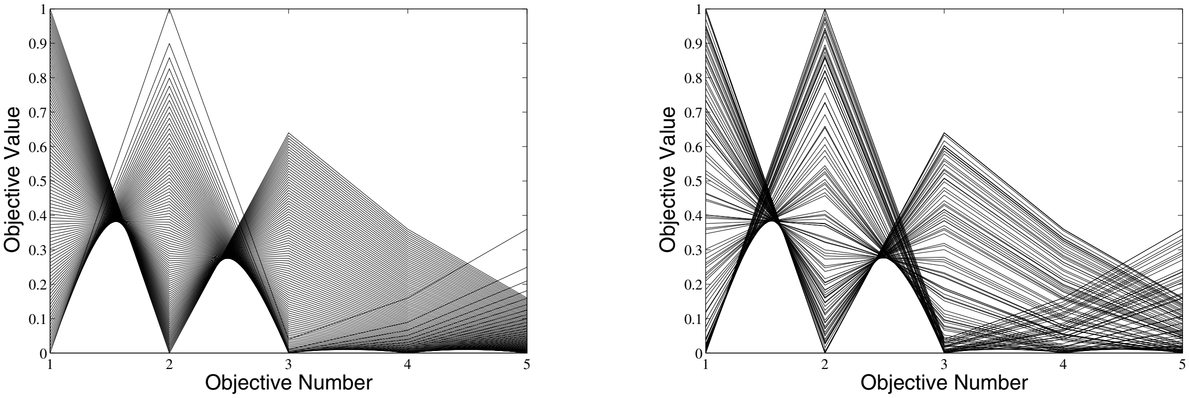
$$\beta_1(y_{3:n}) = \frac{1}{|J_1|}\sum_{j \in J_1} y_j^2$$
$$\beta_2(y_{3:n}) = \frac{1}{|J_2|}\sum_{j \in J_2} y_j^2$$
$$\beta_k(y_{3:n}) = A_k \sum_{j=3}^{n} a_{kj} y_j^2 \quad \forall\ k = 3, 4, \ldots, M.$$

where $y_j = x_j - x_1^{0.5(1.0+\frac{3(j-2)}{n-2})}$, $J_1 = \{j|2 \leq j \leq n,$ and $j-1$ is a multiple of $2\}$ and $J_2 = \{j|2 \leq j \leq n,$ and j is a multiple of $2\}$. A_k and a_{ij} are randomly generated constants. Its PF is curve in the objective space.

We conducted NSGA-II-OE on P1 with the different dimensionality $M = 5, 10, 20$. For all the test instances, the parameter settings of the algorithm are as follows:

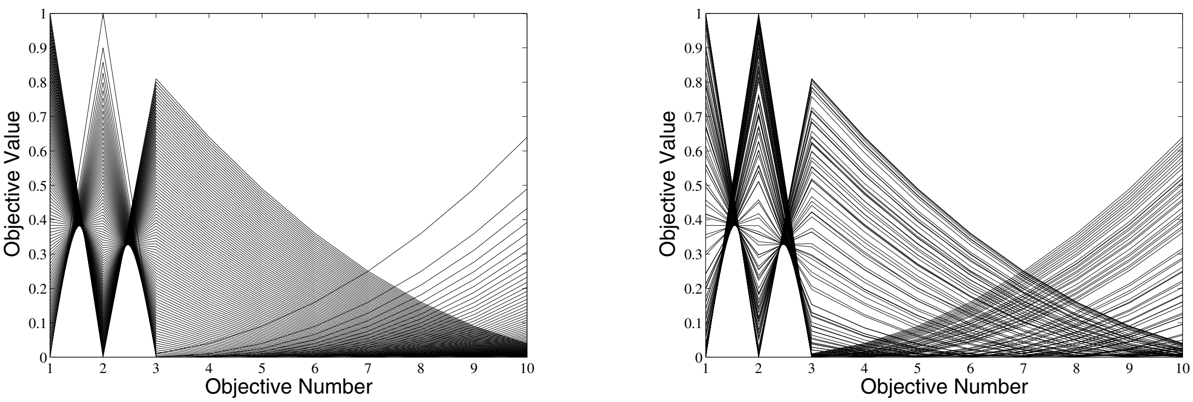
- The algorithm stop after 200 generations for all the test instances.
- The number of reductions O is set at 20 during the search.
- The crossover and mutation operators with the same control parameters in [4] are used.
- The population size N is 100.

Figs. 2, 3, 4 show the final population obtained by the NSGA-II-OE on partial subspace of objective space with $M = 5$, 10 and 20, respectively. It can be seen that we can obtain an approximate Pareto solutions for the test instance. Moreover, Fig. 5 shows the final solutions obtained by NSGA-II-OE in the subspace of x_1 versus x_2 on the test instance with the different values of M. Again, the algorithm can obtain an approximate solution for the test instance. Figs. 6(b), 7(b) and 8(b) give the parallel coordinate plots of the final solutions on the test instance with $M = 5, 10$ and 20, respectively. For comparison, we also give the parallel coordinate plots of 100 points which are uniformly distributed in the objective space along the PF in Figs. 6(a), 7(a) and 8(a). By comparing the experimental results with the ideal one, the solutions obtained by the proposed algorithm are promising, from which it can be seen that the proposed objective reduction method is able to extract the essential objectives accurately.



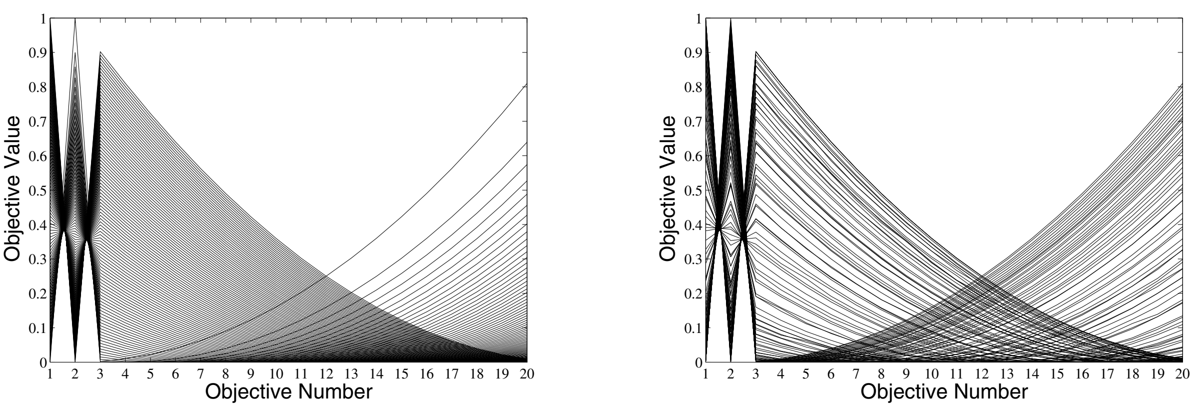
(a) Analytical(Ideal) (b) NSGA-II with the Proposed Objective Reduction Method

Fig. 6. Parallel coordinate plots after 200 generations for 5-objective from a single run.



(a) Analytical(Ideal) (b) NSGA-II with the Proposed Objective Reduction Method

Fig. 7. Parallel coordinate plots after 200 generations for 10-objective from a single run.



(a) Analytical(Ideal) (b) NSGA-II with the Proposed Objective Reduction Method

Fig. 8. Parallel coordinate plots after 200 generations for 20-objective from a single run.

VI. Conclusion

In this paper, we have proposed an online objective reduction method for the many-objective optimization problems. It formulates the essential objective as a linear combination of the original objectives with the weights determined based on the correlations of the essential objectives. Subsequently, we have integrated it into a state-of-the-art EMO, i.e. NSGA-II, to deal with many-objective optimization problems. Experiments have demonstrated the performance of the proposed method on one test instance with the different dimensionality. The promising results have shown that the proposed objective reduction method can extract the essential objectives accurately.

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