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Multiplicative Consistency for Interval Additive Reciprocal Preference Relations

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***Abstract*— The multiplicative consistency (MC) property of interval additive reciprocal preference relations (IARPRs) is explored, and then the consistency index is quantified by the multiplicative consistency estimated IARPR. The MC property is used to measure the level of consistency of the information provided by the experts and also to propose the consistency index induced ordered weighted averaging (CI-IOWA) operator. The novelty of this operator is that it aggregates individual IARPRs in such a way that more importance is put on the most consistent ones. Finally, an approach for group decision making problems with IARPRs is proposed.**

I. Introduction

DECISION MAKERS (DMs) usually need to compare a finite set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$ with respect to a single criterion, and construct preference relations. In general, there are two basic preference relations: multiplicative preference relations [1] and additive reciprocal preference relations [2].

In both cases, the preference relations elements represent the dominance of one alternative over another and take the form of exact numerical values. Both preference relations have been shown to be mathematically equivalent [3]. Thus, in this paper we just focus on the second type of preference relation and we will refer to it as simply reciprocal preference relation (RPR).

However, many decision making processes take place in an environment in which the information is not precisely known [4][5][6][7][8][9][10]. As a consequence, the DMs may feel more comfortable to use an interval number rather than an exact crisp numerical value to represent their preference. Therefore, interval additive reciprocal preference relations (IARPRs) can be considered an appropriate representation format to capture experts' uncertain preference information [11][13][14]. Indeed, the use of IARPRs in GDM problems under uncertain environments has recently attracted the attention of many researchers [15][16].

One key issue that needs to be addressed in this type of decision making environment is that of "consistency". The problem has been extensively studied in the case of reciprocal preference relations (RPRs). For RPRs, consistency is based on the concept of transitivity, which is modelled in many different ways (see for example [17]). Tanino [18] proposed a multiplicative transitivity property for RPRs, which was proved to be the most appropriate one for modelling cardinal consistency of such type of preference relations [19]. Recall that RPRs are particular cases of IARPRs, and therefore the same conclusion can be applied to them. Consequently, the first aim of this article is to formalise the multiplicative transitivity property for IARPRs. Furthermore, we investigate the multiplicative consistency estimated IARPR and quantify the level of consistency or consistency index (CI) of an IARPR.

Because consistent information is considered more relevant or important than inconsistent information, a novel aggregation operator that associates higher weights with more consistent information will be developed. In other words, a new consistency index induced ordered weighted averaging (CI-IOWA) operator is defined and proposed to compute the collective IARPR. Finally, an approach for group decision making problems with IARPRs is proposed.

II. Consistency of Interval Additive Reciprocal Preference Relations

Given three alternatives x_i, x_j, x_k such that x_i is preferred to x_j and x_j to x_k, the question whether the degree or strength of preference of x_i over x_j exceeds, equals, or is less than the degree or strength of preference of x_j over x_k cannot be answered by the classical preference modelling [19]. The introduction of the concept of fuzzy set as an extension of the classical concept of set when applied to a binary relation leads to the concept of a fuzzy relation. The adapted definition of a fuzzy reciprocal preference relation (FRPR) is the following one [20]:

Definition 1: A fuzzy preference relation (FRPR) P on a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ is characterised by a membership function $\mu_P \colon X \times X \longrightarrow [0,1]$, with $\mu_P(x_i, x_j) = p_{ij}$, verifying

$$\forall i, j \in \{1, \ldots, n\} : \; p_{ji} = 1 - p_{ij}. \tag{1}$$

As mentioned before, in this article and because there is no risk of confussion, we will call this type of preference relation as simply reciprocal preference relation (RPR)

Membership functions are subject to uncertainty arising from various sources [21]. Klir and Folger comment [22]:

> "... it may seem problematical, if not paradoxical, that a representation of fuzziness is made using membership grades that are themselves precise real

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numbers. Although this does not pose a serious problem for many applications, it is nevertheless possible to extend the concept of the fuzzy set to allow the distinction between grades of membership to become blurred."

Here Klir and Folger described blurring a fuzzy set to form an *interval valued fuzzy set* [23][24]:

Definition 2: Let $INT([0,1])$ be the set of all closed subintervals of $[0,1]$ and X be an universe of discourse. An interval valued fuzzy set (IVFS) $\widetilde{A}$ on X is characterised by a membership function $\mu_{\widetilde{A}}: X \to INT([0,1])$. An IVFS $\widetilde{A}$ on X can be expressed as follows:

$$A = \{(x, \mu_{\widetilde{A}}(x));\ \mu_{\widetilde{A}}(x) \in INT([0,1])\ \forall x \in X\}. \tag{2}$$

Given two interval numbers $\widetilde{a}_1 = [a_1^-, a_1^+]$ and $\widetilde{a}_2 = [a_2^-, a_2^+]$, the main interval arithmetic operations can be expressed in terms of the lower and upper bounds of the resulting intervals as follows [25]:

1) $\widetilde{a}_1 + \widetilde{a}_2 = [a_1^-, a_1^+] + [a_2^-, a_2^+] = [a_1^- + a_2^-, a_1^+ + a_2^+]$.
2) $\widetilde{a}_1 - \widetilde{a}_2 = [a_1^-, a_1^+] - [a_2^-, a_2^+] = [a_1^- - a_2^+, a_1^+ - a_2^-]$.
3) $\widetilde{a}_1 \cdot \widetilde{a}_2 = [a_1^-, a_1^+] \cdot [a_2^-, a_2^+] = [(a_1a_2)^-, (a_1a_2)^+]$,

$$(a_1a_2)^- = \min\{a_1^-a_2^-, a_1^-a_2^+, a_1^+a_2^-, a_1^+a_2^+\};$$

$$(a_1a_2)^+ = \max\{a_1^-a_2^-, a_1^-a_2^+, a_1^+a_2^-, a_1^+a_2^+\}.$$

4) $\widetilde{a}_1/\widetilde{a}_2 = [(a_1/a_2)^-, (a_1/a_2)^+]$,

$$(a_1/a_2)^- = \min\{a_1^-/a_2^-, a_1^-/a_2^+, a_1^+/a_2^-, a_1^+/a_2^+\};$$

$$(a_1/a_2)^+ = \max\{a_1^-/a_2^-, a_1^-/a_2^+, a_1^+/a_2^-, a_1^+/a_2^+\},$$

provided that $0 \notin [a_2^-, a_2^+]$.

Note that the real number $a \in \mathbb{R}$ can be represented in interval form as $[a,a]$. Two interval numbers $\widetilde{a}_1 = [a_1^-, a_1^+]$ and $\widetilde{a}_2 = [a_2^-, a_2^+]$ are equal if and only if $a_1^- = a_2^-$ and $a_1^+ = a_2^+$. An interval number $\widetilde{a} = [a^-, a^+]$ is positive when $a^- \geq 0$. The product and division of positive interval numbers can be simplified as follows:

3) $\widetilde{a}_1 \cdot \widetilde{a}_2 = [a_1^-, a_1^+] \cdot [a_2^-, a_2^+] = [a_1^-a_2^-, a_1^+a_2^+]$.
4) $\widetilde{a}_1/\widetilde{a}_2 = [a_1^-, a_1^+]/[a_2^-, a_2^+] = [a_1^-/a_2^+, a_1^+/a_2^-]$, provided that $a_2^- > 0$.

The application of the concept of IVFS to a RPR leads to the concept of interval additive reciprocal preference relation (IARPR) [11][13]:

Definition 3: An interval additive reciprocal preference relation (IARPR) $\widetilde{P}$ on a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ is characterised by a membership function $\mu_{\widetilde{P}}: X \times X \longrightarrow INT([0,1])$, with $\mu_{\widetilde{P}}(x_i, x_j) = \widetilde{p}_{ij} = [p_{ij}^-, p_{ij}^+]$, verifying

$$\forall i,j \in \{1,\ldots,n\}:\ \widetilde{p}_{ji} = 1 - \widetilde{p}_{ij}. \tag{3}$$

The above definition of IFPR can be expressed in terms of the lower and upper bound of the interval valued preference values as follows:

$$\forall i,j = 1,2,\ldots n:\ p_{ij}^- + p_{ji}^+ = p_{ij}^+ + p_{ji}^- = 1. \tag{4}$$

Given two interval numbers $\widetilde{a}_1 = [a_1^-, a_1^+]$ and $\widetilde{a}_2 = [a_2^-, a_2^+]$, Xu [26] proposed the following possibility degree (PD) to measure the degree up to which the ordering relation $\widetilde{a}_1 \succ \widetilde{a}_2$ holds:

$$P(\tilde{a}_1 \succ \tilde{a}_2) = \max\left\{1 - \max\left\{\frac{a_2^+ - a_1^-}{a_1^+ - a_1^- + a_2^+ - a_2^-}, 0\right\}, 0\right\} \tag{5}$$

III. Consistency of Interval Additive Reciprocal Preference Relations

Consistency of RPRs is based on the notion of transitivity, in the sense that if alternative x_i is preferred to alternative x_j ($p_{ij} \geq 0.5$) and this one to x_k ($p_{jk} \geq 0.5$), then alternative x_i should be preferred to x_k ($p_{ik} \geq 0.5$). This transitivity notion is normally referred to as *weak stochastic transitivity* [27]. Later, Tanino [18] introduced the multiplicative transitivity of RPRs as follows:

Definition 4: A RPR $P = (p_{ij})$ on a finite set of alternatives X is multiplicative transitive if and only if

$$\frac{p_{ji}}{p_{ij}} = \frac{p_{jk}}{p_{kj}}\frac{p_{ki}}{p_{ik}} \quad \forall i,k,j \in \{1,2,\ldots n\} \tag{6}$$

is verified by non zero preference values.

Obviously, multiplicative transitivity property extends weak stochastic transitivity, and therefore extends the classical transitivity property of crisp preference relations. Furthermore, Chiclana et al. [19] proved that $p_{ij} \cdot p_{jk} \cdot p_{ki} = p_{ik} \cdot p_{kj} \cdot p_{ji} \quad \forall i,k,j$ is equivalent to $p_{ij} \cdot p_{jk} \cdot p_{ki} = p_{ik} \cdot p_{kj} \cdot p_{ji} \quad \forall i < j < k$, and ultimately characterised the formulation of the cardinal consistency of RPRs via representable uninorms. Because the cardinal consistency with the conjunctive representable cross ratio uninorm is equivalent to Tanino's multiplicative transitivity property, and any two representable uninorms are order-isomorphic, it was proved that multiplicative transitivity is the most appropriate property to model consistency of RPRs. This is captured in the following definition [19]:

Definition 5: A RPR $P = (p_{ij})$ on a finite set of alternatives X is consistent if and only if

$$U(p_{ik}, p_{kj}) = \begin{cases} 0, & (p_{ik}, p_{kj}) \in \{(0,1),(1,0)\} \\ \dfrac{p_{ik}p_{kj}}{p_{ik}p_{kj} + (1-p_{ik})(1-p_{kj})}, & otherwise \end{cases} \tag{7}$$

Because RPRs are particular types of IARPRs, we can extend the notion of multiplicative transitivity of RPRs to the case of IARPRs as per the following definition:

Definition 6: An IARPR $\widetilde{P} = (\widetilde{p}_{ij})_{n \times n}$ on a finite set of alternatives X is multiplicative transitive if and only if

$$\frac{\widetilde{p}_{ji}}{\widetilde{p}_{ij}} = \frac{\widetilde{p}_{ki}}{\widetilde{p}_{ik}}\frac{\widetilde{p}_{jk}}{\widetilde{p}_{kj}}, \quad i < k < j \tag{8}$$

Note that reciprocity of preferences and division of interval number yield:

$$\frac{\widetilde{p}_{ji}}{\widetilde{p}_{ij}} = \frac{[p_{ji}^-, p_{ji}^+]}{[p_{ij}^-, p_{ij}^+]} = \left[\frac{p_{ji}^-}{p_{ij}^+}, \frac{p_{ji}^+}{p_{ij}^-}\right] = \left[\frac{1}{p_{ij}^+} - 1, \frac{1}{p_{ij}^-} - 1\right].$$

Applying the product of positive interval numbers and the equality of interval numbers, we have:

$$\frac{1}{p_{ij}^{+}}-1=\left(\frac{1}{p_{ik}^{+}}-1\right)\cdot\left(\frac{1}{p_{kj}^{+}}-1\right);$$

$$\frac{1}{p_{ij}^{-}}-1=\left(\frac{1}{p_{ik}^{-}}-1\right)\cdot\left(\frac{1}{p_{kj}^{-}}-1\right).$$

The above expressions can be rewritten as follows:

$$p_{ij}^{-}=\frac{p_{ik}^{-}p_{kj}^{-}}{p_{ik}^{-}p_{kj}^{-}+(1-p_{ik}^{-})(1-p_{kj}^{-})};$$

$$p_{ij}^{+}=\frac{p_{ik}^{+}p_{kj}^{+}}{p_{ik}^{+}p_{kj}^{+}+(1-p_{ik}^{+})(1-p_{kj}^{+})}.$$

Finally, because function $f(x)=x/(x+a)$ is monotone increasing when $a>0$, then it is clear that

$$0\le p_{ij}^{-}\le p_{ij}^{+}\le 1.$$

Therefore, we have proved the following result:

Theorem 1: If an IARPR $\widetilde{P}$ is multiplicative transitive, then we have

1) $$p_{ij}^{-}=\frac{p_{ik}^{-}p_{kj}^{-}}{p_{ik}^{-}p_{kj}^{-}+(1-p_{ik}^{-})(1-p_{kj}^{-})},i<k<j$$
2) $$p_{ij}^{+}=\frac{p_{ik}^{+}p_{kj}^{+}}{p_{ik}^{+}p_{kj}^{+}+(1-p_{ik}^{+})(1-p_{kj}^{+})},i<k<j$$

The following definition is therefore justified:

Definition 7: An IARPR $\widetilde{P}=(\widetilde{p}_{ij})_{n\times n}$ on a finite set of alternatives X is consistent if and only if

$$\widetilde{U}(\widetilde{p}_{ik},\widetilde{p}_{kj})=\begin{cases}0, & (\widetilde{p}_{ik},\widetilde{p}_{kj})\in\{(0,1),(1,0)\}\\ \left[p_{ij}^{-},p_{ij}^{+}\right] & ,otherwise\end{cases}\tag{9}$$

The consistency property (9) can be used to compute consistency based estimated valued of the elements of a given IARPR. Indeed, given an IARPR $\widetilde{P}=(\widetilde{p_{ij}})$, the interval preference value $\widetilde{p_{ij}}$ can be partially estimated using an intermediate alternative x_k $(i<k<j)$ as follows:

$$\widetilde{up}_{ij}^{k}=\widetilde{U}(\widetilde{p}_{ik},\widetilde{p}_{kj}).\tag{10}$$

Then, the global consistency based estimated value can be computed as the average of the partially estimated values obtained using all possible intermediate alternatives:

$$up_{ij}^{-}=\frac{\sum_{k=i+1}^{j-1}up_{ij}^{k-}}{j-i-1};\quad up_{ij}^{+}=\frac{\sum_{k=i+1}^{j-1}up_{ij}^{k+}}{j-i-1}.$$

Therefore, given an IARPR, $\widetilde{P}=(\widetilde{p_{ij}})$, the following multiplicative consistency estimated IARPR, $\widetilde{UP}=(\widetilde{up_{ij}})_{n\times n}=\left(\left[up_{ij}^{-},up_{ij}^{+}\right]\right)_{n\times n}$ can be constructed:

$$up_{ij}^{-}=\begin{cases}p_{ij}^{-}, & i\le j\le i+1\\ \dfrac{\sum_{k=i+1}^{j-1}up_{ij}^{k-}}{j-i-1}, & i+1\le k\le j\\ 1-up_{ji}^{+}, & i>j\end{cases}\tag{11}$$

and

$$up_{ij}^{+}=\begin{cases}p_{ij}^{+}, & i\le j\le i+1\\ \dfrac{\sum_{k=i+1}^{j-1}up_{ij}^{k+}}{j-i-1}, & i+1\le k\le j\\ 1-up_{ji}^{-}, & i>j\end{cases}\tag{12}$$

Because the multiplicative consistency property is the only consistency property used in the the paper, the symbol $\widetilde{U}$ will not be used unless it is necessary.

A. Consistency Indexes of IARPRs

If the information provided in an IARPR is completely consistent, then it is $\widetilde{p}_{ij}=\widetilde{up}_{ij}$. However, in a real decision making problem experts are not always fully consistent. As a result, it is necessary to measure their degree of inconsistency. The distance between the values $\widetilde{p}_{ij}$ and $\widetilde{up}_{ij}$ can be used in measuring the level of consistency of an IARPR at its three different levels: pair of alternatives, alternatives and relation. One such distance is the Hamming distance, which we propose to use [11]:

$$d(\widetilde{p}_{ij},\widetilde{up}_{ij})=\frac{1}{2}\Big(|p_{ij}^{-}-up_{ij}^{-}|+|p_{ij}^{+}-up_{ij}^{+}|\Big).$$

Definition 8 (Pair of Alternatives Consistency Index): Let $\widetilde{P}$ be an IARPR and $\widetilde{UP}$ be its corresponding multiplicative consistency estimated IARPR. The consistent index at the pair of alternatives (x_i,x_j), CI_{ij}, is:

$$CI_{ij}=1-d(\widetilde{p}_{ij},\widetilde{up}_{ij}).\tag{13}$$

The higher the value of CI_{ij}, the more consistent is $\widetilde{p}_{ij}$ with respect to the rest of preference values involving alternatives x_i and x_j. Note that we always have that $CI_{ij}=CI_{ji}$.

The consistency index at the level of alternatives is obtained by aggregating all the consistency index values of its corresponding pair of alternatives:

Definition 9 (Alternatives Consistency Index): The consistency index associated to the alternative x_i is

$$CI_i=\frac{\sum_{\substack{j=1\\ i\neq j}}^{n}CI_{ij}}{(n-1)}\tag{14}$$

When $CI_i=1$, then all the preference values involving alternative x_i are fully consistent.

The global consistency index of an IARPR is defined as the aggregated value of all individual alternatives consistency indexes:

Definition 10 (IARPR Consistency Index): The consistency index of an IARPR $\widetilde{P}$ is defined as follows:

$$CI=\frac{\sum_{i=1}^{n}CI_i}{n}\tag{15}$$

Note that $CI=1$ if and only if $\sum_{i,j=1,i\neq j}^{n}CI_{ij}=n\cdot(n-1)$. Because $CI_{ij}\in[0,1]$, then we have that $\sum_{i,j=1,i\neq j}^{n}CI_{ij}=n\cdot(n-1)$ if and only if $CI_{ij}=1\forall i\neq j$, i.e. $d(\widetilde{p}_{ij},\widetilde{up}_{ij})=0$ $(\forall i\neq j)$, which means that the IARPR, $\widetilde{P}$, and its corresponding multiplicative consistency based estimated IARPR, $\widetilde{UP}$, coincide. Therefore, we have that the concept of consistent IARPR based on the multiplicative transitivity property is well defined as per the following:

Definition 11: An IARPR $\widetilde{P}$ is consistent if and if only if $CI = 1$.

Example 1. Computation of Consistency Indexes. Suppose four different experts $\{e_1, e_2, e_3, e_4\}$ provide the following IARPRs over a set of four alternatives $\{x_1, x_2, x_3, x_4\}$:

$$\widetilde{P}^1 = \begin{pmatrix} - & [0.3, 0.5] & [0.4, 0.6] & [0.5, 0.7] \\ [0.5, 0.7] & - & [0.5, 0.8] & [0.5, 0.6] \\ [0.4, 0.6] & [0.2, 0.5] & - & [0.4, 0.6] \\ [0.3, 0.5] & [0.4, 0.5] & [0.4, 0.6] & - \end{pmatrix}$$

$$\widetilde{P}^2 = \begin{pmatrix} - & [0.4, 0.5] & [0.3, 0.4] & [0.4, 0.6] \\ [0.5, 0.6] & - & [0.5, 0.6] & [0.6, 0.7] \\ [0.6, 0.7] & [0.4, 0.5] & - & [0.4, 0.5] \\ [0.4, 0.6] & [0.3, 0.4] & [0.5, 0.6] & - \end{pmatrix}$$

$$\widetilde{P}^3 = \begin{pmatrix} - & [0.3, 0.6] & [0.4, 0.5] & [0.2, 0.3] \\ [0.4, 0.7] & - & [0.3, 0.5] & [0.3, 0.4] \\ [0.5, 0.6] & [0.5, 0.7] & - & [0.6, 0.7] \\ [0.7, 0.8] & [0.6, 0.7] & [0.3, 0.4] & - \end{pmatrix}$$

$$\widetilde{P}^4 = \begin{pmatrix} - & [0.4, 0.6] & [0.5, 0.8] & [0.5, 0.8] \\ [0.4, 0.6] & - & [0.4, 0.5] & [0.5, 0.7] \\ [0.2, 0.5] & [0.5, 0.6] & - & [0.4, 0.5] \\ [0.2, 0.5] & [0.3, 0.5] & [0.5, 0.6] & - \end{pmatrix}$$

The consistency based estimated IARPRs are:

$$\widetilde{UP}^1 = \begin{pmatrix} - & [0.30, 0.50] & [0.30, 0.80] & [0.30, 0.65] \\ [0.50, 0.70] & - & [0.50, 0.80] & [0.40, 0.86] \\ [0.20, 0.70] & [0.20, 0.50] & - & [0.40, 0.60] \\ [0.35, 0.70] & [0.14, 0.60] & [0.40, 0.60] & - \end{pmatrix}$$

$$\widetilde{UP}^2 = \begin{pmatrix} - & [0.40, 0.50] & [0.40, 0.60] & [0.36, 0.55] \\ [0.50, 0.60] & - & [0.50, 0.60] & [0.40, 0.60] \\ [0.40, 0.60] & [0.40, 0.50] & - & [0.40, 0.50] \\ [0.45, 0.64] & [0.40, 0.60] & [0.50, 0.60] & - \end{pmatrix}$$

$$\widetilde{UP}^3 = \begin{pmatrix} - & [0.30, 0.60] & [0.16, 0.60] & [0.33, 0.60] \\ [0.40, 0.70] & - & [0.30, 0.50] & [0.39, 0.70] \\ [0.40, 0.84] & [0.50, 0.70] & - & [0.60, 0.70] \\ [0.40, 0.67] & [0.30, 0.61] & [0.30, 0.40] & - \end{pmatrix}$$

$$\widetilde{UP}^4 = \begin{pmatrix} - & [0.40, 0.60] & [0.31, 0.60] & [0.40, 0.79] \\ [0.40, 0.60] & - & [0.40, 0.50] & [0.31, 0.50] \\ [0.40, 0.69] & [0.50, 0.60] & - & [0.40, 0.50] \\ [0.21, 0.60] & [0.50, 0.69] & [0.50, 0.60] & - \end{pmatrix}$$

The experts' consistency indexes are: $CI^1 = 0.924, CI^2 = 0.943, CI^3 = 0.903, CI^4 = 0.926$.

IV. Aggregation based on CI-IOWA operator

The collective preferences are obtained by fusing all the individuals' preferences using the consistency index induced ordered weighted averaging (CI-IOWA) operator [28][29], which extends the induced ordered weighted averaging (IOWA) operator proposed in [30]:

Definition 12: An IOWA operator of dimension m is a function Φ_W, to which a set of weights or weighting vector is associated, $W = (w_1, \ldots, w_m)$, such that $w_i \in [0, 1]$ and $\Sigma_i w_i = 1$, and with the following expression:

$$\Phi_W \left(\langle u_1, p_1 \rangle, \ldots, \langle u_m, p_m \rangle \right) = \sum_{i=1}^{m} w_i \cdot p_{\sigma(i)},$$

being σ a permutation such that $u_{\sigma(i)} \geq u_{\sigma(i+1)}, \ \forall i = 1, \ldots, m-1$.

In the present decision-making context, each expert can always be associated his/her IARPR consistency index value. The more consistent the preferences provided by an expert are, the more importance should be placed on that expert. In other words, we propose to use the consistency indexes to establish the ordering of the preference values to be aggregated, in which case we would be implementing the concept of consistency in the aggregation process of the proposed decision-making model [28]:

Definition 13 (CI-IOWA operator): Let a set of experts, $E = \{e_1, \ldots, e_m\}$, provide preferences about a set of alternatives, $X = \{x_1, \ldots, x_n\}$, using the IARPRs, $\{R^1, \ldots, R^m\}$. A MC-IOWA operator of dimension m, Φ_W^C, is an IOWA operator whose set of order inducing values is the set of consistency index values, $\{CI^1, \ldots, CI^m\}$, associated with the set of experts.

Then, the collective IARPR, $R^c = (r_{ij}^c) = (\langle p_{ij}^{c-}, p_{ij}^{c+} \rangle)$, is computed as follows:

$$p_{ij}^{c-} = \Phi_W^C \left(\langle CI^1, p_{ij}^{1-} \rangle, \cdots, \langle CI^m, p_{ij}^{m-} \rangle \right) = \sum_{h=1}^{m} \gamma_{\sigma(h)} \cdot p_{ij}^{\sigma(h)-} \tag{16}$$

$$p_{ij}^{c+} = \Phi_W^C \left(\langle CI^1, p_{ij}^{1+} \rangle, \cdots, \langle CI^m, p_{ij}^{m+} \rangle \right) = \sum_{h=1}^{m} \gamma_{\sigma(h)} \cdot p_{ij}^{\sigma(h)+} \tag{17}$$

with $CI^{\sigma(h-1)} \geq CI^{\sigma(h)}$, $\gamma_{\sigma(h-1)} \geq \gamma_{\sigma(h)} \geq 0$ ($\forall h \in \{2, \cdots, m\}$) and $\sum_{h=1}^{m} \gamma_{\sigma(h)} = 1$.

The general procedure for the inclusion of importance weight values in the aggregation process involves the transformation of the values to aggregate, r_{ij}^h, under the importance degree, u_h, to generate a new value, $\bar{r}_{ij}^h$, and then aggregate these new values using an aggregation operator. In the area of quantifier guided aggregations, Yager [31] provided a procedure to evaluate the overall satisfaction of m important criteria (experts) by an alternative x by computing the weighting vector associated to an OWA operator as follows:

$$w_h = Q\left(\frac{S(h)}{S(m)}\right) - Q\left(\frac{S(h-1)}{S(m)}\right) \tag{18}$$

being Q the membership function of the linguistic quantifier, $S(h)=\sum_{k=1}^{h} u_{\sigma(k)}$, and σ the permutation used to produce the ordering of the values to be aggregated. This approach for the inclusion of importance degrees associates a zero weight to those experts with zero importance degree. The linguistic quantifier is a Basic Unit-interval Monotone (BUM) function $Q:[0,1]\rightarrow[0,1]$ such that $Q(0)=0$, $Q(1)=1$ and if $x>y$ then $Q(x)\geq Q(y)$.

This procedure was extended to the case of IOWA operator in [30]. In this case, each component in the aggregation consists of a triple (r_{ij}^{h}, u_h, v_h) where r_{ij}^{h} is the argument value to aggregate, u_h is the importance weight value associated to r_{ij}^{h}, and v_h is the order inducing value. The same expression as above is used, and σ is the permutation such that $v_{\sigma(h)}$ is the $h-\mathrm{th}$ largest value in the set $\{v_1,\ldots,v_m\}$.

In the context subject of this paper, we propose to use the consistency values associated with each of the expert both as an importance weight associated to the argument and as the order inducing values, i.e. the following is assumed: $p_{ij}^{h-}=p_{ij}^{h+}=CI^{h}$. Thus, the ordering of the preference values is first induced by the ordering of the experts from the most to the least consistent, and the weights of the MC-IOWA operator is obtained by applying the above expression (18), which reduces to

$$\gamma_{\sigma(h)}=Q\left(\frac{S(\sigma(h))}{S(\sigma(m))}\right)-Q\left(\frac{S(\sigma(h-1))}{S(\sigma(m))}\right) \tag{19}$$

with $S(\sigma(h))=\sum_{k=1}^{h} CI^{\sigma(k)})$, and $CI^{\sigma(h)}$ is the $h-\mathrm{th}$ largest value of set $\{CI^{1},\ldots,CI^{m}\}$.

The BUM function guarantees that all individuals contribute to the final aggregated value because it is a strictly increasing function. To guarantee that the higher the consistency index, the higher the weighting value associated with it, i.e. for the following to be verified

$$CI^{\sigma(m-1)}\geq CI^{\sigma(m)}\geq 0\Rightarrow \gamma_{\sigma(m-1)}\geq\gamma_{\sigma(m)}\geq 0$$

additional constraints are to be imposed to the BUM function. In [19], it was proven that it is sufficient for the BUM function to be concave for the above to be true.

Example 2. (Example 1 continuation) Using the consistency levels we have: $\sigma(1)=2$, $\sigma(2)=4$, $\sigma(3)=1$ and $\sigma(4)=3$. Using the concave BUM function $Q=r^{1/2}$, which can be used to represent the linguistic majority 'most of', we obtain the following weights:

$$\lambda_{\sigma(1)}=0.50,\ \lambda_{\sigma(2)}=0.21,\ \lambda_{\sigma(3)}=0.16,\ \lambda_{\sigma(4)}=0.13.$$

The collective IARPR is:

$$UR^{c}=\begin{pmatrix} \langle 0.50,0.50\rangle & \langle 0.37,0.53\rangle & \langle 0.37,0.53\rangle & \langle 0.41,0.62\rangle \\ \langle 0.47,0.63\rangle & \langle 0.50,0.50\rangle & \langle 0.45,0.60\rangle & \langle 0.52,0.65\rangle \\ \langle 0.47,0.63\rangle & \langle 0.40,0.55\rangle & \langle 0.50,0.50\rangle & \langle 0.42,0.55\rangle \\ \langle 0.38,0.59\rangle & \langle 0.35,0.48\rangle & \langle 0.45,0.58\rangle & \langle 0.50,0.50\rangle \end{pmatrix}$$

Using expression (5), we obtain the following possibility degree matrix:

$$P=\begin{pmatrix} 0.50 & 0.19 & 0.19 & 0.43 \\ 0.81 & 0.50 & 0.33 & 1.00 \\ 0.81 & 0.67 & 0.50 & 0.61 \\ 0.57 & 0.00 & 0.39 & 0.50 \end{pmatrix}$$

We can easily obtain from this matrix the following ordering of the alternatives

$$x_2\succ x_3\succ x_4\succ x_1.$$

Making x_2 as the group choice as per the original IARPRs provided by the experts.

V. Conclusions

This paper investigates the multiplicative transitivity property IARPRs. Then, the consistency index (CI) is investigated to measure the level of consistency of the information provided by the experts. The bigger the CI, the more consistent the expert. Therefore, it also can be regarded as a reliability index to aggregate individual IARPRs into a collective one. To do that, a new induced ordered weighted averaging (CI-IOWA) operator is proposed, which aggregates individual IARPRs in such a way that more importance is put on the most consistent ones. A selection process based on the ordering of the alternatives based on the use of the possibility degree is also presented.

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