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Stability Region Analysis for Polynomial Fuzzy Systems by Polynomial Lyapunov Functions

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***Abstract*—This paper presents a sum-of-squares (SOS) based methodology to obtain inner bounds on the region-of-attraction (ROA) for nonlinear systems represented by polynomial fuzzy systems. The methodology searches a polynomial Lyapunov function to guarantee the local stability and the invariant subset of the ROA is presented as the level set of the polynomial Lyapunov function. At first the methodology checks whether the considered system can be guaranteed to be locally asymptotically stable. After confirming that the system is guaranteed to be locally asymptotically stable, the methodology enlarges the invariant subset of the ROA as much as possible. The constraints for both of checking stability and enlarging contractively invariant set are represented in terms of bilinear SOS optimization problems. The path-following method is applied to solve the bilinear SOS optimization problems in the methodology.**

I. Introduction

IDENTIFYING the region-of-attraction (ROA) for a locally asymptotically stable nonlinear system is a topic of significant importance. However, computing the exact ROA for a nonlinear system is often a tough task. Therefore, researchers have focused on finding invariant subsets of ROAs represented by level sets of Lyapunov functions [1]-[3]. By applying the sum-of-squares (SOS) technique and SOS optimization tools (e.g. SOSTOOLS [4] and SOSOPT [5]), it is possible to search polynomial Lyapunov functions to enlarge the inner estimate of the ROA for nonlinear systems represented by polynomial vector fields [6], [7]. However, these studies [1]-[3], [6], [7] deals with only polynomial systems. Unfortunately, most practical systems are non-polynomial systems.

By applying the well-known sector nonlinearity approach [8] or the Taylor series approach [9], it has been shown that a non-polynomial nonlinear system can be exactly represented by the polynomial fuzzy system (globally or semi-globally) [9], [10]. Using the sum-of-squares (SOS) technique, some studies have been done to analyze the stability for polynomial fuzzy systems [9]-[13]. However, no methodology has been proposed to estimate the ROA for locally asymptotically stable polynomial fuzzy systems.

In this study, a methodology is proposed to estimate the invariant subset of the ROA for non-polynomial nonlinear systems represented by polynomial fuzzy systems. The methodology applies SOS programing to search a polynomial Lyapunov function to guarantee the local stability of the polynomial fuzzy system. Furthermore, the invariant subset of the ROA is represented by the level set of the polynomial Lyapunov function. At first the methodology checks if the polynomial fuzzy system can be guaranteed to be locally asymptotically stable. After confirming the stability, the methodology enlarges the estimated contractively invariant set as much as possible. The constraints for both of checking stability and enlarging contractively invariant set are represented in terms of bilinear SOS optimization problems. However, bilinear SOS optimization problems cannot be directly solved by SOS optimization tools (e.g. SOSTOOLS and SOSOPT). Therefore, the path following method [14] that has been demonstrated to be effective for bilinear semi-definite optimization problems [15], [16] is utilized with SOSOPT to solve the bilinear SOS optimization problems in the methodology.

Throughout this paper, the following definitions are adopted [17]. A monomial in $\boldsymbol{x}(t) = [x_1(t)\ x_2(t)\ \cdots\ x_n(t)]^T$ is a function of the form $x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$ where d_i, $i = 1,\ 2,\ \cdots,\ n$, are nonnegative integers. The degree of a monomial is $d = \sum_{i=1}^{n} d_i$. A polynomial $q(\boldsymbol{x}(t))$ is defined as a finite linear combination of monomials with real coefficients. A polynomial $q(\boldsymbol{x}(t))$ is SOS, if there exist polynomials $f_1(\boldsymbol{x}(t)),\ f_2(\boldsymbol{x}(t)),\ \cdots,\ f_m(\boldsymbol{x}(t))$ such that $q(\boldsymbol{x}(t)) = \sum_{i=1}^{m} f_i^2(\boldsymbol{x}(t))$. It is obvious that $q(\boldsymbol{x}(t))$ being SOS naturally implies $q(\boldsymbol{x}(t)) \geq 0$ for all $\boldsymbol{x}(t) \in \mathbb{R}^n$.

II. System Description and Problem Formulation

Consider the following (non-polynomial) nonlinear system:

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t)) \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ and f is a smooth nonlinear function with $f(\mathbf{0}) = 0$. Applying the well-known sector nonlinearity approach [8] or a Taylor series approach [9], it has been show that (1) can be exactly represented by the following polynomial fuzzy system (globally or semi-globally) [10]:

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} h_i(\boldsymbol{z}(t)) \boldsymbol{A}_i(\boldsymbol{x}(t)) \hat{\boldsymbol{x}}(\boldsymbol{x}(t)) \tag{2}$$

where $\boldsymbol{A}_i(\boldsymbol{x}(t)) \in \mathbb{R}^{n \times N}$ are polynomial system matrices in $\boldsymbol{x}(t)$; $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) \in \mathbb{R}^N$ is a column vector whose entries are all monomials in $\boldsymbol{x}(t)$ such that $\hat{\boldsymbol{x}}(\boldsymbol{x}(t)) = 0$ iff $\boldsymbol{x}(t) = 0$,

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and $z_j(t)$ is the known premise variable. Moreover, $h_i(\boldsymbol{z}(t))$ are the normalized grades of membership and exhibit the following properties: $\sum_{i=1}^{r} h_i(\boldsymbol{z}(t)) = 1$ and $h_i(\boldsymbol{z}(t)) \geq 0\ \forall i$. In [10] and [12], the stability of the polynomial fuzzy system (2) has been investigated. However, in the two studies [10], [12], the estimate of ROA is not examined when the polynomial fuzzy system (2) is locally asymptotically stable.

This study aims to estimate the invariant subset of the ROA for locally asymptotically stable (non-polynomial) nonlinear systems represented by the polynomial fuzzy system (2). By searching a polynomial Lyapunov function, the contractively invariant set are represented by the level set of the polynomial Lyapunov function. In what follows the dependence on the time t will be omitted for ease of notation.

Let $V(\boldsymbol{x})$ be a continuous differentiable function and define $\Omega_V = \{\boldsymbol{x} : V(\boldsymbol{x}) \leq 1\}$. The equilibrium $\boldsymbol{x} = \boldsymbol{0}$ of the polynomial fuzzy system (2) is asymptotically stable and the level set Ω_V is a contractively invariant set if

$$V(\boldsymbol{x}) \text{ is positive definite} \tag{3}$$

$$\Omega_V \text{ is bounded} \tag{4}$$

$$\dot{V}(\boldsymbol{x}) = \sum_{i=1}^{r} h_i(\boldsymbol{z}) \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) < 0 \quad \forall \boldsymbol{x} \in \Omega_V - \{\boldsymbol{0}\}. \tag{5}$$

If the conditions (3)-(5) are fulfilled, it is expected to enlarge the estimated contractively invariant set Ω_V as much as possible. For this purpose, a variable sized region is defined as $P_\beta = \{\boldsymbol{x} : p(\boldsymbol{x}) \leq \beta\}$, where $p(\boldsymbol{x})$ is a fixed positive definite polynomial, and β is maximized while imposing the constraint $P_\beta \subseteq \Omega_V$ along with constraints (3)-(5).

III. SOS-Based Methodology for Estimating the ROA

This section presents the SOS-based methodology for estimating the invariant subset of the ROA. By applying the SOS technique and the polynomial Lyapunov function, the methodology firstly checks if the polynomial fuzzy system (2) can be guaranteed to be asymptotically stable. After the stability is confirmed, the methodology tries to enlarge the estimated contractively invariant set as much as possible.

Sufficient conditions represented in terms of SOS constraints for the conditions (3)-(5) are given in the following theorem.

Theorem 1: The equilibrium $\boldsymbol{x} = \boldsymbol{0}$ of the polynomial fuzzy system (2) is asymptotically stable and the level set Ω_V is a contractively invariant set if there exists a polynomial function $V(\boldsymbol{x})$, polynomials $\mu_i(\boldsymbol{x})$ and a scalar $\alpha < 0$ such that

$$V(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) \quad \text{is SOS} \tag{6}$$

$$-\frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) + \alpha V(\boldsymbol{x}) - \mu_i(\boldsymbol{x})(1 - V(\boldsymbol{x})) \quad \text{is SOS},\ i = 1, \cdots, r, \tag{7}$$

$$\mu_i(\boldsymbol{x}) \quad \text{is SOS},\ i = 1, \cdots, r \tag{8}$$

where $\epsilon(\boldsymbol{x})$ is a positive definite polynomial satisfying $\epsilon(\boldsymbol{x}) \to \infty$ for $||\boldsymbol{x}|| \to \infty$.

Proof: If (6) holds, $V(\boldsymbol{x})$ is positive definite and radially unbounded that insures the conditions (3) and (4). For $\boldsymbol{x} \in \Omega_V$, it has the following property:

$$\mu_i(\boldsymbol{x})(1 - V(\boldsymbol{x})) \geq 0 \tag{9}$$

where $\mu_i(\boldsymbol{x}) \geq 0$, which is guaranteed by (8), are polynomial functions. When $\boldsymbol{x} \in \Omega_V$,

$$\begin{aligned}
&\dot{V}(\boldsymbol{x}) - \alpha V(\boldsymbol{x}) \\
&\quad = \sum_{i=1}^{r} h_i(\boldsymbol{z}) \left\{ \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) - \alpha V(\boldsymbol{x}) \right\} \\
&\quad \leq \sum_{i=1}^{r} h_i(\boldsymbol{z}) \bigg\{ \frac{\partial V(\boldsymbol{x})}{\partial \boldsymbol{x}} \boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) - \alpha V(\boldsymbol{x}) \\
&\qquad + \mu_i(\boldsymbol{x})(1 - V(\boldsymbol{x})) \bigg\}.
\end{aligned}$$

Therefore, if (7) holds for $\alpha < 0$, then $\dot{V}(\boldsymbol{x}) \leq \alpha V(\boldsymbol{x}) < 0$ for all $\boldsymbol{x} \in \Omega_V - \{\boldsymbol{0}\}$ that satisfies the condition (5). ■

By solving the SOS constraints in Theorem 1, the stability of the polynomial fuzzy system (2) can be confirmed. After confirming the stability, the methodology aims to enlarge the estimated contractively invariant set Ω_V as much as possible. For this purpose, the methodology maximizes β subject to the constraint $P_\beta \subseteq \Omega_V$ along with the constraints in Theorem 1. By Lemma 2 of [7], if there exists a polynomial function $\sigma(\boldsymbol{x})$ such that

$$\sigma(\boldsymbol{x}) \quad \text{is SOS} \tag{10}$$

$$-[(\beta - p(\boldsymbol{x}))\sigma(\boldsymbol{x}) + (V(\boldsymbol{x}) - 1)] \quad \text{is SOS} \tag{11}$$

then $P_\beta \subseteq \Omega_V$. Therefore, by setting $\alpha = -\epsilon_\alpha$, where ϵ_α is an extremely small positive value, the problem can be formulated as the following optimization problem:

$$\max_{V(\boldsymbol{x}),\mu_i(\boldsymbol{x}),\sigma(\boldsymbol{x})} \beta \text{ subject to (6)-(8), (10), (11).} \tag{12}$$

It can be seen that both the problem of solving the constrains in Theorem 1 and the optimization problem (12) are bilinear SOS problems. However, bilinear SOS problems cannot be directly solved by SOS optimization tools (e.g. SOSTOOLS and SOSOPT). Therefore, the methodology applies the path following method [14] with SOSOPT to solve the bilinear SOS problems. The algorithm to solve the constrains in Theorem 1 by the path-following method is presented as follows.

Algorithm 1

Step 1: Let $\eta = 0$ and randomly choose SOS polynomials $\mu_{i0}(\boldsymbol{x})$.

Step 2: Set $\mu_i(\boldsymbol{x}) = \mu_{i\eta}(\boldsymbol{x})$ and solve the following optimization problem:

$$\min_{V(\boldsymbol{x})} \alpha \text{ subject to (6)-(8)} \tag{13}$$

Step 3: For the $V(\boldsymbol{x})$ obtained from step 2, solve the following optimization problem, which is the linearized version of (6)-(8) around $V(\boldsymbol{x})$ and $\mu_i(\boldsymbol{x})$:

$$\min_{\delta V(\boldsymbol{x}),\delta\mu_i(\boldsymbol{x})} \alpha \text{ subject to}$$

$$V(\boldsymbol{x}) + \delta V(\boldsymbol{x}) - \epsilon(\boldsymbol{x}) \quad \text{is SOS} \tag{14}$$

$$\begin{aligned}&-\frac{\partial(V(\boldsymbol{x})+\delta V(\boldsymbol{x}))}{\partial \boldsymbol{x}}\boldsymbol{A}_i(\boldsymbol{x})\hat{\boldsymbol{x}}(\boldsymbol{x}) + \alpha(V(\boldsymbol{x})+\delta V(\boldsymbol{x}))\\ &-(\mu_i(\boldsymbol{x})+\delta\mu_i(\boldsymbol{x}))(1-V(\boldsymbol{x})) + \mu_i(\boldsymbol{x})\delta V(\boldsymbol{x})\\ &\quad \text{is SOS},\ i=1,\ \cdots,\ r,\end{aligned} \tag{15}$$

$$\mu_i(\boldsymbol{x}) + \delta\mu_i(\boldsymbol{x}) \quad \text{is SOS},\ i=1,\ \cdots,\ r \tag{16}$$

$$v_1^T \begin{bmatrix}\epsilon_V V^2(\boldsymbol{x}) & \delta V(\boldsymbol{x})\\ \delta V(\boldsymbol{x}) & 1\end{bmatrix} v_1 \quad \text{is SOS} \tag{17}$$

$$v_2^T \begin{bmatrix}\epsilon_\mu \mu_i^2(\boldsymbol{x}) & \delta\mu_i(\boldsymbol{x})\\ \delta\mu_i(\boldsymbol{x}) & 1\end{bmatrix} v_2 \quad \text{is SOS},\ i=1,\ \cdots,\ r \tag{18}$$

where $\epsilon_V,\ \epsilon_\mu \in [0.1\ 0.01]$ are small positive values for the purpose of small perturbations and $v_1,\ v_2$ are vectors independent of $\boldsymbol{x}$.

Step 4: For the $\delta\mu_i(\boldsymbol{x})$ obtained from step 3, update $\mu_{i\eta}(\boldsymbol{x})$ such that $\mu_{i(\eta+1)}(\boldsymbol{x}) = \mu_{i\eta}(\boldsymbol{x}) + \delta\mu_i(\boldsymbol{x})$; then set $\eta = \eta + 1$, and go to step 2.

The iteration stops when $\alpha < 0$ is obtained in step 2, which means that the solution for the constraints in Theorem 1 is found. The iteration also stops when $\alpha \geq 0$ and α cannot be improved any more with respect to former iterations in step 2. In this case, no solution is found for the constraints in Theorem 1.

Remark 1: The random SOS polynomial $\mu_{i0}(\boldsymbol{x})$ can be generated by setting $\mu_{i0}(\boldsymbol{x}) = \boldsymbol{Z}^T(\boldsymbol{x})\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{Z}(\boldsymbol{x})$, where $\boldsymbol{Z}(\boldsymbol{x})$ is a column vector whose entries are all monomials in $\boldsymbol{x}$ and $\boldsymbol{Q}$ is a randomly chosen square matrix with scalar entries. For instance, if $\mu_{i0}(\boldsymbol{x})$ is determined to be a linear combination of all monomials of degrees from 2 to 4, then $\boldsymbol{Z}(\boldsymbol{x})$ should be a column vector whose entries are all monomials of degrees from 1 to 2.

If a solution is found by Algorithm 1 to satisfy the constraints in Theorem 1, the methodology tries to enlarge the estimated contractively invariant set Ω_V by solving the optimization problem (12). As mentioned before, the optimization problem (12) is a bilinear SOS problem, and therefore the path-following method is utilized to solve it. The algorithm to solve the optimization problem (12) by the path-following method is presented as follows.

Algorithm 2

Step 1: Let $\eta = 0$, $\alpha = -\epsilon_\alpha$ and $\mu_{i0}(\boldsymbol{x})$ be the $\mu_i(\boldsymbol{x})$ of the solution found by Algorithm 1.

Step 2: Set $\mu_i(\boldsymbol{x}) = \mu_{i\eta}(\boldsymbol{x})$ and solve the following optimization problem:

$$\max_{V(\boldsymbol{x}),\sigma(\boldsymbol{x})} \beta \text{ subject to (6)-(8), (10), (11).} \tag{19}$$

Step 3: For the $V(\boldsymbol{x})$ obtained from step 2, solve the following optimization problem, which is the linearized version of (6)-(8), (10), (11) around $V(\boldsymbol{x})$ and $\mu_i(\boldsymbol{x})$:

$$\begin{aligned}&\max_{\delta V(\boldsymbol{x}),\delta\mu_i(\boldsymbol{x}),\sigma(\boldsymbol{x})} \beta \text{ subject to (10), (14)-(18) and}\\ &-[(\beta - p(\boldsymbol{x}))\sigma(\boldsymbol{x}) + (V(\boldsymbol{x})+\delta V(\boldsymbol{x})-1)] \text{ is SOS.}\end{aligned} \tag{20}$$

Step 4: For the $\delta\mu_i(\boldsymbol{x})$ obtained from step 3, update $\mu_{i\eta}(\boldsymbol{x})$ such that $\mu_{i(\eta+1)}(\boldsymbol{x}) = \mu_{i\eta}(\boldsymbol{x}) + \delta\mu_i(\boldsymbol{x})$; then set $\eta = \eta + 1$, and go to step 2.

The iteration stops when the β obtained in step 2 cannot be improved any more with respect to former iterations. In this case, Ω_V for the $V(\boldsymbol{x})$ obtained from step 2 is the estimated contractively invariant set that is enlarged for the polynomial fuzzy system (2).

IV. Examples

In this section, two locally asymptotically stable (non-polynomial) nonlinear systems are considered. The two nonlinear systems are exactly represented by the polynomial fuzzy system (2). Then, for the polynomial system exactly representing the nonlinear system, the asymptotical stability is confirmed by Algorithm 1 and the invariant subset of ROA is estimated by Algorithm 2.

A. Example 1

Consider the following nonlinear system that is a pendulum equation with friction [18]:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -10\sin x_1 - x_2.\end{aligned} \tag{21}$$

Figure 1 shows the phase portrait of the nonlinear system. From Fig. 1, it can be seen that the equilibrium $\boldsymbol{x} = \boldsymbol{0}$ of the nonlinear system (21) is locally asymptotically stable. By applying the Taylor series technique [9], the function $\sin x_1$ can be exactly represented as

$$\sin x_1 = h_1 x_1 + h_2(x_1 - \frac{x_1^3}{6}) \tag{22}$$

where

$$h_1 = \begin{cases}\frac{6(\sin x_1 - x_1)}{x_1^3} + 1, & x_1 \neq 0\\ 0, & x_1 = 0\end{cases},\ h_2 = 1 - h_1. \tag{23}$$

Therefore, with the membership functions (23), the nonlinear dynamics (21) can be exactly represented in the form of (2), where $r = 2$, $\hat{\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{x} = [x_1\ x_2]^T$ and

$$\boldsymbol{A}_1 = \begin{bmatrix}0 & 1\\ -10 & -1\end{bmatrix},\ \boldsymbol{A}_2 = \begin{bmatrix}0 & 1\\ -10 + 10x_1^2/6 & -1\end{bmatrix}.$$

The $p(\boldsymbol{x})$ for the variable sized region P_β is given as

$$p(\boldsymbol{x}) = [x_1\ x_2]\begin{bmatrix}0.4 & 0.2\\ 0.2 & 0.28\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{24}$$

The notation n_V denotes the degree of $V(\boldsymbol{x})$, i.e. $V(\boldsymbol{x})$ is a linear combination of all monomials of degrees from 2 to n_V. Applying Algorithm 1 and Algorithm 2 for $n_V = 2$, the maximized β of P_β is obtained as 1.3408. Figure 2 shows the estimated contractively invariant set Ω_V for $n_V = 2$ and $P_{1.3048}$. Applying Algorithm 1 and Algorithm 2 for $n_V = 4$, the maximized β of P_β is obtained as 1.7222. Figure 3 shows the estimated contractively invariant set Ω_V for $n_V = 4$ and $P_{1.7222}$. Moreover, Fig. 4 shows the comparison of the estimated contractively invariant sets of $n_V = 2$ and $n_V = 4$.

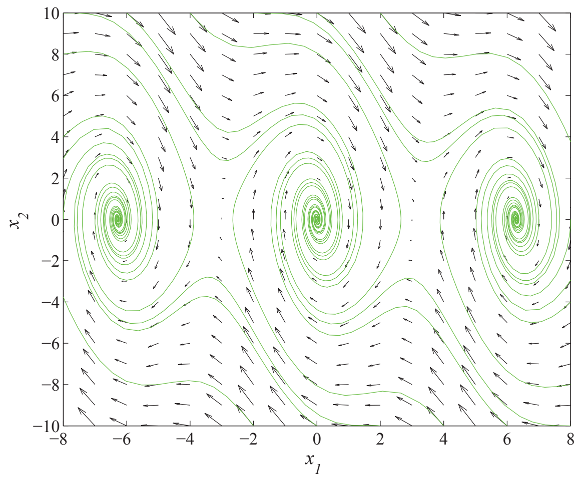

Fig. 1. Phase portrait of the nonlinear system of Example 1.

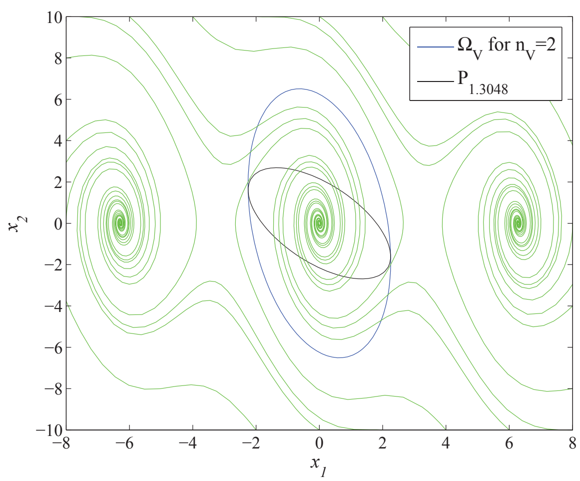

Fig. 2. The estimated contractively invariant set Ω_V and $P_{1.3048}$ for $n_V = 2$ of Example 1.

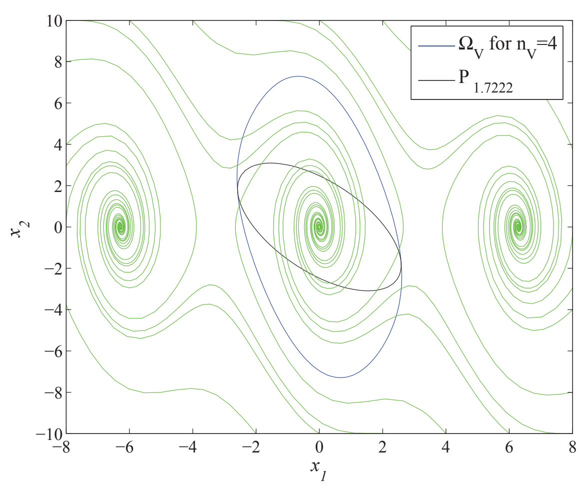

Fig. 3. The estimated contractively invariant set Ω_V and $P_{1.7222}$ for $n_V = 4$ of Example 1.

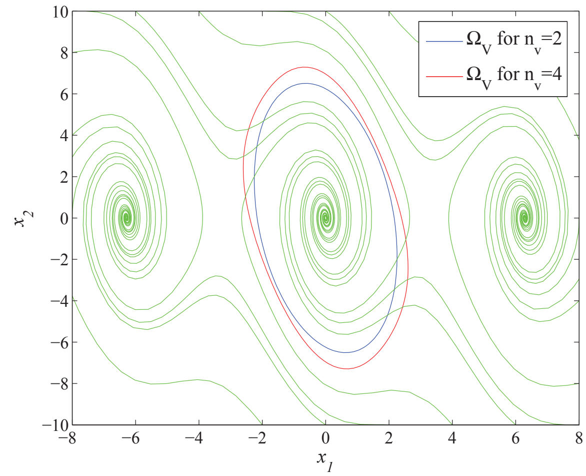

Fig. 4. The comparison of the estimated contractively invariant sets of $n_V = 2$ and $n_V = 4$ for Example 1.

B. Example 2

Consider the following nonlinear dynamics [10]:

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1^2 + x_1^3 + x_1^2 x_2 - x_1 x_2^2 + x_2 \\ \dot{x}_2 &= -\sin(x_1) - x_2 \end{aligned} \tag{25}$$

Figure 5 shows the phase portrait of the nonlinear dynamics. From Fig. 5, it can be seen that the equilibrium $\boldsymbol{x} = \boldsymbol{0}$ of the nonlinear dynamics (25) is locally asymptotically stable. By applying (22) with the membership functions (23), the nonlinear dynamics (25) can be exactly represented in the form of (2), where $r = 2$, $\hat{\boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{x} = [x_1 \ x_2]^T$ and

$$\boldsymbol{A}_1 = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1 x_2 - x_2^2 & 1 \\ -1 & -1 \end{bmatrix}$$
$$\boldsymbol{A}_2 = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1 x_2 - x_2^2 & 1 \\ -1 + x_1^2/6 & -1 \end{bmatrix}.$$

Applying Algorithm 1 and Algorithm 2 with the $p(\boldsymbol{x})$ in (24), Fig. 6 shows the comparison of the estimated contractively invariant sets of $n_V = 2$ and $n_V = 4$.

V. Conclusion

In this paper, a SOS-based methodology has been proposed to estimate the invariant subset of ROA for (non-polynomial) nonlinear systems represented by polynomial fuzzy system. The methodology searches a polynomial Lyapunov function to guarantee the local stability and the contractively invariant set is presented as the level set of the polynomial Lyapunov function. The constraints in the methodology are represented in terms of bilinear SOS problem. Therefore, the algorithms based on the path-following method have been provided for searching the polynomial Lyapunov function along with enlarging the contractively invariant set. Finally, two examples have been given to illustrated the utility and effectiveness of the proposed methodology.

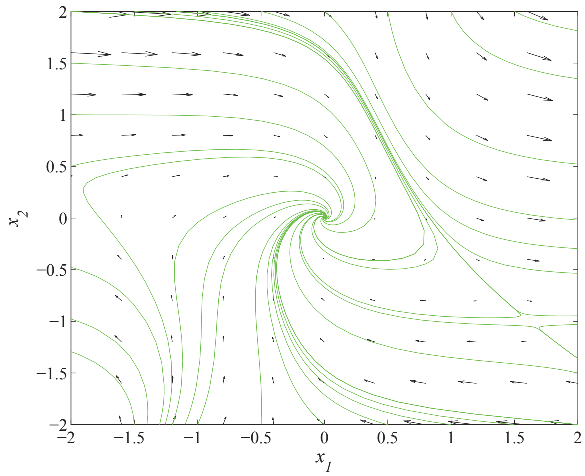

Fig. 5. Phase portrait of the nonlinear dynamics of Example 2.

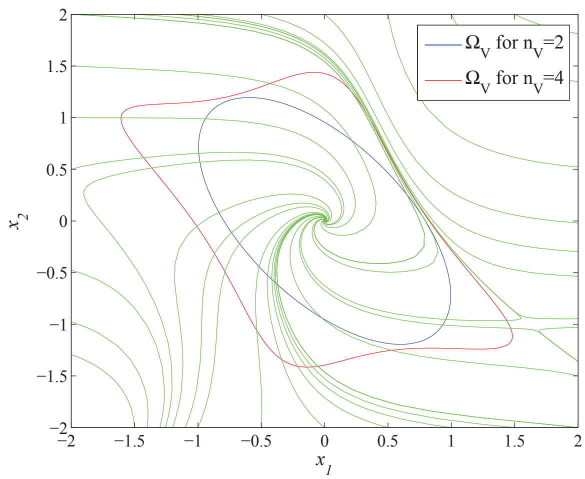

Fig. 6. The comparison of the estimated contractively invariant sets of $n_V = 2$ and $n_V = 4$ for Example 2.

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