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Cauchy-like functional equation based on a class of uninorms

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***Abstract*—Commuting is an important property in any two-step information merging procedure where the results should not depend on the order in which the single steps are performed. In the case of bisymmetric aggregation operators with the neutral elements, Saminger, Mesiar and Dubois, already reduced characterization of commuting n-ary operators to resolving the unary distributive functional equations, but only some sufficient conditions of unary functions distributive over two particular classes of uninorms are given out. Along this way of thinking, in this paper, we will investigate and fully characterize the following functional equation $f(U(x,y)) = U(f(x), f(y))$, where $f\colon [0,1] \to [0,1]$ is an unknown function, a uninorm $U \in \mathcal{U}_{\min}$ has a continuous underlying t-norm T_U and a continuous underlying t-conorm S_U. Our investigation shows the key point is a transformation from this functional equation to the several known ones. Moreover, this equation has non-monotone solutions different completely with those obtained ones.**

I. Introduction

The aggregation of information inherent to the human thinking is viewed as the process of merging all collected data into a concrete representative value [2], [21]. More specifically, the aggregation process is carried out as a two-stepped procedure whereby several local fusion operations are performed in parallel and then the results are merged into a global result [18]. It may happen that in practice the two steps can be exchanged because there is no reason to perform either of the steps first [20]. Thus one would expect the two procedures yield the same results in any sensible approach, and then operations are said to be *commuting*.

In fact, early examples of commuting appear in probability theory for the merging of probability distributions. Suppose two joint probability distributions are merged by combining degrees of probability point-wisely. It is natural that the marginals of the resulting joint probability function are the aggregates of the marginals of the original joint probabilities. To fulfill this requirement the aggregation operation must commute with the addition operation involved in the derivation of the marginals. McConway showed that a weighted arithmetic mean is the only possible aggregation operation for probability functions [12].

After this, the commuting aggregation operators caught more and more attention. For instances, they are used to preserve the transitivity when aggregating preference matrices or fuzzy relations [4], [11], [13], [19] or some form of additivity when aggregating set functions [5]. Specially, when Saminger, Mesiar, and Dubois [20] investigated the property of commuting for aggregation operators in connection with their relationship to bisymmetry, they gave out a full characterization of commuting operators in case that one of them is bisymmetric with some neutral element and further showed that these operators can be attained through functions distributive over the bisymmetric aggregation operator with neutral element involved. Thus they reduced characterization of commuting n-ary operators to resolving the unary distributive functional equations. Note that a full characterization of all bisymmetric aggregation operators with neutral elements, in particular if the neutral elements are from the open interval, is still missing [1], [3] and the characterization of the set of unary functions distributing with such operators is heavily influenced by the structure of the underlying operators [15], [17]. Hence they only focused on several special subclasses of bisymmetric aggregation operators with neutral elements, namely on continuous t-norms, continuous t-conorms and particular classes of uninorms. For classes of uninorms, but only some sufficient conditions of unary functions distributive over two particular classes of uninorms are given out. Indeed, it is very difficult to get the full characterization of these equations because they are bound up with many generalizations of the famous Cauchy functional equation which have not been completely solved so far [1], [14], [16]. Along this way of thinking, in this paper, we will investigate the following two functional equation

$$f(U(x,y)) = U(f(x), f(y)), \qquad (x,y) \in [0,1]^2, \qquad (1)$$

where $f\colon [0,1] \to [0,1]$ is an unknown function, a uninorm $U \in \mathcal{U}_{\min}$ has a continuous underlying t-norm T_U and a continuous underlying t-conorm S_U. Our investigation shows the key point is a transformation from this functional equation to the several known ones. Moreover, this equation has non-monotone solutions different completely with those

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obtained.

The paper is organized as follows. In Section 2, we present some results concerning basic fuzzy logic connectives. In Sections 3 and 4, the main sections of this paper, we will investigate and describe all solutions of Eq. (1). Finally, conclusion is in Section 5.

II. Preliminaries

Definition 1 ([7], [8]) A binary operation $T\colon [0,1]^2 \to [0,1]$ is called a *t-norm* if it is associative, commutative, increasing and has neutral element 1, namely, it holds $T(x,1) = T(1,x) = x$ for all $x \in [0,1]$.

Definition 2 ([10]) A t-norm T is said to be

(i) *continuous*, if for all convergent sequences $(x_n)_{n\in\mathbf{N}}$, $(y_n)_{n\in\mathbf{N}} \in [0,1]^{\mathbf{N}}$, we have $T(\lim\limits_{n\to\infty} x_n, \lim\limits_{n\to\infty} y_n) = \lim\limits_{n\to\infty} T(x_n, y_n)$;

(ii) *Archimedean*, if for every $x, y \in (0,1)$, there exists some $n \in \mathbf{N}$ such that $x_T^n > y$, where $x_T^1 = x$, $x_T^2 = T(x,x)$, $x_T^n = T(x_T^{n-1}, x)$;

(iii) *strict*, if T is continuous and strictly monotone, i.e., $T(x,y) > T(x,z)$ whenever $x \in (0,1]$ and $y > z$;

(iv) *nilpotent*, if T is continuous and if for each $x \in (0,1)$ there exists some $n \in \mathbf{N}$ such that $x_T^n = 0$.

Remark 1 ([8], [9])

(i) A t-conorm T is strict if and only if each continuous additive generator t of T satisfies $t(0) = \infty$.

(ii) A t-conorm T is nilpotent if and only if each continuous additive generator t of T satisfies $t(0) < \infty$.

Theorem 1 ([10]) *T is a continuous t-norm, if and only if one of the following three cases holds.*

(i) $T = \min$,

(ii) *T is continuously Archimedean, i.e., there exists a additive generator, namely, a continuous, strictly decreasing function $t\colon [0,1] \to [0,\infty]$ with $t(1) = 0$, which is uniquely determined up to a positive multiplicative constant, such that*

$$T(x,y) = t^{-1}(\min(t(x)+t(y), t(0))), \quad x, y \in [0,1]. \tag{2}$$

(iii) *There exists a family $\{(a_m, b_m), T_m\}_{m\in A}$ such that T is the ordinal sum of this family denoted by $T = (< a_m, b_m, T_m >)_{m\in A}$. In other words, it holds for all $x, y \in [0,1]$, $T(x,y) =$*

$$\begin{cases} a_m + (b_m - a_m)T_m(\frac{x-a_m}{b_m-a_m}, \frac{y-a_m}{b_m-a_m}) \\ \qquad\qquad\qquad\qquad (x,y) \in [a_m, b_m]^2, \\ \min(x,y) \qquad\qquad\qquad \textit{otherwise}, \end{cases}$$

where $\{(a_m, b_m)\}_{m\in A}$ is a countable family of non-over lapping, open, proper subintervals of $[0,1]$ with each T_m being a continuously Archimedean t-norm, and A is a finite or countable infinite index set. For every $m \in A$, (a_m, b_m) is called an open generating subinterval of T, and T_m is called a correspondingly generating t-norm on (a_m, b_m) (or $[a_m, b_m]$) of T.

Definition 3 ([10]) A binary operation $S\colon [0,1]^2 \to [0,1]$ is called a *t-conorm* if it is associative, commutative, increasing and has neutral element 0, namely, it holds $S(x,0) = S(0,x) = x$ for all $x \in [0,1]$.

Definition 4 ([8]) A t-conorm S is said to be

(i) *continuous*, if for all convergent sequences $(x_n)_{n\in\mathbf{N}}$, $(y_n)_{n\in\mathbf{N}} \in [0,1]^{\mathbf{N}}$, we have $S(\lim\limits_{n\to\infty} x_n, \lim\limits_{n\to\infty} y_n) = \lim\limits_{n\to\infty} S(x_n, y_n)$;

(ii) *Archimedean*, if for every $x, y \in (0,1)$, there exists some $n \in \mathbf{N}$ such that $x_S^n > y$, where $x_S^1 = x$, $x_S^2 = S(x,x)$, $x_S^n = S(x_S^{n-1}, x)$;

(iii) *strict*, if S is continuous and strictly monotone, i.e., $S(x,y) < S(x,z)$ whenever $x \in [0,1)$ and $y < z$;

(iv) *nilpotent*, if S is continuous and if for each $x \in (0,1)$ there exists some $n \in \mathbf{N}$ such that $x_S^n = 1$.

Theorem 2 ([10]) *S is a continuous t-conorm, if and only if one of the following three cases holds*

(i) $S = \max$,

(ii) *S is continuously Archimedean, i.e., there exists a additive generator, namely, a continuous, strictly increasing function $s\colon [0,1] \to [0,\infty]$ with $s(0) = 0$, which is uniquely determined up to a positive multiplicative constant, such that*

$$S(x,y) = s^{-1}(\min(s(x)+s(y), s(1))), \quad x, y \in [0,1]. \tag{3}$$

(iii) *There exists a family $\{(a_m, b_m), S_m\}_{m\in B}$ such that S is the ordinal sum of this family denoted by $S = (< a_m, b_m, S_m >)_{m\in B}$. In other words, it holds for all $x, y \in [0,1]$, $S(x,y) =$*

$$\begin{cases} a_m + (b_m - a_m)S_m(\frac{x-a_m}{b_m-a_m}, \frac{y-a_m}{b_m-a_m}) \\ \qquad\qquad\qquad\qquad (x,y) \in [a_m, b_m]^2, \\ \max(x,y) \qquad\qquad\qquad \textit{otherwise}, \end{cases}$$

where $\{(a_m, b_m)\}_{m\in B}$ is a countable family of non-over lapping, open, proper subintervals of $[0,1]$with each S_m being a continuously Archimedean t-conorm, and B is a finite or countable infinite index set. For every $m \in B$, (a_m, b_m) is called an open generating subinterval of S, and S_m is called a correspondingly generating t-conorm on (a_m, b_m) (or $[a_m, b_m]$) of S.

Definition 5 ([6], [22]) A *uninorm* U is a binary operator $U\colon [0,1] \times [0,1] \to [0,1]$, which is commutative, associative, non-decreasing in each variable and there exists some element $e \in [0,1]$ called neutral element such that $U(e,x) = x$ for all $x \in [0,1]$.

It is clear that the binary operator U becomes a t-norm when $e = 1$ while U a t-conorm when $e = 0$. For any other value $e \in (0,1)$ the operation works as a t-norm in the square $[0,e]^2$, and as a t-conorm in $[e,1]^2$.

Theorem 3 ([6]) *Let $U\colon [0,1]^2 \to [0,1]$ be a uninorm with neutral element $e \in (0,1)$. Then, the sections $x \mapsto (x,1)$ and $x \mapsto (x,0)$ are continuous at each point except perhaps at e if and only if U is given by one of the following formulas.*

(i) *If $U(0,1) = 0$, then $U(x,y) =$*

$$\begin{cases} eT_U(\frac{x}{e}, \frac{y}{e}) & (x,y) \in [0,e]^2, \\ e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & (x,y) \in [e,1]^2, \\ \min\{x,y\} & \textit{otherwise}. \end{cases}$$

(ii) *If* $U(0,1)=1$*, then* $U(x,y)=$

$$\begin{cases} eT_U(\frac{x}{e},\frac{y}{e}) & (x,y)\in[0,e]^2,\\ e+(1-e)S_U(\frac{x-e}{1-e}\frac{y-e}{1-e}) & (x,y)\in[e,1]^2,\\ \max\{x,y\} & \textit{otherwise}.\end{cases}$$

In the following, the set of uninorms as in Case (i) be denoted by $\mathcal{U}_{\min}$ and the set of uninorms as in Case (ii) by $\mathcal{U}_{\max}$. We will denote a uninorm U in $\mathcal{U}_{\min}$ with a continuous underlying t-norm T_U, a continuous underlying t-conorm S_U and neutral element e as $U\equiv\langle T_U,e,S_U\rangle_{\min,cos}$ and in a similar way, a uninorm U in $\mathcal{U}_{\max}$ as $U\equiv\langle T_U,e,S_U\rangle_{\max,cos}$.

Note that the main results in Ref. [16], i.e., Theorem 4.17, only consider the case that S_1 and S_2 are continuous but not Archimedean t-conorms. In fact, they hold for all continuous t-conorms. Therefore, the conditions that S_1 and S_2 are not Archimedean can be dropped. Then, set $I(x,y)=f_x(y)$ and apply this theorem, we can obtain the following characterizations of the Cauchy-like functional equations based on continuous t-conorms.

Theorem 4 ([16]) *Consider two continuous t-conorms* S_1 *and* S_2*, and a unary function* $f\colon[0,1]\to[0,1]$*. The triple of functions* (S_1,S_2,f) *satisfies*

$$f(S_1(x,y))=S_2(f(x),f(y)) \tag{4}$$

for all $x,y\in[0,1]$ *if and only if* f *is increasing, preserves the idempotent property (i.e., if* $S_1(x,x)=x$ *then* $S_2(f(x),f(x))=f(x)$*) and has the following form in every generating subinterval* (α_m,β_m) *of* S_1*,*

(i) *If* S_1 *is strict on its own generating subinterval* (α_m,β_m) *with the additive generator* s_m *and* S_2 *on the generating subinterval* (c_j,d_j) *has the additive generator* s_j*. Then we have one of the following two subcases.*

(a) $f(x)=r$ *when* $f(x)\notin(c_j,d_j)$ *for any* $x\in(\alpha_m,\beta_m)$*. Where* r *is idempotent, i.e.,* $S_2(r,r)=r$*, and satisfies* $f(\alpha_m)\le r\le f(\beta_m)$*.*

(b) *There exists some constant* $c\in(0,\infty)$ *such that* $f(x)=$

$$c_j+(d_j-c_j)\cdot s_j^{-1}(\min(cs_m(\frac{x-\alpha_m}{\beta_m-\alpha_m}),s_j(1))) \tag{5}$$

for any $x\in(\alpha_m,\beta_m)$*, when there exists some* $x_0\in(\alpha_m,\beta_m)$ *for which* $f(x_0)\in(c_j,d_j)$*.*

(ii) *If* S_1 *is nilpotent on its own generating subinterval* (α_m,β_m) *with the additive generator* s_m *and* S_2 *on the generating subinterval* (c_j,d_j) *has the additive generator* s_j*. Then we have one of the following two subcases.*

(a) $f(x)=\beta_m$ *when it holds that* $f(x)\notin(c_j,d_j)$ *for any* $x\in(\alpha_m,\beta_m)$*.*

(b) *There exists some* $x_0\in(\alpha_m,\beta_m)$ *such that* $f(x)$ *has the form Eq. (5) for any* $x\in(\alpha_m,\beta_m)$*, when there exists some* $x_0\in(\alpha_m,\beta_m)$ *for which* $f(x_0)\in(c_j,d_j)$*.*

From now on, we investigate and characterize the functional equation

$$f(U(x,y))=U(f(x),f(y)), \tag{6}$$

where $f\colon[0,1]\to[0,1]$ is an unknown function, a uninorm $U\equiv\langle T_U,e,S_U\rangle_{\min,cos}$, i.e., $U\in\mathcal{U}_{\min}$ has a continuous underlying t-norm T_U and a continuous underlying t-conorm S_U. But our method are also suit for a uninorm $U\equiv\langle T_U,e,S_U\rangle_{\max,cos}$. For the sake of convenience, write $Ran(f)=\{f(x)|x\in[0,1]\}$ and **Id**$(U)=\{x\in[0,1]|U(x,x)=x\}$, respectively.

Lemma 1 ([20]) *Consider a uninorm* $U\equiv\langle T_U,e,S_U\rangle_{\min,cos}$ *with neutral element* $e\in(0,1)$*, and a unary function* $f\colon[0,1]\to[0,1]$*. If* f *satisfies Eq. (6), then all of the following statements hold.*

(i) *If* $x\in$ ***Id***(U)*, then* $f(x)\in$ ***Id***(U)*.*
(ii) *If* $x\in[0,1]$*, then* $U(f(e),f(x))=f(x)$*.*
(iii) *If* $e\in Ran(f)$*, then* $f(e)=e$*.*

Lemma 2 *Consider a uninorm* $U\equiv\langle T_U,e,S_U\rangle_{\min,cos}$ *with neutral element* $e\in(0,1)$*, a unary function* $f\colon[0,1]\to[0,1]$*, and there exists some* $x_0\in[0,e)$ *such that* $f(x_0)<e$*. If* f *satisfies Eq. (6), then it holds that* $f(x)<e$ *for all* $x\in[0,x_0]$*.*

Due to Lemma 2, define $E=\{x\in[0,e)|f(x)<e\}$, then we observe $e\notin E$ and define

$$\alpha=\sup E. \tag{7}$$

It is obvious that $\alpha\le e$ and we can prove that α is an idempotent element of U, namely, $U(\alpha,\alpha)=\alpha$. Next, depending on the order relation between α and e, we need to consider two cases: (I) $\alpha<e$ and (II) $\alpha=e$. We first consider the case $\alpha<e$.

III. CASE: $\alpha<e$

Lemma 3 *Consider a uninorm* $U\equiv\langle T_U,e,S_U\rangle_{\min,cos}$ *with neutral element* $e\in(0,1)$*, a unary function* $f\colon[0,1]\to[0,1]$*, and the above-defined symbol* α *fulfilling* $\alpha<e$*. If* f *satisfies Eq. (6), then all of the following statements hold.*

(i) $f|_{[0,\alpha)}$ *is increasing,* $Ran(f|_{[0,\alpha)})\subseteq[0,e)$*.*
(ii) $f|_{(\alpha,e)}$ *is decreasing,* $Ran(f|_{(\alpha,e)})\subseteq[f(1),1]$*.*
(iii) $f|_{[e,1]}$ *is increasing,* $Ran(f|_{(e,1]})\subseteq[e,f(1)]$*.*

Lemma 4 *Consider a uninorm* $U\equiv\langle T_U,e,S_U\rangle_{\min,cos}$ *with neutral element* $e\in(0,1)$*, a unary function* $f\colon[0,1]\to[0,1]$*, and the above-defined symbol* α *fulfilling* $\alpha<e$*. If* f *satisfies Eq. (6), then one of the following three statements hold.*

(i) *If* $f(\alpha)<e$*, then* $f(\alpha)=\max(Ran(f|_{[0,\alpha]}))$*.*
(ii) *If* $f(\alpha)>e$*, then* $f(\alpha)=\max(Ran(f|_{[0,1]}))$*.*
(iii) $f(\alpha)=e$*.*

Suppose $x,y<\alpha$, define two functions $\phi\colon[0,\alpha]\to[0,1]$ and $\varphi\colon[0,e]\to[0,1]$ by the formulas $\phi(x)=\frac{x}{\alpha}$ and $\varphi(x)=\frac{x}{e}$, respectively. Then there exists some continuous t-norm T_1 such that two sides of Eq. (6) are respectively written as $U(x,y)=\phi^{-1}T_1(\phi(x),\phi(y))$ and $U(f(x),f(y))=$

$\varphi^{-1}T_U(\varphi(f(x)),\varphi(f(y)))$. Therefore, for any $x,y<e$, Eq. (6) can be rewritten as $f(\phi^{-1}T_1(\phi(x),\phi(y)))=\varphi^{-1}T_U(\varphi(f(x)),\varphi(f(y)))$, from which we get $(\varphi_1\circ f\circ\phi^{-1})(T_1(\phi(x),\phi(y)))=T_U(\varphi(f(x)),\varphi(f(y)))$. By routine substitutions, $g=\varphi\circ f\circ\phi^{-1},\ a=\phi(x),\ b=\phi(y)$, we have the Cauchy like functional equation

$$g(T_1(a,b))=T_U(g(a),g(b))\qquad\text{for } a,\ b\in[0,1],\tag{8}$$

where $g\colon[0,1]\to[0,1]$ is an unsolved function. This means that resolving of Eq. (6) when $x,y<\alpha$ is reduced to characterize all solutions of Eq. (8). Fortunately, the full characterization of this case can be obtained by the method of Theorem 4.

Suppose $\alpha<x,y<e$, define yet two functions $\phi'\colon[\alpha,e]\to[0,1]$ and $\varphi'\colon[f(1),1]\to[0,1]$ by the formulas $\phi'(x)=\frac{x-\alpha}{e-\alpha}$ and $\varphi'(x)=\frac{x-f(1)}{1-f(1)}$, respectively. Then there exist a continuous t-norm T_2 and a continuous t-conorm S_1 such that two sides of Eq. (6) are respectively written as $U(x,y)=(\phi')^{-1}T_2(\phi'(x),\phi'(y))$ and $U(f(x),f(y))=(\varphi')^{-1}S_1(\varphi'(f(x)),\varphi'(f(y)))$. Hence, for any $\alpha<x,y<e$, Eq. (6) is rewritten as $f((\phi')^{-1}T_2(\phi'(x),\phi'(y)))=(\varphi')^{-1}S_1(\varphi'(f(x)),\varphi'(f(y)))$, from which we get $(\varphi'\circ f\circ(\phi')^{-1})(T_2(\phi'(x),\phi'(y)))=S_1(\varphi'(f(x)),\varphi'(f(y)))$. By routine substitutions, $g'=\varphi'\circ f\circ(\phi')^{-1},\ a'=\phi'(x),\ b'=\phi'(y)$, we have the Cauchy like functional equation

$$g'(T_2(a',b'))=S_1(g'(a'),g'(b')),\qquad\text{for } a',\ b'\in[0,1],\tag{9}$$

where $g'\colon[0,1]\to[0,1]$ is an unsolved function. This means that resolving of Eq. (6) when $\alpha<x,y<e$ is reduced to characterize all solutions of Eq. (9). Fortunately, the full characterization of this case can be obtained by the method of Theorem 4.

Suppose $x,y>e$, define also two functions $\psi\colon[e,1]\to[0,1]$ and $\omega\colon[e,f(1)]\to[0,1]$ by the formulas $\phi(x)=\frac{x-e}{1-e}$ and $\omega(x)=\frac{x-e}{f(1)-e}$ respectively. Then there exists some continuous t-norm T_2' such that two sides of Eq. (6) are respectively written as $U(x,y)=\psi^{-1}S_U(\psi(x),\psi(y))$ and $U(f(x),f(y))=\omega^{-1}T_2'(\omega(f(x)),\omega(f(y)))$. Therefore, for any $(x,y)\in(e,1]^2$, Eq. (6) can be rewritten as $f(\psi^{-1}S_U(\psi(x),\psi(y)))=\omega^{-1}T_2'(\omega(f(x)),\omega(f(y)))$, from which we get $(\omega\circ f\circ\psi^{-1})(S_U(\psi(x),\psi(y)))=T_2'(\omega(f(x)),\omega(f(y)))$. By routine substitutions, $h=\omega\circ f\circ\psi^{-1},\ c=\psi(x),\ d=\psi(y)$, we have the Cauchy like functional equation

$$h(S_U(c,d))=T_2'(h(c),h(d)),\qquad\text{for } c,\ d\in[0,1],\tag{10}$$

where $h\colon[0,1]\to[0,1]$ is an unsolved function. This means that resolving of Eq. (6) when $(x,y)\in[e,1]^2$ is reduced to characterize all solutions of Eq. (10). Fortunately, this case can be obtained by the method of Theorem 4.

According to the above analyses and lemmas, we have the following theorem.

Theorem 5 *Consider a uninorm $U\equiv\langle T_U,e,S_U\rangle_{\min,cos}$ with neutral element $e\in(0,1)$, a unary function $f\colon[0,1]\to[0,1]$, and the above-defined symbols $\alpha,\ g,\ g',\ h$ fulfilling $\alpha<e$. Then f satisfies Eq. (6) if and only if all of the following statements hold.*

(i) *It holds that $f(x)\in\textbf{Id}(U)$ for all $x\in\textbf{Id}(U)$.*
(ii) *It holds that $U(f(e),f(x))=f(x)$ for all $x\in[0,1]$.*
(iii) *$f|_{[0,\alpha)}$ is increasing, $Ran(f|_{[0,\alpha)})\subseteq[0,e)$, g satisfies Eq. (8).*
(iv) *$f|_{(\alpha,e)}$ is decreasing, $Ran(f|_{(\alpha,e)})\subseteq[f(1),1]$, g' satisfies Eq. (9).*
(v) *$f|_{(e,1]}$ is increasing, $Ran(f|_{(e,1]})\subseteq[e,f(1)]$, h satisfies Eq. (10).*
(vi) *One of the following three statements hold:*
   a) *If $f(\alpha)<e$, then $f(\alpha)=\max(Ran(f|_{[0,\alpha]}))$,*
   b) *If $f(\alpha)>e$, then $f(\alpha)=\max(Ran(f|_{[0,1]}))$,*
   c) *$f(\alpha)=e$.*

IV. Case: $\alpha=e$

In this section, we discuss the case $\alpha=e$. We first assume that there exists some $y_0\in(e,1]$ such that $f(y_0)<e$, but it is not essential.

Lemma 5 *Consider a uninorm $U\equiv\langle T_U,e,S_U\rangle_{\min,cos}$ with neutral element $e\in(0,1)$, a unary function $f\colon[0,1]\to[0,1]$, and the above-defined symbol α fulfilling $\alpha=e$ and there exists some $y_0\in(e,1]$ such that $f(y_0)<e$. If f satisfies Eq. (6), then it holds that $f(y)<e$ for all $y\in[y_0,1]$.*

Due to Lemma 5, define $F=\{x\in(e,1]|f(x)<e\}$, then we observe $e\notin F$ and define

$$\beta=\inf F.\tag{11}$$

It is obvious that $e\le\beta$ and we can prove that β is an idempotent element of U, namely, $U(\beta,\beta)=\beta$. Next, depending on the order relation between β and e, we need to consider two subcases: (i) $\beta=e$ and (ii) $\beta>e$. At first, let us consider the subcase $\beta=e$.

A. Subcase: $\beta=e$

Lemma 6 *Consider a uninorm $U\equiv\langle T_U,e,S_U\rangle_{\min,cos}$ with neutral element $e\in(0,1)$, a unary function $f\colon[0,1]\to[0,1]$, and the above-defined symbols α and β fulfilling $\alpha=\beta=e$. If f satisfies Eq. (6), then the following two statements hold.*

(i) *$f|_{[0,e)}$ is increasing, $Ran(f|_{[0,e)})\subseteq[0,f(1)]$.*
(ii) *$f|_{(e,1]}$ is decreasing, $Ran(f|_{(e,1]})\subseteq[f(1),e)$.*

Suppose $x,y\in[0,e)$, define two functions $\phi_1\colon[0,e]\to[0,1]$ and $\varphi_1\colon[0,f(1)]\to[0,1]$ by the formulas $\phi_1(x)=\frac{x}{e}$ and $\varphi_1(x)=\frac{x}{f(1)}$ respectively. Then there exists some continuous t-norm T_3 such that both sides of Eq. (6) can be written as $U(x,y)=\phi_1^{-1}T_U(\phi_1(x),\phi_1(y))$ and $U(f(x),f(y))=\varphi_1^{-1}T_3(\varphi_1(f(x)),\varphi_1(f(y)))$. Thus, for $x,y\in[0,e)$, Eq. (6) can be rewritten as $f(\phi_1^{-1}T_U(\phi_1(x),\phi_1(y)))=\varphi_1^{-1}T_3(\varphi_1(f(x)),\varphi_1(f(y)))$, from which we get $(\varphi_1\circ f\circ\phi_1^{-1})(T_U(\phi_1(x),\phi_1(y)))=T_3(\varphi_1(f(x)),\varphi_1(f(y)))$.

By routine substitution, $g_1 = \varphi_1 \circ f \circ \phi_1^{-1}$, $a_1 = \phi_1(x)$, $b_1 = \phi_1(y)$, we have the Cauchy like functional equation

$$g_1(T_U(a_1, b_1)) = T_3(g_1(a_1), g_1(b_1)), \quad \text{for } a_1,\ b_1 \in [0,1], \tag{12}$$

where $g_1\colon [0,1] \to [0,1]$ is an unsolved function. This means that resolving of Eq. (6) when $x, y \in [0, e]$ is reduced to characterize all solutions of Eq. (12). Fortunately, the full characterization of this case can be obtained by the method of Theorem 4.

Suppose $x, y > e$, define two functions $\psi_1\colon [e,1] \to [0,1]$ and $\omega_1\colon [f(1), e] \to [0,1]$ by the formulas $\psi_1(x) = \frac{x-e}{1-e}$ and $\omega_1(x) = \frac{x-f(1)}{e-f(1)}$ respectively. Then there exists some continuous t-norm T_4 such that two sides of Eq. (6) are respectively written as $U(x,y) = \psi_1^{-1} S_U(\psi_1(x), \psi_1(y))$ and $U(f(x), f(y)) = \omega_1^{-1} T_4(\omega_1(f(x)), \omega_1(f(y)))$. Thus, for any $(x,y) \in [e,1]^2$, Eq. (6) can be rewritten as $f(\psi_1^{-1} S_U(\psi_1(x), \psi_1(y))) = \omega_1^{-1} T_4(\omega_1(f(x)), \omega_1(f(y)))$, from which we get $(\omega_1 \circ f \circ \psi_1^{-1}) S_U(\psi_1(x), \psi_1(y))) = T_4(\omega_1(f(x)), \omega_1(f(y)))$. By routine substitution, $h_1 = \omega_1 \circ f \circ \psi_1^{-1}$, $c_1 = \psi_1(x)$, $d_1 = \psi_1(y)$, we have the Cauchy like functional equation

$$h_1(S_U(c_1, d_1)) = T_4(h_1(c_1), h_1(d_1)), \quad \text{for } c_1,\ d_1 \in [0,1], \tag{13}$$

where $h_1\colon [0,1] \to [0,1]$ is an unsolved function. This means that resolving of Eq. (6) when $(x,y) \in [e,1]^2$ is reduced to characterize all solutions of Eq. (13). Fortunately, the full characterization of this case can be obtained by the method of Theorem 4.

Theorem 6 *Consider a uninorm $U \equiv \langle T_U, e, S_U \rangle_{\min, cos}$ with neutral element $e \in (0,1)$, a unary function $f\colon [0,1] \to [0,1]$, and $g_1,\ h_1,\ \alpha,\ \beta$ are the above-defined symbols fulfilling $\alpha = \beta = e$. Then f satisfies Eq. (6) if and only if all of the following statements hold.*

- (i) *It holds that $f(x) \in \boldsymbol{Id}(U)$ for all $x \in \boldsymbol{Id}(U)$.*
- (ii) *It holds that $U(f(e), f(x)) = f(x)$ for all $x \in [0,1]$.*
- (iii) *$f|_{[0,e)}$ is increasing, $Ran(f|_{[0,e)}) \subseteq [0, f(1)]$, g_1 satisfies Eq. (12).*
- (iv) *$f|_{(e,1]}$ is decreasing, $Ran(f|_{(e,1]}) \subseteq [f(1), e)$, h_1 satisfies Eq. (13).*

Remark 1 Take $f(1) = f(e)$ in Theorem 6, then we get a part of Proposition 33 in Ref. [20].

Next, consider the remaining case $\beta > e$.

B. Subcase: $\beta > e$

Lemma 7 *Consider a uninorm $U \equiv \langle T_U, e, S_U \rangle_{\min, cos}$ with neutral element $e \in (0,1)$, a unary function $f\colon [0,1] \to [0,1]$, and the above-defined symbols α and β fulfilling $\alpha = e < \beta$. If f satisfies Eq. (6), then all of the following statements hold.*

- (i) *$f|_{[0,e)}$ is increasing, $Ran(f|_{[0,e)}) \subseteq [0, f(1)]$.*
- (ii) *$f|_{(e,\beta)}$ is increasing, $Ran(f|_{(e,\beta)}) \subseteq [e, 1]$.*
- (iii) *$f|_{(\beta,1]}$ is decreasing, $Ran(f|_{(\beta,1]}) \subseteq [f(1), e)$.*

Lemma 8 *Consider a uninorm $U \equiv \langle T_U, e, S_U \rangle_{\min, cos}$ with neutral element $e \in (0,1)$, a unary function $f\colon [0,1] \to [0,1]$, and the above-defined symbol α and β fulfilling $\alpha = e < \beta$. If f satisfies Eq. (6), then one of the following statements holds.*

- (i) *If $f(\beta) < e$, then $f(\beta) = \max(Ran(f|_{[\beta,1]}))$.*
- (ii) *If $f(\beta) > e$, then $f(\beta) = \max(Ran(f|_{[0,1]}))$.*
- (iii) *$f(\beta) = e$.*

Suppose $x, y \in [0, e)$, define two functions $\phi_2\colon [0,e] \to [0,1]$ and $\varphi_2\colon [0, f(1)] \to [0,1]$ by the formulas $\phi_2(x) = \frac{x}{e}$ and $\varphi_2(x) = \frac{x}{f(1)}$ respectively. Then there exists some continuous t-norm T_5 such that both sides of Eq. (6) can be written as $U(x,y) = \phi_2^{-1} T_U(\phi_2(x), \phi_2(y))$ and $U(f(x), f(y)) = \varphi_2^{-1} T_5(\varphi_2(f(x)), \varphi_2(f(y)))$. Hence, for $x, y \in [0, e)$, Eq. (6) can be rewritten as $f(\phi_2^{-1} T_U(\phi_2(x), \phi_2(y))) = \varphi_2^{-1} T_5(\varphi_2(f(x)), \varphi_2(f(y)))$, from which we get $(\varphi_2 \circ f \circ \phi_2^{-1})(T_U(\phi_2(x), \phi_2(y))) = T_5(\varphi_2(f(x)), \varphi_2(f(y)))$. By routine substitution, $g_2 = \varphi_2 \circ f \circ \phi_2^{-1}$, $a_2 = \phi_2(x)$, $b_2 = \phi_2(y)$, we have the Cauchy like functional equation

$$g_2(T_U(a_2, b_2)) = T_5(g_2(a_2), g_2(b_2)), \quad \text{for } a_2,\ b_2 \in [0,1], \tag{14}$$

where $g_2\colon [0,1] \to [0,1]$ is an unsolved function. This means that resolving of Eq. (6) when $x, y \in [0, e]$ is reduced to characterize all solutions of Eq. (14). Fortunately, the full characterization of this case can be obtained by the method of Theorem 4.

Suppose $x, y \in (e, \beta)$, define two functions $\phi_2'\colon [e, \beta] \to [0,1]$ and $\varphi_2'\colon [e,1] \to [0,1]$ by the formulas $\phi_2'(x) = \frac{x-e}{\beta-e}$ and $\varphi_2'(x) = \frac{x-e}{1-e}$ respectively. Then there exists some continuous t-conorm S_3 such that two sides of Eq. (6) can be written as $U(x,y) = (\phi_2')^{-1} S_3(\phi_2'(x), \phi_2'(y))$ and $U(f(x), f(y)) = (\varphi_2')^{-1} S_U(\varphi_2'(f(x)), \varphi_2'(f(y))$. Therefore, for $x, y \in (e, \beta)$, Eq. (6) can be rewritten as $f((\phi_2')^{-1} S_3(\phi_2'(x), \phi_2'(y))) = (\varphi_2')^{-1} S_U(\varphi_2'(f(x)), \varphi_2'(f(y))$, from which we have $(\varphi_2' \circ f \circ (\phi_2')^{-1})(S_3(\phi_2'(x), \phi_2'(y))) = S_U(\varphi_2'(f(x)), \varphi_2'(f(y)))$. By routine substitution, $g_2' = \varphi_2' \circ f \circ (\phi_2')^{-1}$, $a_2' = \phi_2'(x)$, $b_2' = \phi_2'(y)$, we have the Cauchy like functional equation

$$g_2'(S_3(a_2', b_2')) = S_U(g_2'(a_2'), g_2'(b_2')), \quad \text{for } a_2',\ b_2' \in [0,1], \tag{15}$$

where $g_2'\colon [0,1] \to [0,1]$ is an unsolved function. This means that resolving of Eq. (6) when $x, y \in [e, \beta)$ is reduced to characterize all solutions of Eq. (15). Fortunately, the full characterization of this case can be obtained by the method of Theorem 4.

Suppose $x, y \in (\beta, 1]$, define two functions $\psi_2\colon [\beta, 1] \to [0,1]$ and $\omega_2\colon [f(1), e] \to [0,1]$ by the formulas $\psi_2(x) = \frac{x-\beta}{1-\beta}$ and $\omega_2(x) = \frac{x-f(1)}{e-f(1)}$ respectively. Then there exist a continuous t-conorm S_4 and a continuous t-norm T_6 such that two sides of Eq. (6) are respectively written as $U(x,y) = \psi_2^{-1} S_4(\psi_2(x), \psi_2(y))$ and $U(f(x), f(y)) = \omega_2^{-1} T_6(\omega_2(f(x)), \omega_2(f(y)))$. Therefore, for any $(x,y) \in [\beta, 1]^2$, Eq. (6) can be rewritten as $f(\psi_2^{-1} S_4(\psi_2(x), \psi_2(y))) = \omega_2^{-1} T_6(\omega_2(f(x)), \omega_2(f(y)))$, from which we have $(\omega_2 \circ f \circ \psi_2^{-1})(S_4(\psi_2(x), \psi_2(y))) = T_6(\omega_2(f(x)), \omega_2(f(y)))$. By routine substitution, $h_2 = \omega_2 \circ f \circ \psi_2^{-1}$, $c_2 = \psi_2(x)$, $b_2 =$

$\psi_2(y)$, we have the Cauchy like functional equation

$$h_2(S_4(c_2, d_2)) = T_6(h_2(c_2), h_2(d_2)), \quad \text{for } c_2,\ d_2 \in [0,1], \tag{16}$$

where $h_2\colon [0,1] \to [0,1]$ is an unsolved function. This means that resolving of Eq. (6) when $(x,y) \in [\beta,1]^2$ is reduced to characterize all solutions of Eq. (16). Fortunately, the full characterization of this case can be obtained by the method of Theorem 4.

Theorem 7 *Consider a uninorm $U \equiv \langle T_U, e, S_U\rangle_{\min,cos}$ with neutral element $e \in (0,1)$, a unary function $f\colon [0,1] \to [0,1]$, all of $\alpha,\ \beta,\ g_2,\ g_2',\ h_2$ are the-above defined symbols fulfilling $\alpha = e < \beta$. Then f satisfies Eq. (6) if and only if all of the following statements hold.*

(i) *It holds that $f(x) \in \textbf{Id}(U)$ for all $x \in \textbf{Id}(U)$.*

(ii) *It holds that $U(f(e), f(x)) = f(x)$ for all $x \in [0,1]$.*

(iii) *$f|_{[0,e)}$ is increasing, $Ran(f|_{[0,e)}) \subseteq [0, f(1)]$, g_2 satisfies Eq. (14).*

(iv) *$f|_{(e,\beta)}$ is increasing, $Ran(f|_{(e,\beta)}) \subseteq [e,1]$, g_2' satisfies Eq. (15).*

(v) *$f|_{(\beta,1]}$ is decreasing, $Ran(f|_{(\beta,1]}) \subseteq [f(1), e)$, h_2 satisfies Eq. (16).*

(vi) *One of the following three statements hold:*

a) *If $f(\beta) < e$, then $f(\beta) = \max(Ran(f|_{[\beta,1]}))$.*

b) *If $f(\beta) > e$, then $f(\beta) = \max(Ran(f|_{[0,1]}))$.*

c) *$f(\beta) = e$.*

V. Example

Example 1 Consider the following uninorm U with neutral element $e = \frac{1}{2}$,

$$U(x,y) = \begin{cases} 8x+8y-8xy-7, & (x,y) \in [\frac{7}{8},1]^2, \\ 7x+7y-8xy-\frac{21}{4}, & (x,y) \in [\frac{3}{4},\frac{7}{8}]^2, \\ \max(x,y), & \\ \qquad (x,y) \in [\frac{1}{2},1]^2 \setminus ([\frac{7}{8},1]^2 \cup [\frac{3}{4},\frac{7}{8}]^2), & \\ \frac{1}{8}(8x-1)(8y-1)+\frac{1}{8}, & (x,y) \in [\frac{1}{8},\frac{1}{4}]^2, \\ 8xy, & (x,y) \in [0,\frac{1}{8}]^2, \\ \min(x,y), & \text{otherwise.} \end{cases}$$

Then we know

$$T_U(x,y) = \begin{cases} 4xy, & (x,y) \in [0,\frac{1}{4}]^2, \\ \frac{1}{4}(4x-1)(4y-1)+\frac{1}{4}, & (x,y) \in [\frac{1}{4},\frac{1}{2}]^2, \\ \min(x,y), & \text{otherwise,} \end{cases}$$

and

$$S_U(x,y) = \begin{cases} 8(x+1)+8(y+1)-4(x+1)(y+1)-15, & \\ \qquad\qquad (x,y) \in [\frac{3}{4},1]^2, & \\ 7(x+1)+7(y+1)-4(x+1)(y+1)-\frac{23}{2}, & \\ \qquad\qquad (x,y) \in [\frac{1}{2},\frac{3}{4}]^2, & \\ \max(x,y), & \\ \qquad\qquad \text{otherwise.} & \end{cases}$$

In fact, T_U and S_U are two ordinal sums with twice the product as summands and twice the probabilistic sum as summands respectively. Let us recall that the product $T_P = xy$ has a additive generator $y = -\ln x$ while the probabilistic sum $S_U = x+y-xy$ has a additive generator $y = -\ln(1-x)$.

(i) Take $\alpha = \frac{1}{8}$, then we know from Theorem 5 that

$$f(x) = \begin{cases} \frac{1}{8}, & x \in [0,\frac{1}{8}], \\ 1, & x \in (\frac{1}{8},\frac{1}{2}), \\ \frac{1}{2}, & x = \frac{1}{2}, \\ \frac{3}{4}, & x \in (\frac{1}{2},1], \end{cases}$$

is a solution of Eq. (6).

(ii) Take $\alpha = e = \beta = \frac{1}{2}$, then we know from Theorem 6 that

$$f(x) = \begin{cases} \frac{1}{8}, & x \in [0,\frac{1}{2}), \\ \frac{1}{2}, & x = \frac{1}{2}, \\ \frac{3}{4}, & x \in (\frac{1}{2},1], \end{cases}$$

is a solution of Eq. (6).

(iii) Take $\alpha = e = \frac{1}{2},\ \beta = \frac{7}{8}$, then we know from Theorem 7 that

$$f(x) = \begin{cases} \frac{1}{8}, & x \in [0,\frac{1}{2}), \\ \frac{3}{4}, & x \in [\frac{1}{2},\frac{7}{8}], \\ \frac{1}{4}, & x \in (\frac{7}{8},1], \end{cases}$$

is a solution of Eq. (6).

VI. Conclusion

To investigate property of commuting for bisymmetric aggregation operators with neutral element, according to Saminger, Mesiar and Dubois's suggestion [20], in this paper, we have investigated and fully characterized the following functional equation $f(U(x,y)) = U(f(x), f(y))$, where $f\colon [0,1] \to [0,1]$ is an unknown function, a uninorm Consider a uninorm $U \equiv \langle T_U, e, S_U\rangle_{\min,cos}$ with neutral element $e \in (0,1)$. Our investigation shows the key point is a transformation from this functional equation to the several known ones. Moreover, this equation has non-monotone solution different completely with those obtained ones. These results are an important step toward obtaining complete characterization of the mentioned-above other unary distributive functional equations. Obviously, there are several unary distribute functions not to be consider in this direction. Thus, future work will be devoted to deal with $f(U(x,y)) = U(f(x), f(y))$, where $f\colon [0,1] \to [0,1]$ is an unknown function and U comes from the other kind of special uninorms.

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