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On the cross-migrativity of triangular subnorms

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***Abstract*—In this paper, the sufficient and necessary conditions that a triangular subnorm M is (α, T)-cross-migrative are given out, where T stands for any of the three prototype triangular norms, that is, $T \in \{T_P,\ T_M,\ T_L\}$, M is a triangular subnorm with a continuously additive generator. By comparison with results of the (α, T)-cross-migrativity of triangular norms obtained by Fodor, it is in some sense that they are compatible.**

I. Introduction

For a two-stepped information aggregation procedure, it may happen that in practice the two steps can be exchanged because there is no reason to perform either of the steps first. So, one would expect the two processes yield the same results in any sensible approach. The important property of this procedure is said to be *commuting* [14]. To study the commuting of aggregation operators used usually to model this procedure, within the class of triangular norms (t-norms for short), it is [5] that shows the classical commuting equation

$$T_1(T_2(x, y), T_2(u, v)) = T_2(T_1(x, u), T_1(y, v)) \qquad (1)$$

has only the trivial solution $T_1 = T_2$. Then, by fixing $u = 1$ and writing $y = \alpha$ in Eq. (1), a weaker functional equation

$$T_1(T_2(\alpha, x), y) = T_2(x, T_1(\alpha, y)) \qquad (2)$$

has been obtained and such a pair (T_1, T_2) satisfying Eq. (2) is said to be *α-cross-migrative* (or, equivalently, that T_1 is *α-cross-migrative* with respect to T_2, or T_1 is *(α, T_2)-cross-migrative*) [5]. Finally, the sufficient and necessary conditions that a t-norm T_1 is (α, T_2)-cross-migrative had been given out [5], where T_2 belongs to any of the three prototype t-norms, that is, $T_2 \in \{T_P,\ T_M,\ T_L\}$.

On the other hand, to investigate the convex combination of a continuous t-norm T and the drastic product T_D, which indeed is a special form of the open problem– the convex combination of t-norms [1], the α-migrativity of a t-norm is introduced and defined as follows.

Definition 1.1 (see [4]) Let $\alpha \in (0, 1)$, a continuous t-norm T is said to be *α-migrative*, if for all $(x, y) \in [0, 1]^2$, we have

$$T(\alpha x, y) = T(x, \alpha y). \qquad (3)$$

The interest of the α-migrativity property comes from its applications, for instances in decision making processes [11], when a repeated, partial information needs to be fused in a global result, or in image processing, since in this context migrativity expresses the invariance of a given property under a proportional rescaling of some part of the image. Eq. (3) and its generalizations have extendedly been studied for t-norms, t-subnorms, semicopulas, quasi-copulas, copulas and aggregation functions (see [2], [3]). But so far, one of the most influential and impressive results about this topic are probably given out by Fodor and Rudas (see [6], [7], [8], [9]).

Recently, it is [15] that also investigates α-migrativity of a triangular subnorm (t-subnorm for short) M with a continuously additive generator w.r.t. any of the three prototype t-norms. Along this line, in this paper, we would like to extend the α-cross-migrativity into continuous t-subnorms. But note that any continuous t-subnorm is an ordinal sum of continuously Archimedean t-norms and at most one continuously Archimedean proper t-subnorm. Thus the α-cross-migrativity of continuous t-subnorms can ultimately be come down to the (α, T)-cross-migrativity of a continuous t-subnorm or t-norm. Hence, in this paper, according to [5], we are mainly going to investigate the following functional equation.

Definition 1.2 Let $\alpha \in [0, 1]$, M and T be a t-subnorm and a t-norm respectively. The pair (M, T) is said to be *α-cross-migrative* (or, equivalently, that M is *α-cross-migrative* with respect to T, or M is *(α, T)-cross-migrative*) if the following functional equation

$$M(T(x, \alpha), y) = T(x, M(\alpha, y)) \qquad (4)$$

holds for all $(x, y) \in [0, 1]^2$.

Although only then absence of one word between α-cross-migrativity and α-migrativity, they are actually quite different. These are the reasons that α-migrativity is completely determined by two lines $T_1(\alpha, x)$ and $T_2(\alpha, x)$(see [6], [7]), meanwhile α-cross-migrativity is completely determined by two graphs of T_1 and T_2 restriction on $[0, \alpha]^2$ and their neighboring regions [5]. It is in this sense that α-cross-migrativity and α-migrativivt are not dual, even the α-cross-migrativity are more complex. In addition, by comparison with [15] from the above analysis or the following results and proofs, this paper is very different.

The paper is organized as follows. In Section II, we recall some results concerning t-norms and t-subnorms. In Sections III, IV and V, we will give all the sufficient and necessary conditions that a t-subnorm M is α-cross-migrative with respect to T_M, T_P and T_L, respectively. Finally, Conclusion is in Section VI.

II. Triangular Norms and Subnorms

Definition 2.1 (see [10]) A binary operation $T\colon [0,1]^2 \to [0,1]$ is called a *t-norm* if it is associative, commutative, increasing and has the neutral element 1, namely, it holds $T(x,1) = T(1,x) = x$ for all $x \in [0,1]$.

Clearly, Definition 2.1 implies that for all $x, y \in [0,1]$, it holds

$$T(x,y) \leq \min\{x,y\}. \tag{5}$$

A t-norm is said to be *continuous* if it is continuous as a binary function. The three basic continuous t-norms the minimum T_M, the product T_P and the Lukasiewicz t-norm T_L are given by, respectively: $T_M(x,y) = \min\{x,y\}$; $T_P(x,y) = x \cdot y$; $T_L(x,y) = \max\{0, x+y-1\}$.

Definition 2.2 (see [12]) A binary operation $M\colon [0,1]^2 \to [0,1]$ is called a *t-subnorm* if it is associative, commutative, increasing and satisfies the inequation (5) mentioned above.

Observe that each t-norm is a t-subnorm but not vice versa. Thus a t-subnorm is said to be *proper* if it is not a t-norm, that is, there exists some $x \in [0,1]$ such that $M(x,1) < x$. Mesiarová [12] proved that a continuous t-subnorm M is proper if and only if $M(1,1) < 1$. One also use the additive generators to construct t-subnorms [12]. To be more specific, let $f\colon [0,1] \to [0,\infty]$ be a continuously non-increasing mapping, then the operation $M\colon [0,1]^2 \to [0,1]$ given by $M(x,y) = f^{(-1)}(f(x)+f(y))$ is a left-continuous t-subnorm, where $f^{(-1)}\colon [0,\infty] \to [0,1]$ is the pseudo-inverse of f, and is given by $f^{(-1)}(x) = f^{-1}(\min\{x, f(0)\})$. Moreover, Mesiarová [12] also proved that M is a continuously proper t-subnorm if and only if $f(1) > 0$ and $f|_{[0,f^{(-1)}(2f(1))]}$ is strictly decreasing. In addition, $f^{(-1)}$ is strictly decreasing on $[f(1), f(0)]$.

Lemma 2.3 *Let T and M be a t-norm and a subnorm respectively, $\alpha \in (0,1]$, and (M,T) be α-cross-migrative. Then the following statements hold.*

(i) *(M,M) is α-cross-migrative;*
(ii) *(M,T) is α-cross-migrative if and only if (T,M) is α-cross-migrative;*
(iii) *(M,T) is α-cross-migrative, then for all $(x,y) \in [0,1]^2$, we have*

$$M(T(x,\alpha), T(y,\alpha)) = T(T(x,y), M(\alpha,\alpha)). \tag{6}$$

Proof. Here it is enough to prove (iii) since (i) and (ii) obviously hold. If (M,T) is α-cross-migrative, repeating Definition I twice, then for any $(x,y) \in [0,1]^2$ and some $\alpha \in (0,1]$, it holds

$$\begin{aligned} M(T(x,\alpha), T(y,\alpha)) &= T(x, M(\alpha, T(y,\alpha))) \\ &= T(x, M(T(y,\alpha),\alpha)) \\ &= T(x, T(y, M(\alpha,\alpha))) \\ &= T(T(x,y), M(\alpha,\alpha)). \end{aligned}$$

Therefore (iii) is proved.

Remark 2.4 Note that any t-subnorm is 0-cross-migrative with respect to any t-norm. In another word, Eq. (4) always holds when $\alpha = 0$. Hence, only the case $\alpha \in (0,1]$ is considered in the following.

Remark 2.5 Obviously, the slight differences between t-subnorms and t-norms lie in their neutral elements. But these make t-subnorms more complicated. For example, any continuously Archimedean t-norm has a continuously additive generator but not for a continuously Archimedean t-subnorm. Indeed, some continuously Archimedean t-subnorms have no any continuously additive generator, even through some non-continuous t-subnorms have continuously additive generators. Hence, in the whole paper we only deal with a subclass of t-subnorms with continuously additive generators.

III. (α, T_M)-cross-migrative t-subnorms

Now, let us characterize all t-subnorms which are (α, T_M)-cross-migrative.

Theorem 3.1 *Let $\alpha \in (0,1]$, M be a t-subnorm with a continuously non-increasing additive generator $f\colon [0,1] \to [0,\infty]$. Then (M, T_M) is α-cross-migrative if and only if $f(\alpha) + f(1) \geq f(0)$.*

As a special case of continuously Archimedean t-subnorms, any continuously Archimedean t-norm has a continuous and strictly decreasing additive generator $t : [0,1] \to [0,\infty]$ with $t(1) = 0$. Hence, we have the following corollary.

Corollary 3.2 *Let $\alpha \in (0,1]$ and T be a continuously Archimedean t-norm, then (T, T_M) is not α-cross-migrative.*

Proof. Suppose that T is a continuously Archimedean t-norm with an additive generator t, by Theorem 3.1, then (T, T_M) is α-cross-migrative is equivalent with $t(\alpha) + t(1) \geq t(0)$. Note that $t(1) = 0$ and t is non-increasing, hence $t(\alpha) + t(1) \geq t(0)$ is also equivalent with $t(\alpha) = t(0)$, which contradicts with the fact that t is strictly decreasing.

Remark 3.3

(i) According to Theorem 3.8 in [5], we can also obtain Corollary III. This shows our results are compatible with those ones in [5].
(ii) For a t-subnorm M with a continuously additive generator $f\colon [0,1] \to [0,\infty]$, [15] has proven that (M, T_M) is α-migrative if and only if either $f(\alpha) + f(1) \geq f(0)$ or $f(\alpha) = f(1)$, while our results show that (M, T_M) is α-cross-migrative if and only if $f(\alpha) + f(1) \geq f(0)$. Even through the duality of viewpoint, there is no contradiction because $f(\alpha) + f(1) \geq f(0)$ includes $f(\alpha) = f(0)$.

Example 3.4 Let $f(x) = 5 - 3x,\ x \in [0,1]$, then the t-subnorm M generated by f is given by $M(x,y) = \max\{x + y - \frac{5}{3}, 0\}$ for all $x,\ y \in [0,1]$. Take $\alpha = \frac{1}{4}$, then it holds that both $f(0) = 5 < \infty$ and $f(\frac{1}{4}) + f(1) \geq f(0)$. Then for all $x \in [0, \frac{1}{4}],\ y \in [0,1]$ we have $M(x,y) = 0$, $M(T_M(x, \frac{1}{4}), y) \leq M(\frac{1}{4}, y) = 0$ and $T_M(x, M(\frac{1}{4}, y)) = T_M(x, 0) = 0$. Hence (M, T_M) is $\frac{1}{4}$-cross-migrative.

Example 3.5 Let $f(x) = -\ln(\max\{0, x - 0.2\}),\ x \in [0,1]$. Then we know $f(0) = \infty$, and then the t-subnorm M generated by f is given by $M(x,y) =$

$$\begin{cases} 0.2 + (x - 0.2)(y - 0.2) & (x,y) \in (0.2, 1]^2, \\ 0 & \text{otherwise.} \end{cases}$$

Take $\alpha = 0.2$, then we have that $f(0) = f(0.2) = \infty$, $f(0.2) + f(1) \geq f(0)$ and $M(x,y) = 0$ for all $x \in [0, 0.2], y \in [0,1]$. Thus it follows that $M(T_M(x, 0.2), y) \leq M(0.2, y) = 0$ and $T_M(x, M(0.2, y)) = T_M(x, 0) = 0$. So (M, T_M) is 0.2-cross-migrative.

IV. (α, T_P)-CROSS-MIGRATIVE T-SUBNORMS

In this section, we will characterize the sufficient and necessary conditions that (M, T_P) is α-cross-migrative.

Theorem 4.1 *Let $\alpha \in (0,1]$ and M be a t-subnorm with a continuously non-increasing additive generator $f\colon [0,1] \to [0, \infty]$, then (M, T_P) is α-cross-migrative if and only if one of the following two conditions holds.*

(i) $f(\alpha) + f(1) \geq f(0)$.
(ii) *there exist $\delta \in (-\infty, 0)$, $c \in (0, \infty)$ and a continuously non-increasing function $g\colon [\alpha, 1] \to [0, \delta \log c\alpha]$ with $g(\alpha) = \delta \log c\alpha$ such that*

$$f(x) = \begin{cases} \delta \log(cx) & \text{if } x \in [0, \alpha], \\ g(x) & \text{if } x \in [\alpha, 1]. \end{cases} \tag{7}$$

Corollary 4.2 *Let $\alpha \in (0,1)$ and T be a continuously Archimedean t-norm with a continuously non-increasing additive generator $t\colon [0,1] \to [0, \infty]$, then (T, T_P) is α-cross-migrative if and only if there exist $\delta \in (-\infty, 0)$, $c \in (0, \infty)$ and a continuously non-increasing function $g\colon [\alpha, 1] \to [0, \delta \log c\alpha]$ with $g(\alpha) = \delta \log c\alpha$ such that*

$$t(x) = \begin{cases} \delta \log(cx) & \text{if } x \in [0, \alpha], \\ g(x) & \text{if } x \in [\alpha, 1]. \end{cases} \tag{8}$$

Remark 4.3

(i) If a t-subnorm M has a continuously non-increasing additive generator f satisfying $f(\alpha) + f(1) \geq f(0)$, then M is α-cross-migrative with respect to any t-norm or any t-subnorm (See Example III and III).
(ii) Indeed, Corollary 4.2 can also be obtained from Theorem 4.2 in [5]. This indirectly and again shows that our results are compatible with those results given out by Fodor in [5].

Example 4.4 Let

$$f(x) = \begin{cases} -3 \log_2(0.5x) & x \in [0, 0.5], \\ 6.5 - x & x \in [0.5, 1]. \end{cases}$$

Then the t-subnorm M generated by f is given by

$$M(x,y) = \begin{cases} \frac{1}{2}xy & x \in [0, 0.5], y \in [0, 0.5], \\ 2^{\frac{1}{3}(y-6.5)}x & x \in [0, 0.5], y \in [0.5, 1], \\ 2^{\frac{1}{3}(x-6.5)}y & x \in [0.5, 1], y \in [0, 0.5], \\ 2^{-\frac{10}{3} + \frac{x+y}{3}} & x \in [0.5, 1], y \in [0.5, 1]. \end{cases}$$

Fix $\alpha \in [0, 0.5]$, then we easily check that (M, T_P) is α-cross-migrative. In fact, if $x \in [0, 0.5], y \in [0, 0.5]$, then $M(\alpha x, y) = \frac{1}{2}\alpha xy$ and $xM(\alpha, y) = x\frac{1}{2}\alpha y$. If $x \in [0, 0.5], y \in [0.5, 1]$, then $M(\alpha x, y) = 2^{\frac{1}{3}(y-6.5)}\alpha x$ and $xM(\alpha, y) = x2^{\frac{1}{3}(y-6.5)}\alpha$. If $x \in [0.5, 1], y \in [0, 0.5]$, then $M(\alpha x, y) = \frac{1}{2}\alpha xy$ and $xM(\alpha, y) = x\frac{1}{2}\alpha y$. If $x \in [0.5, 1], y \in [0.5, 1]$, then $M(\alpha x, y) = 2^{\frac{1}{3}(y-6.5)}\alpha x$ and $xM(\alpha, y) = x2^{\frac{1}{3}(y-6.5)}\alpha$.

V. (α, T_L)-CROSS-MIGRATIVE T-SUBNORMS

In this section, we will characterize the sufficient and necessary conditions that (M, T_L) is α-cross-migrative.

Theorem 5.1 *Let $\alpha \in (0,1]$ and M be a t-subnorm with a continuously non-increasing additive generator $f\colon [0,1] \to [0, \infty]$, then (M, T_L) is α-cross-migrative if and only if one of the following two conditions holds.*

(i) $f(\alpha) + f(1) \geq f(0)$.
(ii) *$f(0) = m < \infty$ and there exist a constant $d \in (0, \frac{m}{\alpha})$ and a continuously non-increasing function $g\colon [\alpha, 1] \to [0, m - d\alpha]$ with $g(\alpha) = m - d\alpha$ such that*

$$f(x) = \begin{cases} m - dx & x \in [0, \alpha], \\ g(x) & x \in [\alpha, 1]. \end{cases}$$

Corollary 5.2 Let $\alpha \in (0,1)$ and T be a continuously Archimedean t-norm with an additive generator $t\colon [0,1] \to [0, \infty]$. Then (T, T_L) is α-cross-migrative if and only if $t(0) < \infty$ and there exist a constant $d \in (0, \frac{t(0)}{\alpha})$ and a continuously non-increasing function $g : [\alpha, 1] \to [0, m - d\alpha]$ with $g(\alpha) = m - d\alpha$ such that

$$t(x) = \begin{cases} m - dx & x \in [0, \alpha], \\ g(x) & x \in [\alpha, 1]. \end{cases} \tag{9}$$

Example 5.3 Let $f(x) = \begin{cases} 4 - 6x & x \in [0, \frac{1}{2}], \\ \frac{3}{2} - x & x \in [\frac{1}{2}, 1], \end{cases}$ then the t-subnorm M generated by f is given by

$$M(x,y) = \begin{cases} \max\{x + y - \frac{2}{3}, 0\} & x \in [0, \frac{1}{2}], y \in [0, \frac{1}{2}], \\ \max\{x + \frac{y}{6} - \frac{1}{4}, 0\} & x \in [0, \frac{1}{2}], y \in [\frac{1}{2}, 1], \\ \max\{y + \frac{x}{6} - \frac{1}{4}, 0\} & x \in [\frac{1}{2}, 1], y \in [0, \frac{1}{2}], \\ \frac{x+y+1}{6} & x \in [\frac{1}{2}, 1], y \in [\frac{1}{2}, 1]. \end{cases}$$

Fix $\alpha = \frac{1}{2}$, then we easily check that (M, T_L) is α-cross-migrative. Indeed, if $x \in [0, \frac{1}{2}], y \in [0, \frac{1}{2}]$, then $M(\max\{x + \alpha - 1, 0\}, y) = 0$ and $\max\{x + M(\alpha, y) - 1, 0\} = 0$. If $x \in [0, \frac{1}{2}], y \in [\frac{1}{2}, 1]$, then $M(\max\{x + \alpha - 1, 0\}, y) = 0$ and $\max\{x + M(\alpha, y) - 1, 0\} = \max\{x + \frac{\alpha+y+1}{6} - 1, 0\} = 0$. If $x \in [\frac{1}{2}, 1], y \in [0, \frac{1}{2}]$, then $M(\max\{x + \alpha - 1, 0\}, y) = \max\{x + \alpha + y - \frac{5}{3}, 0\}$ and $\max\{x + M(\alpha, y) - 1, 0\} =$

$\max\{x+\max\{\alpha+y-\frac{2}{3},0\}-1,0\} = \max\{x+\alpha+y-\frac{5}{3},0\}$. If $x \in [\frac{1}{2},1], y \in [\frac{1}{2},1]$, then $M(\max\{x+\alpha-1,0\},y) = \max\{x+\alpha+\frac{y}{6}-\frac{5}{4},0\}$ and $\max\{x+M(\alpha,y)-1,0\} = \max\{x+\max\{\alpha+\frac{y}{6}-\frac{1}{4},0\}-1,0\} = \max\{x+\alpha+\frac{y}{6}-\frac{5}{4},0\}$.

VI. Conclusions

In this paper, we studied and characterized a t-subnorm M with a continuously additive generator is α-cross-migrative with respect to a t-norm T, where $T \in \{T_P, T_M, T_L\}$. If we can extend a t-subnorm M with a continuously additive generator into a commonly continuous t-subnorm, which is not obliged to have a continuously additive generator. Then, applying the fully similar methods in [5], we easily obtain that the sufficient and necessary conditions that such a pair of (M,T) is α-cross-migrative. This is a reason that any continuous t-subnorm is an ordinal sum of continuously Archimedean t-norms and at most one continuously Archimedean proper t-subnorm. But for a fixed and commonly continuous t-norm T, it is very difficult to characterize the t-subnorm M which is α-cross-migrative with respect to T. In fact, it has something to do with the conjecture proposed by Fodor et al., in [5]. In further work, we will investigate this case.

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