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Local H_∞ Control and Invariant Set Analysis for Continuous-Time T–S Fuzzy Systems with Magnitude- and Energy-Bounded Disturbances

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***Abstract*— This paper addresses local H_∞ controller design problems for continuous-time Takagi–Sugeno (T–S) systems with magnitude- and energy-bounded disturbances. The design procedure is formulated as optimizations subject to linear matrix inequalities (LMIs) which can be solved by means of convex optimization techniques. The designed controllers not only guarantee the H_∞ performance but also ensure the state not to escape an invariant set that is included by the region where the T–S fuzzy model is defined. Finally, examples are given to illustrate the proposed method.**

I. Introduction

Stability analysis analysis and control design of Takagi–Sugeno (T–S) fuzzy systems have attracted a great deal of attention for decades. Approaches based on Lyapunov stability theory are the most popular way to deal with those problems, since they enable the problems to be described as linear matrix inequalities (LMIs) for which LMI solvers are available [1]–[3]. Among them, the simplest method is the common quadratic Lyapunov function approach [4], [5], which is in general overly conservative because a common Lyapunov matrix should be found for all subsystems of fuzzy systems. To reduce the conservatism, significant efforts have been made to date. Nowadays, there is immense literature addressing the relaxation problem through various approaches, just to name a few:

- slack variable approaches that develop quadratic-Lyapunov-function-based LMI conditions that reflect more information on the properties of the unit simplex [6]–[9];
- LMI conditions that reflect some information on the membership functions' shape [10], [11];
- convergent LMI relaxation techniques that exploit Pólya's theorem [12];
- approaches based on piecewise Lyapunov functions that are piecewise quadratic with respect to some partitions of the state space [13]–[15], fuzzy Lyapunov functions that are linearly dependent on the membership functions [16]–[20], a class of Lyapunov functions using line integral [21], polynomial Lyapunov functions whose dependence on the state variables are expressed as polynomial forms [22]–[25], switching polynomial Lyapunov function [26], polynomial fuzzy Lyapunov functions that are polynomial in the membership functions [27]–[30];
- the so-called k-samples variation approaches which use augmented fuzzy Lypunov functions depending on the states over several samples [31]–[33];
- approaches using new bounding techniques on the time derivative of the membership functions [34]–[36];
- local stability and stabilization approaches [37]–[43] that guarantee the asymptotic stability only in some local region of the state space.

Among the promising results, in this paper, we focus on the local stability approaches. Recently, the local stability methods were extended to deal with the H_∞ control problems in [44], [45] and to address the local stabilization problems with invariant set analysis subject to magnitude- and energy-bounded disturbances in [46]. Especially, [46] developed LMI conditions to design controllers that ensure the state to be confined within the region where the T–S fuzzy model is defined when the magnitude- and energy-bounded disturbances exist.

Motivated by the results in [46], in this paper, we suggest three LMI-based approaches for invariant set analysis and H_∞ control of continuous-time T–S fuzz systems subject to magnitude- and energy-bounded disturbances. First of all, we present an LMI-based procedure to design controllers that locally stabilize the system when the disturbance vanishes and otherwise guarantee that the invariant set of the state is confined within the region where the T–S fuzzy model is defined. Notice that the proposed first LMI approach is not entirely new and can be viewed as a version of Theorem 2 in [46]. However, the proposed LMI condition is derived from a slightly different framework that was used in [46], and we have made some efforts to improve the rigorousness of the proof. Secondly, by using Schur complement several times, the first LMI condition is then modified to linearize some nonlinear search parameter in the first LMI optimization. This might be viewed as an advantage over our first LMI condition and Theorem 2 in [46] since the elimination of the parameter's search procedure significantly reduces the

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computational effort of the overall design procedure. Finally, the first LMI approach is extended to tackle the H_∞ control problem. The designed H_∞ controller also guarantees that the state starting inside some set of the state variables will not escape an invariant set which is confined within the region of T–S fuzzy models. Numerical examples are given to illustrate the proposed methods.

II. Preliminaries

A. Notation

The adopted notation is as follows: A^T: transpose of matrix A; $A \succ 0$ ($A \prec 0$, $A \succeq 0$, and $A \preceq 0$, respectively): symmetric positive definite (negative definite, positive semi-definite, and negative semi-definite, respectively) matrix A; 0_n: origin of $\mathbb{R}^n$; $\mathrm{He}\{A\} := A + A^T$; $*$ inside a matrix: transpose of its symmetric term; $\mathcal{I}_r := \{1, 2, \dots, r\}$; $\mathbb{R}_{\geq 0} := \{t \in \mathbb{R} : t \geq 0\}$; $\mathbb{R}_{>0} := \{t \in \mathbb{R} : t > 0\}$; $\Upsilon(\xi) := \sum_{i=1}^{r} h_i(\xi(t))\Upsilon_i$; I: identity matrix of appropriate dimensions.

B. Problem Formulation

Let us consider the continuous-time T–S fuzzy system

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} h_i(\xi(t))(A_i x(t) + B_{u,i}u(t) + B_{w,i}w(t)) \\ z(t) = \sum_{i=1}^{r} h_i(\xi(t))(C_i x(t) + D_{u,i}u(t) + D_{w,i}w(t)) \end{cases}$$
$$\forall x(t) \in \mathcal{L}, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $w(t) \in \mathbb{R}^p$ is the disturbance, $z(t) \in \mathbb{R}^q$ is the controlled output, $A_i \in \mathbb{R}^{n\times n}$, $B_{u,i} \in \mathbb{R}^{n\times m}$, $B_{w,i} \in \mathbb{R}^{n\times p}$, $C_i \in \mathbb{R}^{q\times n}$, $D_{u,i} \in \mathbb{R}^{q\times m}$, $D_{w,i} \in \mathbb{R}^{q\times p}$ are constant matrices, $i \in \mathcal{I}_r := \{1, 2, \dots, r\}$ is the rule number, $\xi(t) \in \mathbb{R}^s$ is the vector containing premise variables in the fuzzy inference rule, $h_i(\xi(t))$ is the membership function (MF) for each rule, $\mathcal{L} \subseteq \mathbb{R}^n$ is a set of state variables where the T–S fuzzy system is defined, and the vector of the MFs $h(\xi(t)) := [\, h_1(\xi(t)) \ \cdots \ h_r(\xi(t)) \,]^T \in \mathbb{R}^r$ lies in the unit simplex Λ_r for all $(t, x(t)) \in [0, \infty) \times \mathcal{L}$, where $\Lambda_r := \{\alpha \in R^r : \alpha_1 + \cdots + \alpha_r = 1, 0 \leq \alpha_i \leq 1, i \in \mathcal{I}_r\}$. In this paper, we assume that

$$\xi(t) = \mathcal{T}x(t) = \begin{bmatrix} \mathcal{T}_1 \\ \vdots \\ \mathcal{T}_s \end{bmatrix} x(t) \in \mathbb{R}^s, \quad \mathcal{T} \in \mathbb{R}^{s\times n},$$

i.e., the premise variables are linear combinations of the state variables and that $\mathcal{L}$ is described as

$$\mathcal{L} := \{x \in \mathbb{R}^n : \mathcal{T}_i x \in [-\xi_{i,\max}, \xi_{i,\max}], \quad i \in \mathcal{I}_s\},$$

where $\xi_{i,\max} > 0$, $i \in \mathcal{I}_s$ are *a priori* given real numbers. The first problem addressed in this paper can be roughly stated as follows.

Problem 1: (Local stabilization and invariant set analysis). Determine a state-feedback control law $u(t) = K(\xi)x(t)$ such that

1) under $w(t) = 0_p$, the zero equilibrium point of (1) is locally asymptotically stable and estimate an invariant subset of the domain of attraction (DA) [47];
2) under $w(t)^T w(t) \leq \delta, \forall t \in [0, \infty)$ and $\int_0^\infty w(\tau)^T w(\tau) d\tau \leq \varepsilon$, any trajectories starting from some domain do not escape region $\mathcal{L}$ for all $t \in [0, \infty)$.

The second problem can be briefly summarized as follows.

Problem 2: (Local H_∞ control problem). Determine a state-feedback control law $u(t) = K(\xi)x(t)$ such that statements 1), 2) of Problem 1 are fulfilled, and the H_∞ performance $\sqrt{\int_0^\infty z(\tau)^T z(\tau) d\tau / \int_0^\infty w(\tau)^T w(\tau) d\tau} < \gamma^{1/2}$ is satisfied under $x(0) = 0_n$.

Remark 1: Notice that Problem 1 was already addressed in [46] in a systematical manner. Inspired by the ideas in [46], we present an LMI-based optimization procedures which can be viewed as modified versions of Theorem 2 in [46]. In this paper, the proof is based on the ideas in [46] and also follows the lines similar to those in [40], [42], [43] which deal with the local stability problem.

Remark 2: In [41], the local stability analysis was addressed without considering the local controller synthesis problem. On the other hand, in this paper, we consider the local stabilization with the local H_∞ performance provided in Problem 2. Moreover, in [41], different definitions and sets of the state variables were used.

C. Main Result

In this section, we present LMI-based optimization procedures to solve the Problems 1 and 2. For $P_i \succ 0$, $i \in \mathcal{I}_r$, let

$$\begin{aligned} V(x(t)) :=& x(t)^T \left(\sum_{i=1}^{r} h_i(\xi(t))P_i \right)^{-1} x(t) \\ =& x(t)^T P(\xi)^{-1} x(t) \end{aligned}$$

be a candidate of Lyapunov functions. In addition, let us consider the so-called non-parallel distributed compensation state-feedback control law [18]:

$$\begin{aligned} u(t) =& \left(\sum_{i=1}^{r} h_i(\xi(t))F_i \right) \left(\sum_{i=1}^{r} h_i(\xi(t))P_i \right)^{-1} x(t) \\ =& F(\xi)P(\xi)^{-1}x(t), \end{aligned}$$

where $F(\xi) := \sum_{i=1}^{r} h_i(\xi(t))F_i$. Combining the above control law with (1), the resulting closed-loop system is given by

$$\begin{cases} \dot{x}(t) = (A(\xi) + B_u(\xi)F(\xi)P(\xi)^{-1})x(t) + B_w(\xi)w(t); \\ z(t) = (C(\xi) + D_u(\xi)F(\xi)P(\xi)^{-1})x(t) + D_w(\xi)w(t); \end{cases}$$
$$\forall x(t) \in \mathcal{L}. \tag{2}$$

For the development of the main results, we also need to define the following sets:

1) $\mathcal{H}(b) := \{x \in \mathcal{L} : |h_i(\xi)| \leq b, \xi = \mathcal{T}x, i \in \mathcal{I}_r\}$;
2) $\Omega(\gamma) := \{x \in \mathcal{L} : V(x) \leq \gamma\}$;

3) $\mathcal{W}(\delta, \varepsilon)$: set of continuous functions $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^q$ such that $w(t)^T w(t) \leq \delta, \forall t \in [0, \infty)$ and $\int_0^\infty w(\tau)^T w(\tau) d\tau \leq \varepsilon$.
4) $\mathcal{V}$: set of vertices of $\{v \in \mathbb{R}^r : -b \leq v_i \leq b, i \in \mathcal{I}_r, v_1 + v_2 + \cdots + v_r = 0\}$;
5) $\mathcal{G}_i$: set of vertices of a polytope that includes $\partial h_i(\xi)/\partial\xi$ for all $\xi = \mathcal{T}_i x, x \in \mathcal{L}$.

Remark 3: Note that the assumption of magnitude-bounded and energy-bounded disturbances was already used in [46]. In addition, the assumption of magnitude-bounded disturbances was also used in [44], [45]. Compared to [46], we add the continuity assumption of $w(t)$ to improve rigorousness of the proof.

Remark 4: A remark on how to calculate set $\mathcal{V}$ is in order. In case $r = 2$, set $\mathcal{V}$ is simply $\mathcal{V} = \left\{ \begin{bmatrix} b \\ -b \end{bmatrix}, \begin{bmatrix} -b \\ b \end{bmatrix} \right\}$. For the general case, first note that set $\mathcal{M} := \{v \in \mathbb{R}^r : -b \leq v_i \leq b, i \in \mathcal{I}_r, v_1 + v_2 + \cdots + v_r = 0\}$ is a polyhedron which is the intersection of polytope $\{v \in \mathbb{R}^r : -b \leq v_i \leq b, i \in \mathcal{I}_r\}$ and hyper plane $\{v \in \mathbb{R}^r : v_1 + v_2 + \cdots + v_r = 0\}$. The problem of computing vertices of the polyhedron $\mathcal{M}$ is known as the vertex enumeration problem in computational geometry and can be solved using program `Polyhedron` in Multi-Parametric Toolbox 3.0 [49].

Remark 5: If one does not want to put additional computational efforts to compute set $\mathcal{V}$, then the slack variables approache in [20] or the bounding techniques developed in [37]–[39] can be applied alternatively.

To proceed further, we need to recall the following relaxation lemma presented in [7].

Lemma 1 ([7]): Given symmetric matrices Υ_{ij}, $(i, j) \in \mathcal{I}_r^2$, $\sum_{i=1}^r \sum_{j=1}^r h_i(\xi(t)) h_j(\xi(t)) \Upsilon_{ij} \prec 0$ holds for all $x(t) \in \mathcal{L}$ if LMIs $\Upsilon_{ii} \prec 0, \forall i \in \mathcal{I}_r$ and $(2/(r-1))\Upsilon_{ii} + \Upsilon_{ij} + \Upsilon_{ji} \prec 0$, $i \neq j$, $\forall (i, j) \in \mathcal{I}_r^2$ are fulfilled.

We are now in position to establish LMI-based optimization procedure that solves Problem 1.

Theorem 1: Let parameters $(b, \delta, \eta, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0}$ be given. Suppose that there exist matrices $P_i = P_i^T \in \mathbb{R}^{n \times n}$, $F_i \in \mathbb{R}^{n \times m}$, $i \in \mathcal{I}_r$, and a number $\varepsilon \in \mathbb{R}_{>0}$ such that the following optimization problem is satisfied:

$$\max_{P_i, F_i, \varepsilon} \varepsilon \,\text{subject to}$$

$$\begin{bmatrix} -\frac{1}{1+\eta\varepsilon} P_i & * \\ \mathcal{T}_j P_i & -\xi_{j,\max}^2 \end{bmatrix} \prec 0, \quad \forall (i, j) \in \mathcal{I}_r \times \mathcal{I}_s, \tag{3}$$

$$\Upsilon_{ii}(g) \prec 0, \quad \forall i \in \mathcal{I}_r, \quad \forall g \in \mathcal{G}_k, \quad \forall k \in \mathcal{I}_r, \tag{4}$$

$$\frac{2}{r-1} \Upsilon_{ii}(g) + \Upsilon_{ij}(g) + \Upsilon_{ji}(g) \prec 0,$$
$$(i, j) \in \forall \{(i, j) \in \mathcal{I}_r \times \mathcal{I}_r : i \neq j\}, \quad \forall g \in \mathcal{G}_k, \quad \forall k \in \mathcal{I}_r, \tag{5}$$

$$\begin{bmatrix} -\beta^{-1} I_n & * \\ I_n & -P_i \end{bmatrix} \prec 0, \quad \forall i \in \mathcal{I}_r, \tag{6}$$

$$\Psi_{ii}(v) \prec 0, \quad \forall i \in \mathcal{I}_r, \quad \forall v \in \mathcal{V}, \tag{7}$$

$$\frac{2}{r-1} \Psi_{ii}(v) + \Psi_{ij}(v) + \Psi_{ji}(v) \prec 0,$$
$$\forall (i, j) \in \{(i, j) \in \mathcal{I}_r \times \mathcal{I}_r : i \neq j\}, \quad \forall v \in \mathcal{V}, \tag{8}$$

where

$$\Upsilon_{ij}(g) := \begin{bmatrix} -\frac{1}{1+\eta\varepsilon+\delta} \begin{bmatrix} P_i & 0 \\ 0 & I \end{bmatrix} & * \\ g\mathcal{T} \begin{bmatrix} A_i P_j + B_{u,i} F_j & B_{w,i} \end{bmatrix} & -b^2 I \end{bmatrix},$$

$$\Psi_{ij}(v) := \begin{bmatrix} \mathrm{He}\{(A_i P_j + B_{u,i} F_j)\} - \sum_{k=1}^r P_k v_k & * \\ B_{w,i}^T & -\eta I \end{bmatrix}.$$

Then, the following statements are true:

1) $\Omega(1 + \eta\varepsilon) \subset \mathcal{L}$;
2) If $w(t) \in \mathcal{W}(\delta, \varepsilon)$ and $x(0) \in \Omega(1 + \eta\varepsilon)$, then $\Omega(1 + \eta\varepsilon) \subset \mathcal{H}(b)$;
3) $\{x \in \mathbb{R}^n : x^T x \leq \beta\} \subset \Omega(1)$;
4) If $w(t) = 0_p$, then closed-loop system (2) is locally asymptotically stable and an invariant subset of the DA for the closed-loop system is given by $\Omega(1 + \eta\varepsilon)$;
5) If $w(t) \in \mathcal{W}(\delta, \varepsilon)$ and $x(0) \in \Omega(1)$, then all the future trajectories will remain within $\Omega(1 + \eta\varepsilon)$, i.e., $x(t) \in \Omega(1 + \eta\varepsilon)$, $\forall t \in [0, \infty)$.

Proof: To begin with, note that using Lemma 1 and relation $\partial h_i(\xi)/\partial\xi \in \mathcal{G}_i$, $\forall x(t) \in \mathcal{L}$, $\forall i \in \mathcal{I}_r$, LMIs (3)-(8) guarantee

$$\begin{bmatrix} -\frac{1}{1+\eta\varepsilon} P(\xi) & * \\ \mathcal{T}_j P(\xi) & -\xi_{j,\max}^2 \end{bmatrix} \prec 0, \quad \forall (x(t), j) \in \mathcal{L} \times \mathcal{I}_s, \tag{9}$$

$$\begin{bmatrix} -\frac{1}{1+\eta\varepsilon+\delta} \begin{bmatrix} P(\xi) & 0 \\ 0 & I \end{bmatrix} & * \\ \frac{\partial h_i(\xi)}{\partial\xi} \mathcal{T} \begin{bmatrix} A(\xi)P(\xi) + B_u(\xi)F(\xi) & B_w(\xi) \end{bmatrix} & -b^2 I \end{bmatrix}$$
$$\prec 0, \quad \forall (x(t), i) \in \mathcal{L} \times \mathcal{I}_r, \tag{10}$$

$$\begin{bmatrix} -\beta^{-1} I & * \\ I & -P(\xi) \end{bmatrix} \prec 0, \quad \forall x(t) \in \mathcal{L}, \tag{11}$$

$$\begin{bmatrix} \mathrm{He}\{(A(\xi)P(\xi) + B_u(\xi)F(\xi))\} - \dot{P}(\xi) & * \\ B_w(\xi)^T & -\eta I \end{bmatrix}$$
$$\prec 0, \quad \forall x(t) \in \mathcal{H}(b). \tag{12}$$

Then, the proof consists of several parts.

Proof for statement 1): Let $\zeta(t) := \begin{bmatrix} P(\xi)^{-1} \\ \xi_{j,\max}^{-2} \mathcal{T}_j \end{bmatrix} x(t)$. We multiply (9) by $\zeta(t)^T$ on the left and $\zeta(t)$ on the right to obtain

$$\xi_{j,\max}^{-2} \xi_j(t)^2 < \frac{1}{1+\eta\varepsilon} V(x(t)), \quad \forall (x(t), j) \in \mathcal{L} \backslash \{0_n\} \times \mathcal{I}_s,$$

which implies $1 + \eta\varepsilon < V(x(t))$, $\forall x(t) \in \partial\mathcal{L}$, where $\partial\mathcal{L}$ is the boundary of $\mathcal{L}$. This means $\partial\mathcal{L} \cap \partial\Omega(1 + \eta\varepsilon) = \emptyset$. It is important to note that the level set is defined only in $\mathcal{L}$. Also, it is easy to see that conditions (9)-(12) ensure that $V(x(t))$ is positive definite in $\mathcal{L}$. At this stage, there may be no guarantee that any level set $\Omega(\sigma)$, $\sigma \in \mathbb{R}_{>0}$ inside $\mathcal{L}$ is connected because there is no guarantee that $V(x(t))$ is a Lyapunov function in $\mathcal{L}$. However, since $V(0_n) = 0$ and $V(x(t))$ is continuous in $\mathcal{L}$, we know that at least $\Omega(1 + \eta\varepsilon)$ is nonempty, and from $\partial\mathcal{L} \cap \partial\Omega(1 + \eta\varepsilon) = \emptyset$, it can be seen that $\Omega(1 + \eta\varepsilon)$ is strictly included by $\mathcal{L}$, i.e., $\Omega(1 + \eta\varepsilon) \subset \mathcal{L}$.

Proof for statement 2): Pre- and post-multiplying (10) by $\mathrm{diag}(P(\xi)^{-1}, I, I)$ and applying Schur complement yield

$$\begin{aligned}&b^{-2}\left[\begin{array}{cc}A(\xi)+B_u(\xi)F(\xi)P(\xi)^{-1} & B_{wz}(\xi)\end{array}\right]^T\\&\times\mathcal{T}^T\frac{\partial h_i(\xi(t))}{\partial\xi(t)}^T\frac{\partial h_i(\xi(t))}{\partial\xi(t)}\mathcal{T}\\&\times\left[\begin{array}{cc}A(\xi)+B_u(\xi)F(\xi)P(\xi)^{-1} & B_w(\xi)\end{array}\right]\\&-\frac{1}{1+\eta\varepsilon+\delta}\left[\begin{array}{cc}P(\xi)^{-1} & 0\\0 & I\end{array}\right]\prec 0,\quad\forall x(t)\in\mathcal{L}.\end{aligned}$$

Once again, pre- and post-multiply the above inequality by $\left[\begin{array}{cc}x(t)^T & w(t)^T\end{array}\right]$ and its transpose, respectively, and use relation $\dot{h}_i(\xi(t))=(\partial h_i(\xi(t))/\partial\xi(t)\mathcal{T}\dot{x}(t)$ to obtain

$$\frac{1}{b^2}\dot{h}_i(\xi)^2<\frac{1}{1+\eta\varepsilon+\delta}(x(t)^TP(\xi)^{-1}x(t)+w(t)^Tw(t)),$$

$\forall x(t)\in\mathcal{L}\backslash\{0_n\}$. Since $w(t)\in\mathcal{W}(\delta,\varepsilon)$, we have $(1/b^2)\dot{h}_i(\xi(t))^2<(1+\eta\varepsilon+\delta)^{-1}(x(t)^TP(\xi)^{-1}x(t)+\delta)$, $\forall x(t)\in\mathcal{L}\backslash\{0_n\}$, which implies $(1/b^2)\dot{h}_i(\xi(t))^2<(1+\eta\varepsilon+\delta)^{-1}(1+\eta\varepsilon+\delta)=1$, $\forall x(t)\in\mathcal{L}\backslash\{0_n\}\cap\Omega(1+\eta\varepsilon)\backslash\{0_n\}$. From statement 1), $\Omega(1+\eta\varepsilon)\subset\mathcal{L}$ and hence, we have $(1/b^2)\dot{h}_i(\xi(t))^2<1, \forall x(t)\in\Omega(1+\eta\varepsilon)\backslash\{0_n\}\Rightarrow\Omega(1+\eta\varepsilon)\backslash\{0_n\}\subset\mathcal{H}(b)\Rightarrow\Omega(1+\eta\varepsilon)\subset\mathcal{H}(b)$.

Proof for statement 3): Pre- and post-multiplying (11) by $\left[\begin{array}{cc}x(t)^T & x(t)^T\end{array}\right]$ and its transpose, respectively, we have that inequality (11) ensures

$$\begin{aligned}&V(x(t))<\beta^{-1}x(t)^Tx(t),\quad\forall x(t)\in\mathcal{L}\backslash\{0_n\}\\&\Leftrightarrow V(x(t))-1<\beta^{-1}x(t)^Tx(t)-1,\quad\forall x(k)\in\mathcal{L}\backslash\{0_n\}\\&\Leftrightarrow V(x(t))-1<\beta^{-1}(x(t)^Tx(t)-\beta),\quad\forall x(k)\in\mathcal{L}\backslash\{0_n\}\\&\Rightarrow\{x\in\mathbb{R}^n:x^Tx\le\beta\}\subset\Omega(1).\end{aligned}$$

Proof for statement 4): From the first block diagonal of (12), it follows that

$$\mathrm{He}\{(A(\xi)P(\xi)+B_u(\xi)F(\xi))\}-\dot{P}(\xi)\prec 0,\quad\forall x(t)\in\mathcal{H}(b).$$

Left- and right-multiplying the above inequality by $P(\xi)^{-1}$ and using relation $-P(\xi)^{-1}\dot{P}(\xi)P(\xi)^{-1}=d(P(\xi)^{-1})/dt$, we have

$$\begin{aligned}&\mathrm{He}\{P(\xi)^{-1}(A(\xi)+B_u(\xi)F(\xi)P(\xi)^{-1})\}\\&+d(P(\xi)^{-1})/dt\prec 0,\quad\forall x(t)\in\mathcal{H}(b).\end{aligned}$$

Under $w(t)=0_p$, the last inequality implies $\dot{V}(x(t))<0, \forall x(t)\in\mathcal{H}(b)\backslash\{0_n\}\Rightarrow\mathcal{H}(b)\backslash\{0_n\}\subseteq\{x\in\mathcal{L}:\dot{V}(x)<0\}$. On the other hand, from statement 2), one has $\Omega(1+\eta\varepsilon)\subset\mathcal{H}(b), \forall w(t)\in\mathcal{W}(\delta,\varepsilon)\Rightarrow\Omega(1+\eta\varepsilon)\subset\mathcal{H}(b), \forall w(t)=0_p$, so together with $\Omega(1+\eta\varepsilon)\subset\mathcal{H}(b)$, $\mathcal{H}(b)\backslash\{0_n\}\subseteq\{x\in\mathcal{L}:\dot{V}(x)<0\}$ implies $\Omega(1+\eta\varepsilon)\subset\{x\in\mathcal{L}:\dot{V}(x)<0\}$. By Lyapunov theory [47], (2) with $w(t)=0_p$ is locally asymptotically stable and $\Omega(1+\eta\varepsilon)$ is an invariant subset of the DA [47].

Proof for statement 5): Pre- and post-multiply (12) by $\left[\begin{array}{cc}x(t)^TP(\xi) & w(t)^T\end{array}\right]$ and its transpose, respectively, and use relation $-P(\xi)^{-1}\dot{P}(\xi)P(\xi)^{-1}=d(P(\xi)^{-1})/dt$ to obtain

$$\dot{V}(x(t))-\eta w(t)^Tw(t)<0,\quad\forall x(t)\in\mathcal{H}(b)\backslash\{0_n\}.\quad(13)$$

Now the proof is done by contradiction. Suppose that (13) holds but $x(t)$ starting from $x(0)\in\Omega(1)$ satisfies

$$\begin{cases}x(t)\in\Omega(1+\eta\varepsilon)\backslash\partial\Omega(1+\eta\varepsilon), & t\in[0,\tau_1);\\x(\tau_1)\in\partial\Omega(1+\eta\varepsilon),\end{cases}$$

for some $\tau_1\in\mathbb{R}_{>0}$, where $\partial\Omega(1+\eta\varepsilon)$ is the boundary of $\Omega(1+\eta\varepsilon)$. From statement 2), $\Omega(1+\eta\varepsilon)\subset\mathcal{H}(b)$ holds, and from which we know that $x(t)\in\mathcal{H}(b)$ holds for all $t\in[0,\tau_1]$. This means that (13) is satisfied for all $t\in[0,\tau_1]$. Therefore, we can integrate the left-hand side of (13) from 0 to τ_1 to obtain

$$V(x(\tau_1))<V(x(0))+\eta\int_0^{\tau_1}w(\tau)^Tw(\tau)d\tau.$$

By assumptions $w(t)\in\mathcal{W}(\delta,\varepsilon)$ and $x(0)\in\Omega(1)$, it is true that

$$\int_0^{\tau_1}w(\tau)^Tw(\tau)d\tau<\int_0^{\infty}w(\tau)^Tw(\tau)d\tau\le\varepsilon.$$

and $V(x(0))\le 1$. Hence, it follows from the last two inequalities that

$$\begin{aligned}V(x(\tau_1))&<V(x(0))+\eta\int_0^{\tau_1}w(\tau)^Tw(\tau)d\tau\\&<V(x(0))+\eta\varepsilon\le 1+\eta\varepsilon,\end{aligned}$$

so $x(\tau_1)\notin\partial\Omega(1+\eta\varepsilon)$ which gives a contradiction. This implies that state $x(t)$ will not reach the boundary of $\Omega(1+\eta\varepsilon)$. From the continuity assumption of $w(t)$, $x(t)$ is also continuous, and $x(t)$ starting from $x(0)\in\Omega(1)$ will not escape domain $\Omega(1+\eta\varepsilon)$. This completes the proof. ■

Remark 6: The optimization problem of Theorem 1 is an one-dimensional maximization problem subject to LMI constraints, and for fixed ε, conditions (3)-(8) are LMIs tractable via LMI solvers [1]–[3]. Thus, the optimization problem can be solved by means of a sequence of LMI optimizations, i.e. a line search or a bisection process over ε.

In Theorem 1, (δ,β) are considered as prescribed design parameters chosen by the designer depending on the required performances of the controller, while (b,η) are considered as parameters to be searched over $\mathbb{R}_{>0}\times\mathbb{R}_{>0}$ as in [46]. In this paper, a version of Theorem 1 is developed, and it can be shown that η can be incorporated into the LMIs as a linear decision variable. It is established in the following theorem.

Theorem 2: Let parameters $(b,\delta,\beta)\in\mathbb{R}_{>0}\times\mathbb{R}_{>0}\times\mathbb{R}_{>0}$ be given. Suppose that there exist matrices $P_i=P_i^T\in\mathbb{R}^{n\times n}$, $F_i\in\mathbb{R}^{n\times m}$, $i\in\mathcal{I}_r$, and numbers $\varepsilon\in\mathbb{R}_{>0}$, $\phi\in\mathbb{R}_{>0}$ such that the following optimization problem is satisfied:

$$\max_{P_i,F_i,\varepsilon,\phi}\ \varepsilon\,\text{subject to}$$

$$\left[\begin{array}{ccc}-P_i & * & *\\\mathcal{T}_jP_i & -\xi_{j,\max}^2 & *\\\mathcal{T}_jP_i & 0 & -\frac{\phi}{\varepsilon}\xi_{j,\max}^2\end{array}\right]\prec 0,\quad\forall(i,j)\in\mathcal{I}_r\times\mathcal{I}_s,\quad(14)$$

$$\text{LMIs}\,(4)-(8)$$

with $\Upsilon_{ij}(g)$ and $\Psi_{ij}(v)$ replaced, respectively, by

$$\Upsilon_{ij}(g) := \begin{bmatrix} -\begin{bmatrix} P_i & 0 \\ 0 & I \end{bmatrix} & * & * \\ g\mathcal{T}\begin{bmatrix} A_iP_j + B_{u,i}F_j & B_{w,i} \end{bmatrix} & -\frac{b^2}{1+\delta}I & * \\ g\mathcal{T}\begin{bmatrix} A_iP_j + B_{u,i}F_j & B_{w,i} \end{bmatrix} & 0 & -\frac{\phi b^2}{\varepsilon}I \end{bmatrix}, \tag{15}$$

$$\Psi_{ij}(v) := \mathrm{He}\{(A_iP_j + B_{u,i}F_j)\} - \sum_{k=1}^{r} P_k v_k + \phi B_{w,i}B_{w,j}^T. \tag{16}$$

Then, statements 1)-5) in Theorem 1 hold with $\eta = \phi^{-1}$.

Proof: Let us start with conditions (9)-(12) in the proof of Theorem 1. Applying Schur complement to (10) and multiplying the resulting condition by $1 + \eta\varepsilon + \delta$ yield

$$-\begin{bmatrix} P(\xi) & 0 \\ 0 & I \end{bmatrix} + \frac{1+\delta}{b^2}\Xi^T\Xi + \frac{\eta\varepsilon}{b^2}\Xi^T\Xi \prec 0,$$

$\forall (x(t), i) \in \mathcal{L} \times \mathcal{I}_r$, where

$$\Xi := \frac{\partial h_i(\xi)}{\partial \xi}\mathcal{T}\begin{bmatrix} A(\xi)P(\xi) + B_u(\xi)F(\xi) & B_w(\xi) \end{bmatrix}.$$

Then, by applying the Schur complement twice to the above inequality with change of variable $\eta^{-1} = \phi$ and by applying Lemma 1, we obtain (4) and (5) with $\Upsilon_{ij}(g)$ replaced by (15). Similar procedures can be performed to derive (14) by using (9). On the other hand, applying Schur complement to (12) with change of variable $\eta^{-1} = \phi$ and using Lemma 1 give (7) and (8) with $\Psi_{ij}(v)$ replaced by (16). This completes the proof. ■

Remark 7: The incorporation of search parameter η into LMIs in Theorem 2 significantly reduces the computational efforts of the entire design procedure. Note also that following the similar lines, Theorem 2 in [46] can be modified in order to remove search parameter η.

Remark 8: Using the similar line, δ can be also incorporated into the LMIs as a linear decision variable. This modification can be used optionally.

Theorem 1 can be then extended to cope with the H_∞ controller design problem. It is established in the following theorem.

Theorem 3: Let parameters $(b, \delta, \eta, \beta, \gamma) \in \mathbb{R}_{>0}\times\mathbb{R}_{>0}\times\mathbb{R}_{>0}\times\mathbb{R}_{>0}\times\mathbb{R}_{>0}$ be given. Suppose that there exist matrices $P_i = P_i^T \in \mathbb{R}^{n\times n}$, $F_i \in \mathbb{R}^{n\times m}$, $i \in \mathcal{I}_r$, and a number $\varepsilon \in \mathbb{R}_{>0}$ such that the following optimization problem is satisfied:

$$\max_{P_i, F_i, \varepsilon} \varepsilon \,\text{subject to}\, (3)-(6)\,\text{and}$$

$$\Phi_{ii}(v) \prec 0, \quad \forall i \in \mathcal{I}_r, \quad \forall v \in \mathcal{V}, \tag{17}$$

$$\frac{2}{r-1}\Phi_{ii}(v) + \Phi_{ij}(v) + \Phi_{ji}(v) \prec 0,$$
$$\forall (i, j) \in \{(i,j) \in \mathcal{I}_r \times \mathcal{I}_r : i \neq j\}, \quad \forall v \in \mathcal{V}, \tag{18}$$

where

$$\Phi_{ij}(v) := \begin{bmatrix} \mathrm{He}\{(A_iP_j + B_{u,i}F_j)\} - \sum_{k=1}^{r} P_k v_k & * & * \\ B_{w,i}^T & -\eta I & * \\ C_iP_j + D_{u,i}F_j & D_{w,i} & -\frac{\gamma}{\eta}I \end{bmatrix}.$$

Then, 1)-5) in Theorem 1 and the following statement hold:

6) If $w(t) \in \mathcal{W}(\delta, \varepsilon)$ and $x(0) = 0_n$, then all the future trajectories will remain within $\Omega(1 + \eta\varepsilon)$ and the H_∞ performance $\sqrt{\int_0^\infty z(\tau)^T z(\tau)d\tau / \int_0^\infty w(\tau)^T w(\tau)d\tau} < \gamma^{1/2}$ is satisfied.

Proof: First of all, it is easy to prove that the feasibility of the optimization problem of Theorem 3 assures the feasibility of the optimization problem of Theorem 1 because the first block diagonal matrix of $\Phi_{ij}(v)$ of dimension $n+p$ in Theorem 3 is equivalent to $\Psi_{ij}(v)$ in Theorem 1. For this reason, proofs for statements 1)-5) are equivalent to those in Theorem 1. Therefore, it suffices to prove just statement 6) here. The first statement of 6) is true because $x(0) = 0_n \in \Omega(1)$ and hence all the future trajectories will remain within $\Omega(1 + \eta\varepsilon)$ from statement 5). For the second statement of 6), suppose that LMIs (17) and (18) hold. From Lemma 1, we have

$$\begin{bmatrix} \mathrm{He}\{A(\xi)P(\xi) + B_u(\xi)F(\xi)\} - \dot{P}(\xi) & * & * \\ B_w^T(\xi) & -\eta I & * \\ C(\xi)P(\xi) + D_u(\xi)F(\xi) & D_w(\xi) & -\frac{\gamma}{\eta}I \end{bmatrix}$$
$$\prec 0, \quad \forall x(t) \in \mathcal{H}(b).$$

Pre- and post-multiply the above inequality by $\mathrm{diag}(P(\xi), I, I)$, use relation $-P(\xi)^{-1}\dot{P}(\xi)P(\xi)^{-1} = d(P(\xi)^{-1})/dt$, apply Schur complement, and pre- and post-multiply the resulting condition by $\begin{bmatrix} x(t)^T & w(t)^T \end{bmatrix}$ and its transpose, respectively to show

$$\dot{V}(x(t)) + (\eta/\gamma)z(t)^T z(t) - \eta w(t)^T w(t) < 0,$$
$$\forall x(t) \in \mathcal{H}(b)\backslash\{0_n\}. \tag{19}$$

From statement 5), $x(t) \in \Omega(1+\eta\varepsilon)$, $\forall t \in [0, \infty)$ and from statement 2), we deduce $\Omega(1 + \eta\varepsilon) \subset \mathcal{H}(b) \Rightarrow x(t) \in \mathcal{H}(b)$, $\forall t \in [0, \infty)$. This ensures that (19) holds for all $t \in [0, \infty)$. Integrating the left-hand side of (19) from 0 to ∞, one gets

$$(\eta/\gamma)\int_0^\infty z(\tau)^T z(\tau)d\tau - \eta\int_0^\infty w(\tau)^T w(\tau)d\tau < 0$$
$$\Leftrightarrow \int_0^\infty z(\tau)^T z(\tau)d\tau \Big/ \int_0^\infty w(\tau)^T w(\tau)d\tau < \gamma.$$

This completes the proof. ■

Remark 9: What has been performed in Theorem 2 can be also done for Theorem 3 to linearize search parameter η. The modified version of Theorem 3 is not presented here due to the lack of space.

Remark 10: A brief outline of the local H_∞ controller synthesis procedure based on Theorem 3 is presented in the sequel. **Step 1**: Given continuous-time nonlinear system $\dot{x}(t) = f(x(t), u(t), w(t))$, $z(t) = g(x(t), u(t), w(t))$, calculate membership functions $h_i(\xi(t))$, the system matrices, premise variables $\xi(t) = \mathcal{T}x(t)$, and $\mathcal{L}$ for T–S fuzzy model

(1). **Step 2**: Calculate $\partial h_i(\xi(t))/\partial\xi(t)$ and its vertices in $\mathcal{G}_i$ for all $i \in \mathcal{I}_r$. **Step 3**: Construct vertex set $\mathcal{V}$ using the methods given in Remark 4. **Step 4**: Solve the optimization of Theorem 3 or the modified optimization of Theorem 3 stated in Remark 9. **Step 5**: If infeasible, Problem 2 cannot be solved via the proposed optimization procedure. Otherwise, if feasible, then Problem 2 can be solved. In other words, statements 1)-5) in Theorem 1 and statement 6) in Theorem 3 hold.

III. EXAMPLES

All numerical examples in the sequel were treated with the help of MATLAB R2012b running on a Windows 7 PC with Intel Core i7-3770 3.4GHz CPU, 32GB RAM. The LMI problems were solved with SeDuMi [2] and Yalmip [3].

Example 1: In this example, we make a comparison between Theorem 1 and Theorem 2 in [46]. For more fair comparisons, a few slight modifications of Theorem 2 in [46] are made as follows:

1) Condition (6) in Theorem 1 is included to guarantee that the one-sublevel set of $V(x(t))$ is larger then $\{x \in \mathbb{R}^n : x^T x \le \beta\}$.
2) In order that the invariant set is included by only the region $\mathcal{L}$ where the T–S fuzzy system is defined, LMIs in (26) of [46] are modified as follows:

$$\begin{bmatrix} \frac{\alpha_q^2}{1+\varepsilon\eta} & L_{(k)}P_i \\ P_i L_{(k)}^T & P_i \end{bmatrix} \succeq 0, \quad (i,\,k) \in \mathcal{I}_r \times \mathcal{I}_p,$$

where p is the number of premise variables in [46].
3) When Theorem 2 in [46] is used in combination with Lemma 2 in [46], to the best of the authors' knowledge, Lemma 2 might need to be slightly modified because Theorem 2 in [46] only assures that the state will remain within $\Omega(1+\eta\varepsilon)$ not $\Omega(1)$. The modified version would be Lemma 2 in [46] with Ψ_{ij} replaced by

$$\Psi_{ij} := \begin{bmatrix} \frac{1}{1+\varepsilon\eta}\left(\frac{\mu_k}{2\lambda_k}\right)^2 & * \\ (A_j P_i + B_j F_i)^T L_{(k)}^T & P_i \end{bmatrix}.$$

Let us consider the following T–S fuzzy model of the chaotic Lorenz system taken from [46]:

$$A_1 = \begin{bmatrix} -\sigma_1 & \sigma_1 & 0 \\ \sigma_2 & -1 & 20 \\ 0 & -20 & -\sigma_3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -\sigma_1 & \sigma_1 & 0 \\ \sigma_2 & -1 & -30 \\ 0 & 30 & -\sigma_3 \end{bmatrix},$$

$$B_{u,1} = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ \sigma_3 & 0 \end{bmatrix}, \quad B_{u,2} = \begin{bmatrix} 0 & -\sigma_1 \\ -\sigma_2 & 0 \\ 0 & \sigma_3 \end{bmatrix},$$

$$B_{w,1} = B_{w,2} = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_1 & 0 \\ 0 & 0 & \sigma_1 \end{bmatrix},$$

$$\mathcal{T} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad \xi_1(t) = \mathcal{T}x(t) = x_1(t),$$

$$\mathcal{L} = \{x \in \mathbb{R}^n : -\xi_{1,\max} \le \mathcal{T}x \le \xi_{1,\max}\}, \quad \xi_{1,\max} = 30,$$

$$h_1(\xi_1(t)) = \frac{-\xi_1(t) + \xi_{1,\max}}{2\xi_{1,\max}}, \quad h_2(\xi_1(t)) = \frac{\xi_1(t) + \xi_{1,\max}}{2\xi_{1,\max}},$$

$$(\sigma_1,\,\sigma_2,\,\sigma_3) = (5,\,30,\,2).$$

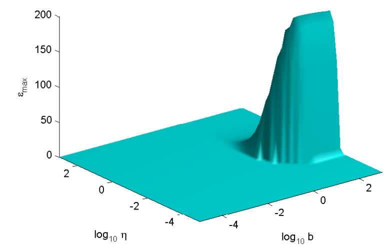


Fig. 1. Example 1. $\varepsilon_{\max}$ obtained using Theorem 1 for all $(b,\,\eta) \in \{10^{-5},\,10^{-5+0.2},\ldots,\,10^3\}^2$.

In this case, we have

$$\frac{\partial h_1(\xi_1(t))}{\partial \xi_1(t)} = \frac{-1}{2\xi_{1,\max}} \in \mathrm{co}\left\{\frac{-1}{2\xi_{1,\max}}\right\}, \quad \forall x(t) \in \mathcal{L},$$

$$\frac{\partial h_2(\xi_1(t))}{\partial \xi_1(t)} = \frac{1}{2\xi_{1,\max}} \in \mathrm{co}\left\{\frac{1}{2\xi_{1,\max}}\right\}, \quad \forall x(t) \in \mathcal{L},$$

where $\mathrm{co}\{\cdot\}$ denotes the convex hull, and hence

$$\mathcal{G}_1 = \left\{\frac{-1}{2\xi_{1,\max}}\right\}, \quad \mathcal{G}_2 = \left\{\frac{1}{2\xi_{1,\max}}\right\}.$$

In addition, $\mathcal{V}$ is the set of vertices of

$$\left\{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 : -b \le v_i \le b,\ i \in \{1,\,2\},\ v_1 + v_2 = 0\right\}$$

and given by $\mathcal{V} = \left\{\begin{bmatrix} b \\ -b \end{bmatrix}, \begin{bmatrix} -b \\ b \end{bmatrix}\right\}$. On the other hand, for Theorem 2 in [46], we set $\begin{cases} \lambda_1 = \lambda_2 = \frac{1}{2\xi_{1,\max}} = \frac{1}{60}; \\ L_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}; \\ \alpha_1 = 30. \end{cases}$

Let us assume that the maximum magnitude of $w(t)^T w(t)$ is $\delta = 10$, and assume that we want for the one-sublevel set $\Omega(1)$ to be larger than $\{x \in \mathbb{R}^n : x^T x \le \beta = 10^2\}$. In this case, in order to obtain the least conservative results, we need to perform exhaustive search of parameters $(b,\,\eta)$ for Theorem 1 and search of parameters $(\mu_1,\,\eta)$ for Theorem 2 in [46]. Figure 1 shows the values of maximum ε, denoted by $\varepsilon_{\max}$, obtained by using Theorem 1 with $(b,\,\eta)$ searched over Δ^2, where $\Delta := \{10^{-5},\,10^{-5+0.2},\,10^{-5+0.4},\,10^{-5+0.6},\ldots,\,10^3\}$. In addition, Figure 2 plots $\varepsilon_{\max}$ computed by using Theorem 2 in [46] with $(\mu_1,\,\eta)$ searched over Δ^2. By comparing the figures, we can see that both approaches produce similar results for this example. For Theorem 1, setting $(b,\,\eta) = (10^2,\,10^{-2})$ gives $\varepsilon_{\max} = 199.1433$, and for Theorem 2 in [46] with $(\mu_1,\,\eta) = (10^2,\,10^{-2})$ gives $\varepsilon_{\max} = 199.2958$. The estimated regions $\Omega(1)$ and $\Omega(1+\eta\varepsilon)$ of both methods are depicted in Figure 3, where the outer blue surface is the cross section of $\Omega(1+\eta\varepsilon)$ and the inner red elliptical sphere is $\Omega(1)$.

On the other hand, if one solves the problem using Theorem 2, then the search of parameters $(b,\,\eta)$ over Δ^2 in Theorem 1 reduces to the search of parameter b over Δ.

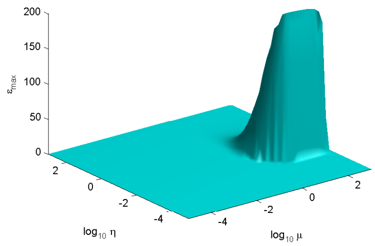
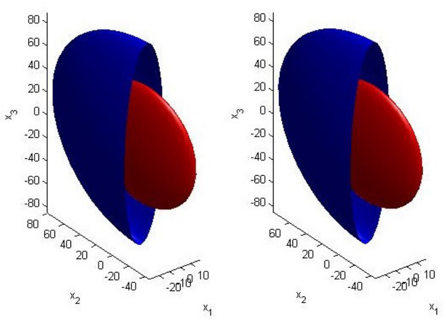


Fig. 2. Example 1. $\varepsilon_{\max}$ obtained using Theorem 2 in [46] for all $(\mu_1, \eta) \in \{10^{-5}, 10^{-5+0.2}, \dots, 10^3\}^2$.



(a) (b)

Fig. 3. Example 1. $\Omega(1)$ (inner red elliptical sphere) and cross section of $\Omega(1 + \eta\varepsilon)$ (outer blue surface) obtained using (a) Theorem 1 and (b) Theorem 2 in [46].

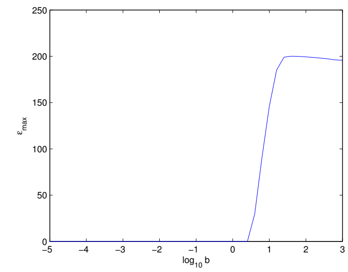


Fig. 4. Example 1. $\varepsilon_{\max}$ obtained using Theorem 2 for all $b \in \{10^{-5}, 10^{-5+0.2}, \dots, 10^3\}$.

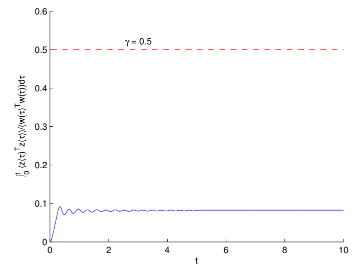


Fig. 5. Example 2. $\frac{\int_0^t z(\tau)^T z(\tau) d\tau}{\int_0^t w(\tau)^T w(\tau) d\tau}$ (blue solid line) and $\gamma = 0.5$ (red dashed line).

Thus, the computational efforts can be significantly reduced. Figure 4 shows $\varepsilon_{\max}$ obtained using Theorem 2 for all $b \in \Delta$ and reveals that Theorem 2 produces results similar to our Theorem 1 and Theorem 2 in [46].

Example 2: In this example, we illustrate the proposed H_∞ control design method. Let us consider the system in Example 1 again with the controlled output matrices $\begin{cases} C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}; \\ D_{u,1} = D_{u,2} = \begin{bmatrix} 0 & 0 \end{bmatrix}; \\ D_{w,1} = D_{w,2} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}. \end{cases}$ Solving the optimization of Theorem 3 with $(b, \eta, \gamma, \delta, \beta) = (10^2, 10^{-2}, 0.5, 10, 10^2)$, we obtained $\varepsilon_{\max} = 139.6874$. Regions $\Omega(1)$ and $\Omega(1 + \eta\varepsilon)$ in this case are very similar to those shown in Figure 3, so omitted here due to space limitations. For simulation, we used the following disturbance signal:

$$\begin{cases} w(t) = \sqrt{\frac{\delta}{3}} \begin{bmatrix} \sin(10t) \\ \sin(10t) \\ \sin(10t) \end{bmatrix}, \quad t \in [0, 5); \\ w(t) = 0_3, \quad t \in [5, \infty), \end{cases}$$

which satisfies $w(t)^T w(t) \leq \delta = 10, \forall t \in [0, \infty)$ and $\int_0^\infty w(\tau)^T w(\tau) d\tau \leq 25.1269 < \varepsilon = 139.6874$. With the zero initial condition, simulation results are shown in Figure 5, where the blue solid line is the values of $\int_0^t z(\tau)^T z(\tau) d\tau \Big/ \int_0^t w(\tau)^T w(\tau) d\tau$ as a H_∞ performance measure and the dashed red line indicates the H_∞ performance bound $\gamma = 0.5$. From the figure, it can be seen that the designed controller satisfies H_∞ performance $\sqrt{\int_0^\infty z(\tau)^T z(\tau) d\tau / \int_0^\infty w(\tau)^T w(\tau) d\tau} < \sqrt{0.5}$.

IV. Conclusions

In this paper, we have proposed LMI-based procedures to design local H_∞ controller for continuous-time T–S systems with magnitude- and energy-bounded disturbances. Examples have illustrated the proposed method. The extension of the proposed strategy to the local H_∞ controller design for discrete-time T–S fuzzy systems will be a possible future research topic.

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