2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)
July 6-11, 2014, Beijing, China

A Revised Procedure to Estimate Missing Values in Incomplete Fuzzy Preference Relations

Yejun Xu[1,2], Feng Ma[1,2], Huimin Wang[1,2]
[1]State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering
[2]Business School
Hohai University
Nanjing, China
Email: xuyejohn@163.com, 514722370@qq.com, hmwang@hhu.edu.cn

***Abstract*—In this paper, we propose a four-way procedure to estimate missing preference values when dealing with acceptable incomplete fuzzy preference relations (IFPRs). The proposed revised procedure can estimate more missing elements in the first iteration and also has more advantages than the existing methods. An illustrative example and comparative analyses are offered to demonstrate the advantages of the proposed method.**

Keywords—Incomplete fuzzy preference relations; estimate missing values; revised procedure.

I. INTRODUCTION

Fuzzy preference relations (FPRs) are commonly used to represent decision makers (DMs)' preferences over a set of possible alternative solutions $X=\{x_1,x_2,\ldots,x_n\}$ ($n\geq 2$) [7, 10-12, 16]. In many cases, since each expert has his/her own experience concerning the problem being studied they could have some difficulties in giving all their preferences. This may be due to an expert not possessing a precise or sufficient level of knowledge of the problem, time pressure, or because that expert is unable to discriminate the degree to which some options are better than others. In such situations, experts are only able to provide IFPRs with some of their values missing or unknown. Over the past decades, IFPRs [3, 8, 9, 18-21, 23, 24]have received more attention. Xu [23] defined the concepts of IFPRs, additive consistent IFPRs and multiplicative consistent IFPRs, then proposed two goal programming models for obtaining the priority vector of an IFPR. Alonso, et al. [2] presented a procedure to find out the missing values of an IFPR using the known values based on additive consistency property. Herrera-Viedma, et al. [8] developed a feedback mechanism to generate advice on how experts should change or complete their preferences in order to reach a solution with high consensus and consistency degrees when dealing with IFPRs. Herrera-Viedma, et al. [9] proposed an iterative procedure based on additive consistency to estimate the missing information in an expert's IFPR. Herrera-Viedma, et al. [8] presented a consensus model for GDM problems with IFPRs. Alonso, et al. [3] presented a procedure to estimate missing preference values which can be applied to incomplete fuzzy, multiplicative, interval-valued and linguistic preference relations. Liu, et al. [13] developed the least square completion and inconsistency repair methods to deal with IFPRs. Xu, et al. [20] gave a reasonable definition of multiplicative consistent for IFPR and presented a logarithmic least squares method (LLSM) to priority for group decision making (GDM) with IFPRs. Xu and Wang [21] extended the eigenvector method (EM) to priority for an IFPRs.

However, in all the above researches, the missing elements in IFPRs after completed may be not always mutually complementary. So in this paper, we will present a revised estimation procedure to estimate missing information for IFPRs which is slightly different from the existing methods. And the estimated values will be complementary by this revised estimation procedure.

To do this, the rest of this paper is set out as follows. Section 2 presents some basic concepts necessary throughout the paper, that is, the definition of IFPRs and additive consistency property. In Section 3, we present an additive consistency based revised procedure to estimate the missing preference values in an IFPR. Section 4 illustrates an example, comparison results are shown to demonstrate its advantages. Section 5 concludes the paper.

II. PRELIMINARIES

Let $X=\{x_1,x_2,\ldots,x_n\}$ be a set of alternatives. In multiple attribute decision making problems, DMs need to rank the alternatives $x_1,x_2,\ldots,x_n$ from the best to the worst according to his or her preferences. A brief description of the FPR is given below.

Definition 1 [12, 14]: An FPR P on a set of alternatives X is a fuzzy set on the product set $X\times X$, i.e. it is characterized by a membership function $\mu_P : X\times X \to [0,1]$.

When cardinality of X is small, the FPR may be conveniently represented by the $n\times n$ matrix $P=(p_{ik})_{n\times n}$, being $p_{ik} = \mu_P(x_i,x_k)$ $(\forall i,k\in\{1,...,n\})$ interpreted as the preference degree or intensity of the alternative x_i over x_k: $p_{ik}=0.5$ indicates indifference between x_i and x_k ($x_i \sim x_k$), $p_{ik}=1$ indicates that x_i is absolutely preferred to x_k, and $0\leq p_{ik} < 0.5$ indicates that x_k is strictly preferred to x_i $(x_i \prec x_k)$.

In the models of solving GDM problems, we also assume that DMs are always able to provide all the preferences required. However, this situation is not always possible to achieve. And there will be missing information appeared. It may be due to experts' lack of knowledge about part of the problems, or simply because they may not be able to quantify the degree preference of one alternative over another. It must be clear then that when an expert e_h is not able to express the particular value p_{ik}, because he/she does not have a clear idea of how better alternative x_i is over alternative x_k, this does not mean that he/she prefers both options with the same intensity.

In order to model these situations, in the following we introduce the definitions of the IFPRs.

Definition 2 [9]: A function $f: X \to Y$ is partial when not every element in the set X necessarily maps onto an element in the set Y. When every element from the set X maps to one element in the set Y then we have a total function.

Definition 3 [9]: An IFPR P on a set of alternatives X is a fuzzy set on the product set $X \times X$ that is characterized by a partial membership function.

The necessary condition of acceptable IFPR P^h is that there exists at least one known element in each row or column of P^h except for the diagonal elements $(p_{ii}, i = 1, 2, ..., n)$, i.e., there needs at least $(n-1)$ judgments.

Now given an IFPR P^h, the following sets are defined:

$$A = \{(i,j) | i, j \in \{1,...,n\}\}$$

$$MV^h = \{(i,j) \in A | p_{ij}^h \text{ is unknown}\}$$

$$KV^h = A \setminus MV^h = \{(i,j) \in A | p_{ij}^h \text{ is known}\}$$

$$KV_i^h = \{(a,b) | (a,b) \in KV^h \wedge (a = i \vee b = i)\}$$

$$\exists i, h | KV_i^h \neq \phi$$

where MV^h is the set pairs of alternatives for which the preference degrees are unknown or missing, KV^h is the set pairs of alternatives for which preference degrees are given by the expert e_h, and KV_i^h is the set of pairs of alternatives involving alternative x_i for which expert e_h provides a preference value.

Consistency is usually characterized by transitivity. There are several possible characterizations for the transitivity property [6, 10]. In this paper, we adopt the additive transitivity property, which for FPRs can be seen as the parallel concept of Saaty's consistency property for multiplicative preference relations. The mathematical formulation of the additive transitivity was given by Tanino [16].

$$(p_{ij} - 0.5) + (p_{jk} - 0.5) = (p_{ik} - 0.5), \ \forall i, j, k \in \{1,...,n\} \quad (1)$$

or equivalently,

$$p_{ik} = p_{ij} + p_{jk} - 0.5, \ \ \forall i, j, k \in \{1,...,n\} \quad (2)$$

In this paper, additive consistency is the only considered property for FPR, and also $p_{ii} = 0.5$. Herrera-Viedma, et al. [9] assumed that p_{ii} is always equal to 0.5 and denoted as "-", p_{ii} is not considered as a known value in their estimation procedure. This difference is depicted in Example 1.

III. A Revised Estimation Procedure to Estimate Missing Values for IFPRS

Because experts are not always able to provide preference degrees between each pair of possible alternatives, missing information will appear. Therefore, it is necessary to estimate the missing values before the application of a selection model. In this section we use an iterative procedure to estimate the missing values in an IFPR, which is based on the additive consistency property.

A. Estimating Missing Values Based on The Additive Consistency

Equation (2) could be used to estimate missing values p_{ik}. However, three other possible ways to estimate missing values can be derived from (2) in fact, the preference value p_{ik} $(i \neq k)$ can be estimated using an intermediate alternative x_j in four different ways:

1) Since $p_{ik} = p_{ij} + p_{jk} - 0.5$, we can estimate p_{ik} by

$$(cp_{ik})^{j1} = p_{ij} + p_{jk} - 0.5 \quad (3)$$

2) Since $p_{jk} = p_{ji} + p_{ik} - 0.5$, we can estimate p_{ik} by

$$(cp_{ik})^{j2} = p_{jk} - p_{ji} + 0.5 \quad (4)$$

3) Since $p_{ij} = p_{ik} + p_{kj} - 0.5$, we can estimate p_{ik} by

$$(cp_{ik})^{j3} = p_{ij} - p_{kj} + 0.5 \quad (5)$$

4) Since $(p_{ik} + p_{kj} - 0.5) + p_{ji} - 0.5 = p_{ii} = 0.5$, we have

$$p_{ik} + p_{kj} = 1.5 - p_{ji} \quad (6)$$

Hence we can estimate p_{ik} by

$$(cp_{ik})^{j4} = 1.5 - p_{ji} - p_{kj} \quad (7)$$

The overall estimated value cp_{ik} of p_{ik} is obtained by all possible $(cp_{ik})^{j1}$, $(cp_{ik})^{j2}$, $(cp_{ik})^{j3}$ and $(cp_{ik})^{j4}$ values.

B. A Revised Procedure to Estimate Missing Values

Based on (3)-(5), Herrera-Viedma et al. [9] proposed a procedure to estimate missing preference values for IFPRs. In the following, we present a revised procedure to estimate missing for IFPRs based on (3)-(5) and (7), it is showed that the revised procedure has some advantages compared with Herrera-Viedma et al. [9]'s.

1. Estimate the missing values in each iteration of the procedure

Given an IFPR P^h, we define the sets H_{ik}^{h1}, H_{ik}^{h2}, H_{ik}^{h3} and H_{ik}^{h4}, respectively, which are used to estimate the missing preference value p_{ik}. Then the subset of missing values MV^h that can be estimated in step t of our procedure is denoted by EMV_t^h (estimated missing values) and defined as follows:

$$EMV_t^h = \{(i,k) \in MV^h \setminus \bigcup_{l=0}^{t-1} EMV_l \,|\, i \neq k \wedge \exists j \in \{(H_{ik}^{h1})^t \cup (H_{ik}^{h2})^t \cup (H_{ik}^{h3})^t \cup (H_{ik}^{h4})^t\}\} \quad (8)$$

With

$$(H_{ik}^{h1})^t = \left\{ j \,\middle|\, (i,j),(j,k) \in \left\{ \mathrm{KV} \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \quad (9)$$

$$(H_{ik}^{h2})^t = \left\{ j \,\middle|\, (j,i),(j,k) \in \left\{ \mathrm{KV} \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \quad (10)$$

$$(H_{ik}^{h3})^t = \left\{ j \,\middle|\, (i,j),(k,j) \in \left\{ \mathrm{KV} \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \quad (11)$$

$$(H_{ik}^{h4})^t = \left\{ j \,\middle|\, (j,i),(k,j) \in \left\{ \mathrm{KV} \bigcup_{l=0}^{h-1} EMV_l \right\} \right\} \quad (12)$$

and $EMV_0^h = \phi$ (by definition); $(H_{ik}^{h1})^t$, $(H_{ik}^{h2})^t$, $(H_{ik}^{h3})^t$, $(H_{ik}^{h4})^t$ are the sets of intermediate alternatives x_j that can be used to estimate the preference value p_{ik}^h $(i \neq k)$. When $EMV_{maxIter}^h = \phi$ with $maxIter > 0$, the procedure will stop as there will not be any more missing values to be estimated. Furthermore, if $\bigcup_{l=0}^{maxIter} EMV_l^h = MV^h$, then all missing values are estimated, and consequently, the procedure is said to be successful in the completion of the IFPR.

2. Estimate a particular value p_{ik}^h in the step t

In iteration t, to estimate a particular value p_{ik}^h with $(i,k) \in EMV_t^h$, the following function estimate $p(h,i,k,t)$ is established:

function estimate_p(h,i,k,t)

1. $\kappa = 0$

2. $cp_{ik}^{h1} = \begin{cases} \left(\sum_{j \in (H_{ik}^{h1})^t} (cp_{ik}^h)^{j1}\right) / \#(H_{ik}^{h1})^t, \kappa{+}{+} & \text{if } \#(H_{ik}^{h1})^t \neq 0 \\ 0 & \text{otherwise} \end{cases}$

3. $cp_{ik}^{h2} = \begin{cases} \left(\sum_{j \in (H_{ik}^{h2})^t} (cp_{ik}^h)^{j2}\right) / \#(H_{ik}^{h2})^t, \kappa{+}{+} & \text{if } \#(H_{ik}^{h2})^t \neq 0 \\ 0 & \text{otherwise} \end{cases}$

4. $cp_{ik}^{h3} = \begin{cases} \left(\sum_{j \in (H_{ik}^{h3})^t} (cp_{ik}^h)^{j3}\right) / \#(H_{ik}^{h3})^t, \kappa{+}{+} & \text{if } \#(H_{ik}^{h3})^t \neq 0 \\ 0 & \text{otherwise} \end{cases}$

5. $cp_{ik}^{h4} = \begin{cases} \left(\sum_{j \in (H_{ik}^{h4})^t} (cp_{ik}^h)^{j4}\right) / \#(H_{ik}^{h4})^t, \kappa{+}{+} & \text{if } \#(H_{ik}^{h4})^t \neq 0 \\ 0 & \text{otherwise} \end{cases}$

6. Calculate $cp_{ik} = \begin{cases} \frac{1}{\kappa}\left((cp_{ik}^{h1}) + (cp_{ik}^{h2}) + (cp_{ik}^{h3}) + (cp_{ik}^{h4})\right), \kappa \neq 0 \\ x, \kappa = 0 \end{cases}$

end function

When we use this function to compute the final estimated value of missing value p_{ik}^h, we should point out that some estimated values might lie outside the interval $[0,1]$, i.e., for some (i,k) we may have $cp_{ik}^h < 0$ or $cp_{ik}^h > 1$. In order to normalize the expression domains in the decision model, we set the following function:

$$f(y) = \begin{cases} 0 & \text{if } y < 0 \\ 1 & \text{if } y > 1 \\ y & \text{otherwise} \end{cases}$$

Then, the complete iterative estimation procedure pseudo code is as follows:

```
0. EMV_0^h = φ
1. t = 1
2. while(EMV_t^h ≠ φ){
3.    for every(i,k) ∈ EMV_t^h {
4.        estimate_p(h,i,k,t)
5.    }
6.    t++
7. }
```

In step t, we can estimate the missing preference value p_{ik} by expression

$$cp_{ik} = \Delta\left(\frac{1}{\kappa} \left(\begin{array}{l} \left(\sum_{j\in\left(H_{ik}^{h1}\right)} \left(cp_{ik}^{h}\right)^{j1}\right) / \#\left(H_{ik}^{h1}\right) + \left(\sum_{j\in\left(H_{ik}^{h2}\right)} \left(cp_{ik}^{h}\right)^{j2}\right) / \#\left(H_{ik}^{h2}\right) \\ + \left(\sum_{j\in\left(H_{ik}^{h3}\right)} \left(cp_{ik}^{h}\right)^{j3}\right) / \#\left(H_{ik}^{h3}\right) + \left(\sum_{j\in\left(H_{ik}^{h4}\right)} \left(cp_{ik}^{h}\right)^{j4}\right) / \#\left(H_{ik}^{h4}\right) \end{array} \right)\right) \tag{13}$$

Remark1. In [1, 8, 9], the authors only used (3)-(5) to estimate the missing values in an IFPR. Actually, in (3), if $p_{ij} = 1 - p_{ji}$, we have (4), if $p_{kj} = 1 - p_{jk}$, we have (5), if $p_{ij} = 1 - p_{ji}$ and $p_{kj} = 1 - p_{jk}$ simultaneously, we have (7). Therefore, (7) is also a way which can be used to estimate the missing values. Furthermore, we have the following result.

Theorem 1. If an IFPR can be completed by (3) -(5) and (7), then the completed missing elements verify $p_{ik} + p_{ki} = 1$.

Proof. If there exist j that satisfies (3), then it also satisfies (7), and we can get $(cp_{ik})^{j1} + (cp_{ki})^{j4} = 1$. Similarly, for each j which satisfies (4) and (5), there will be $(cp_{ik})^{j2} + (cp_{ki})^{j2} = 1$, $(cp_{ik})^{j3} + (cp_{ki})^{j3} = 1$, $(cp_{ik})^{j4} + (cp_{ki})^{j1} = 1$. $(p_{ik}) + (p_{ki})$ is the average value of all the estimated value using (3)-(5) and (7), thus, $p_{ik} + p_{ki} = 1$ holds.

Remark 2. Theorem 1 shows that if (7) is added to estimate the missing values, the reciprocity holds for the missing values, while Herrera-Viedma et al. [9]'s method could not, which also can be seen from their examples. In the following, we illustrate the following example and present an algorithm to show the property of the revised estimate procedure.

IV. Illustrative Example

Let $X = \{x_1, x_2, x_3, x_4\}$ be a set of four alternatives, we should find the optimal alternative from a DM's preference values. Suppose the following IFPR provided by an expert:

$$P = \begin{pmatrix} 0.5 & x & x & x \\ x & 0.5 & 0.4 & 0.6 \\ 0.6 & x & 0.5 & x \\ x & 0.3 & x & 0.5 \end{pmatrix}$$

Case 1: We use (3)-(5) to estimate the missing elements which is proposed by Herrera-Viedma et al. [9]'s where $i \neq j \neq k$. The estimation procedure is as follows:

Iteration 1. The set of elements that can be estimated is:

$$EMV_1\#_3 = \{(2,1),\ (3,4),(4,3)\}$$

- To estimate p_{21}, the procedure is as follows:

$$H_{21}^1 = \{3\} \ \Rightarrow cp_{21}^1 = cp_{21}^{31} = p_{23}^1 + p_{31}^1 - 0.5$$
$$= 0.4 + 0.6 - 0.5 = 0.5$$
$$H_{21}^2 = H_{21}^3 = \phi \Rightarrow cp_{21}^2 = cp_{21}^3 = 0$$
$$\kappa = 1 \Rightarrow cp_{21} = \frac{cp_{21}^1 + cp_{21}^2 + cp_{21}^3}{1} = \frac{0.5 + 0 + 0}{1} = 0.5$$

- To estimate p_{34}, the procedure is as follows:

$$H_{34}^2 = \{2\} \ \Rightarrow cp_{34}^2 = cp_{34}^{22} = p_{24}^2 - p_{23}^2 + 0.5$$
$$= 0.6 - 0.4 + 0.5 = 0.7$$
$$H_{34}^1 = H_{34}^3 = \phi \Rightarrow cp_{34}^1 = cp_{34}^3 = 0$$
$$\kappa = 1 \Rightarrow cp_{34} = \frac{cp_{34}^1 + cp_{34}^2 + cp_{34}^3}{1} = \frac{0 + 0.7 + 0}{1} = 0.7$$

- To estimate p_{43}, the procedure is as follows:

$$H_{43}^1 = \{2\} \ \Rightarrow cp_{43}^1 = cp_{43}^{21} = p_{42}^1 + p_{23}^1 - 0.5$$
$$= 0.3 + 0.4 - 0.5 = 0.2$$
$$H_{43}^2 = \{2\} \ \Rightarrow cp_{43}^2 = cp_{43}^{22} = p_{23}^2 - p_{24}^2 + 0.5$$
$$= 0.4 - 0.6 + 0.5 = 0.3$$
$$H_{43}^3 = \phi \Rightarrow cp_{43}^3 = 0$$
$$\kappa = 2 \Rightarrow cp_{43} = \frac{cp_{43}^1 + cp_{43}^2 + cp_{43}^3}{2} = \frac{02 + 0.3 + 0}{2} = 0.25$$

After these elements have been estimated, we have

$$P\#_3 = \begin{pmatrix} - & x & x & x \\ \mathbf{0.5} & - & 0.4 & 0.6 \\ 0.6 & x & - & \mathbf{0.7} \\ x & 0.3 & \mathbf{0.25} & - \end{pmatrix}$$

In order to show the differences of the estimate procedure proposed by Herrera-Viedma et al. [9]'s and this paper, we only give the first iteration estimation result.

Case 2: We use (3)-(5) and (7) to estimate the missing elements proposed in this paper where $i \neq j \neq k$. The estimation procedure is as follows:

Iteration 1. The set of elements that can be estimated is:

$$EMV_1\#_4 = \{(1,2),(2,1),(3,4),(4,3)\}$$

- To estimate p_{12}, the procedure is as follows:

$$H_{12}^1 = H_{12}^2 = H_{12}^3 = \phi \Rightarrow cp_{12}^1 = cp_{12}^2 = cp_{12}^3 = 0$$
$$H_{12}^4 = \{3\} \Rightarrow cp_{12}^4 = cp_{12}^{34} = 1.5 - p_{31} - p_{23}$$
$$= 1.5 - 0.6 - 0.4 = 0.5$$

- To estimate p_{21}, the procedure is as follows:

$$\kappa = 1 \Rightarrow cp_{12} = \frac{cp_{12}^1 + cp_{12}^2 + cp_{12}^3 + cp_{12}^4}{1}$$
$$= \frac{0 + 0 + 0 + 0.5}{1} = 0.5$$
$$H_{21}^1 = \{3\} \ \Rightarrow cp_{21}^1 = cp_{21}^{31} = p_{23}^1 + p_{31}^1 - 0.5$$
$$= 0.4 + 0.6 - 0.5 = 0.5$$
$$H_{21}^2 = H_{21}^3 = H_{21}^4 = \phi \Rightarrow cp_{21}^2 = cp_{21}^3 = cp_{21}^4 = 0$$

$$\kappa = 1 \Rightarrow cp_{21} = \frac{cp_{21}^1 + cp_{21}^2 + cp_{21}^3) + cp_{21}^4}{1} = \frac{0.5+0+0+0}{1} = 0.5$$

• To estimate p_{34}, the procedure is as follows:

$$H_{34}^2 = \{2\} \Rightarrow cp_{34}^2 = cp_{34}^{22} = p_{24}^2 - p_{23}^2 + 0.5 = 0.6 - 0.4 + 0.5 = 0.7$$

$$H_{34}^1 = H_{34}^3 = \phi \Rightarrow cp_{34}^1 = cp_{34}^3 = 0$$

$$H_{34}^4 = \{2\} \Rightarrow cp_{12}^4 = cp_{12}^{24} = 1.5 - p_{23} - p_{42} = 1.5 - 0.4 - 0.3 = 0.8$$

$$\kappa = 2 \Rightarrow cp_{34} = \frac{cp_{34}^1 + cp_{34}^2 + cp_{34}^3 + cp_{34}^4}{2} = \frac{0+0.7+0+0.8}{2} = 0.75$$

• To estimate p_{43}, the procedure is as follows:

$$H_{43}^1 = \{2\} \Rightarrow cp_{43}^1 = cp_{43}^{21} = p_{42}^1 + p_{23}^1 - 0.5 = 0.3 + 0.4 - 0.5 = 0.2$$

$$H_{43}^2 = \{2\} \Rightarrow cp_{43}^2 = cp_{43}^{22} = p_{23}^2 - p_{24}^2 + 0.5 = 0.4 - 0.6 + 0.5 = 0.3$$

$$H_{43}^3 = H_{43}^4 = \phi \Rightarrow cp_{43}^3 = cp_{43}^4 = (s_0, 0)$$

$$\kappa = 2 \Rightarrow cp_{43} = \frac{cp_{43}^1 + cp_{43}^2 + cp_{43}^3 + cp_{43}^4}{2} = \frac{0.2+0.3+0+0}{2} = 0.25$$

After these values have been estimated, we obtain

$$P\#_4 = \begin{pmatrix} - & \mathbf{0.5} & x & x \\ \mathbf{0.5} & - & 0.4 & 0.6 \\ 0.6 & x & - & \mathbf{0.75} \\ x & 0.3 & \mathbf{0.25} & - \end{pmatrix}$$

Compared with the above results, it is showed that the estimated value p_{34} in the first iteration by Herrera-Viedma et al.'s method and our method is different. Our estimated missing values could preserve the reciprocity property of the IFPR (i.e., $p_{34} + p_{43} = 1$), which meets Theorem 1. Furthermore, in the first iteration, we could estimate the value p_{12} by (7), while Herrera-Viedma et al.'s method could not. Herrera-Viedma et al.'s method would estimate the value p_{12} in the second iteration, which would be lack of accuracy, because the value p_{12} would be estimated by the estimated value in the first iteration, while our method could estimate p_{12} by the known value which is directly provided by the expert.

Case 3. In the above, both procedures require the condition $i \neq j \neq k$, actually, if any two of i, j, k could be equal (i.e., $i = j$, $i = k$ or $j = k$), and we use (3)-(5) and (7), the estimation procedure is as follows:

Iteration 1. The set of elements that can be estimated is:

$$EMV_1 \#_{4'} = \{(1,2),(1,3),(2,1),(3,2),(3,4),(4,3)\}$$

• To estimate the value p_{13}, the procedure is as follows:

$$H_{13}^1 = H_{13}^2 = H_{13}^3 = \phi \Rightarrow cp_{13}^1 = cp_{13}^2 = cp_{13}^3 = 0$$

$$H_{13}^4 = \{3\} \Rightarrow cp_{13}^4 = cp_{13}^{34} = 1.5 - p_{31} - p_{33} = 1.5 - 0.6 - 0.5 = 0.4$$

$$\kappa = 1 \Rightarrow cp_{13} = \frac{cp_{13}^1 + cp_{13}^2 + cp_{13}^3 + cp_{13}^4}{1} = \frac{0+0+0+0.4}{1} = 0.4$$

• To estimate the value p_{32}, the procedure is as follows:

$$H_{32}^1 = H_{32}^2 = H_{32}^3 = \phi \Rightarrow cp_{32}^1 = cp_{32}^2 = cp_{32}^3 = 0$$

$$H_{32}^4 = \{2\} \Rightarrow cp_{13}^4 = cp_{13}^{24} = 1.5 - p_{23} - p_{22} = 1.5 - 0.4 - 0.5 = 0.6$$

$$\kappa = 1 \Rightarrow cp_{32} = \frac{cp_{32}^1 + cp_{32}^2 + cp_{32}^3 + cp_{32}^4}{1} = \frac{0+0+0+0.6}{1} = 0.6$$

After these values are estimated, we obtain

$$P\#_{4'} = \begin{pmatrix} 0.5 & \mathbf{0.5} & \mathbf{0.4} & x \\ \mathbf{0.5} & 0.5 & 0.4 & 0.6 \\ 0.6 & \mathbf{0.6} & 0.5 & \mathbf{0.75} \\ x & 0.3 & \mathbf{0.25} & 0.5 \end{pmatrix}$$

Obviously, if any two of i, j, k could be equal, we could estimate more values in the first estimation.

As we know, if the missing values are estimated directly by the known values, they are more accurately than they are estimated in the second or later iterations, because in the second or later iterations the missing elements are estimated by the estimated values. The third procedure is our revised procedure, which allow any two of i, j, k could be equal in (3) -(5) and (7). And it can estimate the missing elements more quickly than the second procedure and its estimated missing values could preserve the reciprocity property of the FPR.

V. CONCLUSION

In this paper, We present a revised procedure to estimate missing values based on additive consistency to deal with IFPRs. The revised procedure is a four-way estimation method while Herrera-Viedma et al.'s is a three-way method. From the illustrative example, we know that the revised procedure can estimate more missing values in the first iteration in some cases, and thus need less iterations. It also can preserve the reciprocity property for the estimated missing values.

In the future, we will investigate this procedure to deal with consensus problems with IFPRs [8], incomplete linguistic preference relations [1, 4, 5, 17, 22], or dynamic GDM problems [15].

ACKNOWLEDGMENT

This work was partially supported by the National Natural Science Foundation of China (Nos.71101043), the major project of the national Social Science Fund of China (No.12&ZD214), Program for Excellent Talents in Hohai University.

REFERENCES

[1] S. Alonso, F. Cabrerizo, F. Chiclana, F. Herrera, and E. Herrera‐Viedma, "Group decision making with incomplete fuzzy linguistic preference relations," *Int. J. Intell. Syst.,* vol. 24, no. 2, pp. 201-222, 2009.

[2] S. Alonso, F. Chiclana, F. Herrera, and E. Herrera-Viedma, "A learning procedure to estimate missing values in fuzzy preference relations based on additive consistency," in *MDAI 2004, LNAI 3131*, V. Torra and Y. Narukawa, Eds., ed, 2004, pp. 227-238.

[3] S. Alonso, F. Chiclana, F. Herrera, E. Herrera‐Viedma, J. Alcalá‐Fdez, and C. Porcel, "A consistency‐based procedure to estimate missing pairwise preference values," *Int. J. Intell. Syst.,* vol. 23, no. 2, pp. 155-175, 2008.

[4] F. J. Cabrerizo, I. J. Perez, and E. Herrera-Viedma, "Managing the consensus in group decision making in an unbalanced fuzzy linguistic context with incomplete information," *Knowl-Based Syst,* vol. 23, no. 2, pp. 169-181, 2010.

[5] T. H. Chang, S. C. Hsu, T. C. Wang, and C. Y. Wu, "Measuring the success possibility of implementing ERP by utilizing the Incomplete Linguistic Preference Relations," *Appl. Soft. Comput,* vol. 12, no. 5, pp. 1582-1591, 2012.

[6] F. Chiclana, E. Herrera-Viedma, S. Alonso, and F. Herrera, "Cardinal consistency of reciprocal preference relations: A characterization of multiplicative transitivity," *IEEE T Fuzzy Syst,* vol. 17, no. 1, pp. 14-23, 2009.

[7] F. Chiclana, F. Herrera, and E. Herrera-Viedma, "Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations," *Fuzzy Set Syst,* vol. 97, no. 1, pp. 33-48, 1998.

[8] E. Herrera-Viedma, S. Alonso, F. Chiclana, and F. Herrera, "A consensus model for group decision making with incomplete fuzzy preference relations," *IEEE Trans. Fuzzy Syst.,* vol. 15, no. 5, pp. 863-877, 2007.

[9] E. Herrera-Viedma, F. Chiclana, F. Herrera, and S. Alonso, "Group decision-making model with incomplete fuzzy preference relations based on additive consistency," *IEEE Trans. Syst., Man, Cybern. B, Cybern.,* vol. 37, no. 1, pp. 176-189, 2007.

[10] E. Herrera-Viedma, F. Herrera, F. Chiclana, and M. Luque, "Some issues on consistency of fuzzy preference relations," *Eur. J. Oper. Res.,* vol. 154, no. 1, pp. 98-109, 2004.

[11] F. Herrera and E. Herrera-Viedma, "Choice functions and mechanisms for linguistic preference relations," *Eur. J. Oper. Res.,* vol. 120, no. 1, pp. 144-161, 2000.

[12] J. Kacprzyk, "Group decision making with a fuzzy linguistic majority," *Fuzzy Sets Syst.,* vol. 18, no. 2, pp. 105-118, 1986.

[13] X. W. Liu, Y. W. Pan, Y. J. Xu, and S. Yu, "Least square completion and inconsistency repair methods for additively consistent fuzzy preference relations," *Fuzzy Set Syst,* vol. 198, no. pp. 1-19, 2012.

[14] S. Orlovsky, "Decision-making with a fuzzy preference relation," *Fuzzy Sets Syst.,* vol. 1, no. 3, pp. 155-167, 1978.

[15] I. J. Pérez, F. J. Cabrerizo, and E. Herrera-Viedma, "A mobile decision support system for dynamic group decision-making problems," *IEEE Trans. Syst., Man, Cybern. A, Syst. Humans,* vol. 40, no. 6, pp. 1244-1256, 2010.

[16] T. Tanino, "Fuzzy preference orderings in group decision making," *Fuzzy Sets Syst.,* vol. 12, no. 2, pp. 117-131, 1984.

[17] T. C. Wang and Y. H. Chen, "Incomplete fuzzy linguistic preference relations under uncertain environments," *Inform Fusion,* vol. 11, no. 2, pp. 201-207, 2010.

[18] Y. J. Xu, Q. L. Da, and L. H. Liu, "Normalizing rank aggregation method for priority of a fuzzy preference relation and its effectiveness," *Int. J. Approx. Reason.,* vol. 50, no. 8, pp. 1287-1297, 2009.

[19] Y. J. Xu, Q. L. Da, and H. M. Wang, "A note on group decision-making procedure based on incomplete reciprocal relations," *Soft Comput.,* vol. 15, no. 7, pp. 1289-1300, 2011.

[20] Y. J. Xu, R. Patnayakuni, and H. M. Wang, "Logarithmic least squares method to priority for group decision making with incomplete fuzzy preference relations," *Appl. Math. Model.,* vol. 37, no. 4, pp. 2139-2152, 2013.

[21] Y. J. Xu and H. M. Wang, "Eigenvector method, consistency test and inconsistency repairing for an incomplete fuzzy preference relation," *Appl. Math. Model.,* vol. 37, no. 7, pp. 5171-5183, 2013.

[22] Z. S. Xu, "An approach to group decision making based on incomplete linguistic preference relations," *Int. J. Inform Tech & Decision Making,* vol. 4, no. 01, pp. 153-160, 2005.

[23] Z. S. Xu, "Goal programming models for obtaining the priority vector of incomplete fuzzy preference relation," *Int. J. Approx. Reason.,* vol. 36, no. 3, pp. 261-270, 2004.

[24] Z. S. Xu and J. Chen, "Group decision-making procedure based on incomplete reciprocal relations," *Soft Comput,* vol. 12, no. 6, pp. 515-521, 2008.