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Two Consensus Models based on the Minimum Cost and the Maximum Return

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***Abstract*—This paper proposes two kinds of minimum cost models regarding all the individuals and regarding with one particular individual respectively, shows the economic significance of these two models by exploring their dual models based on the primal-dual linear programming theories, and builds the conditions when these two models have the same optimal consensus opinion.**

I. Introduction

In group decision making (GDM), most decision makers(DMs) may eventually arrive at a certain degree of consensus associated with the most relevant alternatives after wide and full discussing, much negotiating. The consensus decision making [5], [9], [16], [18], [19], [21] is the base of group choice. In recent years, abundant achievements have been made in the fields of consensus measure and consensus modeling. (1) Consensus measure research. (2) Consensus modeling. Usually, the consensus process needs a moderator who represents collective interest to help to reach consensus, and he/she has determined and effective leadership and strong interpersonal communication and negotiation skills [1], [2], [3], [6], [7], [10], [11], [12], [13], [15], [14], [17], [21].

This paper discusses two kinds of consensus decision makings. The first is that when all individuals are taken into account as a whole, a primal problem of minimum cost and its dual problem of maximum return for reaching the greatest consensus regarding with all the individuals are developed. Secondly, when most individuals opinions do not exceed the tolerated error (or mathematically, in the neighborhood) of consensus opinion suggested by the moderator, they accept the consensus opinion but expect nothing about the return; while only a few moderator insists on their opinions unless the moderator pay more to them, this means that they accept the consensus opinion conditionally. For convenience, we suppose that there is only one individual needs to be paid. Hence, a primal problem of minimum cost and its dual problem of maximum return for reaching the greatest consensus regarding with one individual is also investigated.

This paper is structured as follows. Section 2 analysis the principle of the minimum cost consensus problem and the maximum return consensus problem. Section 3 constructs the primal-dual models based on the minimum cost and the maximum return regarding with all individuals, and discusses the economic significance of these models. Similarly, Section 4 establishes the primal-dual models based on the minimum cost and the maximum return only regarding with one individual, and investigates the economic significance of these models. Section 5 builds the conditions when these two kinds of primal-dual models have the same optimal consensus opinion. Lastly, conclusion is provided in Section 6.

II. PROBLEM DESCRIPTION

Suppose there are m decision makers (DMs) $D = \{d_1, \ldots, d_m\}$, that take part in a GDM. Let o_i, $o_i \in R$ represent the opinion of DM d_i, $i \in M = \{1, 2, \ldots, m\}$ in GDM. Without loss of generality, we always suppose that $o_1 \leq o_2 \leq \ldots \leq o_m$. In group decision making, the ideal state is where there exists an ideal opinion $\tilde{o}$ such that $o_1 = o_2 = \ldots = o_m = \tilde{o}$. When such an ideal opinion is derived, we get a full and unanimous agreement or a Utopian consensus. However, such an ideal opinion is so difficult to obtain that the moderator has to suggest a relatively satisfactory opinion o' to meet the most individuals' preferences. We call such an o' as an acceptable consensus opinion (or, simply, an consensus opinion). Firstly, we suppose that o' exists. In fact, it can be solved by a programming model constructed later: Let $f_i(o') = |o' - o_i|$ be the deviation between the opinion o_i of individual d_i, $i \in M$ and the consensus opinion o'. Obviously, the smaller $f_i(o')$ is, the closer the individual's opinion is to the consensus opinion. Let w_i denote a unit cost that paid by the moderator to persuade individual d_i, $i \in M$, to change his/her opinion. Then $w_i f_i(o')$ denotes the cost that paid by the moderator to persuade individual d_i, $i \in M$. The smaller this value is, the closer the distance between an individual's opinion and the consensus opinion, and the lower the cost to individual d_i, $i \in M$ is.

For all individuals, as they are required to present valuable opinions, they also have to dynamically adjust opinions to conform to the consensus opinion o'. So they expect to gain a unit return according to each unit opinion o_i or each unit change $o_i - o'$. Then $y_i o_i$ or $y_i(o_i - o')$ denotes the total return that d_i expects to obtain from changing his/her opinion. For each individual, the greater the value $o_i y_i$ or the value

$y_i(o_i - o')$ is, the higher the total return is expected by the individual. We prove later that although individual d_i expects as a high unit return y_i as possible, it is hard to arrive at consensus if the unit return is too high. Hence, we suppose that all individuals are rational, and they only need an appropriate value of unit return y_i that will also contribute to reaching consensus. We also prove that this unit return y_i is actually a shadow price in an economic sense.

From the viewpoint of the moderator, he/she hopes to achieve the greatest consensus while paying the minimum cost to all the individuals. And from the viewpoint of each individual, he/she anticipates to gain the maximum compensation for his/her changing opinions. Mathematically, these two goals are dual to each other. Next, we construct two consensus models of mathematical programming, and explore the relationship between these models.

III. PRIMAL PROBLEM OF MINIMUM COST AND ITS DUAL PROBLEM OF MAXIMUM RETURN FOR REACHING THE GREATEST CONSENSUS

In Section 2, if we add all the costs $w_i f_i(o')$ paid by the moderator to persuade the individuals d_i, $i \in M$, then we get a weighted arithmetic mean value $\sum_{i=1}^{m} w_i f_i(o')$. It denotes the total cost paid by the moderator to persuade all the individual DMs for arriving at the consensus. For the moderator, the smaller this value is, the closer the distance between an individual's opinion and the consensus opinion, and the lower total cost to all individuals.

If we add all the returns $y_i o_i$ or $y_i(o_i - o')$ expected by d_i for changing his/her opinion, then we also get a weighted arithmetic mean $\sum_{i=1}^{m} y_i o_i$ or $\sum_{i=1}^{m} y_i(o_i - o')$. It denotes the total return that all individuals are expecting for changing their opinions. For all these individuals, the greater the value $\sum_{i=1}^{m} o_i y_i$ or the value $\sum_{i=1}^{m} y_i(o_i - o')$ is, the higher the total return expected by all individuals.

$\sum_{i=1}^{m} w_i f_i(o')$ is regarded as the total cost (resource) paid by the moderator to obtain consensus. The smaller the value $\sum_{j=1}^{m} w_i f_i(o')$ is, the greater degree of consensus will be. Thus, we construct an nonlinear optimization model $P(w)$ under the premise that there is a consensus opinion such that the total cost to obtain the consensus is the minimum:

$$
\begin{aligned}
P(w): \qquad & min\ \phi = \sum_{j=1}^{m} w_j f_j(o') \\
& s.t. \left\{\ o' \in O \right.
\end{aligned}
\tag{1}
$$

If we let $u_i = [|o' - o_i| + (o' - o_i)]/2$, $v_i = [|o' - o_i| - (o' - o_i)]/2$, then the linear programming format $LP(w)$ of the nonlinear model $P(w)$ is as follows:

$$
\begin{aligned}
LP(w): \qquad & min\ \phi = \sum_{j=1}^{m} (w_j u_j + w_j v_j) \\
& s.t. \left\{ \begin{array}{l} o' - u_i + v_i = o_i,\ i \in M \\ o' \geq 0,\ u_i \geq 0,\ v_i \geq 0,\ i \in M \end{array} \right.
\end{aligned}
\tag{2}
$$

We call LP(w) model the weighted linear consensus problem.

It is easy to prove that the set of feasible solutions $\bar{X} = (o',\ u_1,\ v_1, \ldots,\ u_i,\ v_i, \ldots,\ u_m,\ v_m)^T$ to Model (2) is nonempty. It can also be shown that the number of basic feasible solution of Model (2) is finite and the optimal solution to Model (2) can be solved eaisly [8].

In Model (2), the objective function $\phi = \sum_{j=1}^{m} (w_j u_j + w_j v_j)$ can be considered to be the minimum total cost for obtaining the greatest consensus. Next, we further explore the specific meaning in economics by discussing the dual problem of Model (2). In the primal-dual theory of linear programming, the dual problem of Model (2) is as follows:

$$
\begin{aligned}
DLP(w): \qquad & max\ \psi = \sum_{i=1}^{m} o_i y_i \\
& s.t. \left\{ \begin{array}{l} \sum_{i=1}^{m} y_i \leq 0 \\ |y_i| \leq w_i,\ i \in M \end{array} \right.
\end{aligned}
\tag{3}
$$

Model (3) is also a dual problem of Model (2). It is easy to prove that the set of feasible solutions $\bar{Y} = (\ y_1,\ \ldots,\ y_m)^T$ to Models (3) is nonempty. The number of basic feasible solutions is finite, and $|y_i| \leq w_i,\ i \in M$ is bounded. We would like to further mention that if there exists an optimal solution $\bar{X}^* = (o^*,\ u_1^*,\ v_1^*, \ldots,\ u_i^*,\ v_i^*, \ldots,\ u_m^*,\ v_m^*)^T$, to Model (2), then obviously $o^* > 0$.

We call Model (2) the primary problem ($LP(w)$), and Model (3) the dual problem ($DLP(w)$). The optimal objective function $\psi = \sum_{j=1}^{m} o_i y_i$ is considered to be the total return that is expected by all the individuals for changing their opinions toward the consensus. It is obvious that what all the individuals want is for the return to be as large as possible. The value $max\ \psi$ denotes the maximum return of all the individuals. Because the "return" is actually a shadow price, its real meaning is referred to as the "expected return," but not the true return.

IV. RELATION BETWEEN THE PRIMAL PROBLEM $LP(w)$ AND ITS DUAL PROBLEM $DLP(w)$

The following theorems are fundamental theorems of linear programming. We omit the relevant proofs.

Theorem 1 (Weak Duality) [8] Let X be a primal feasible solution of the primal problem $LP(w)$, and $\phi(X)$ the corresponding value of the primal function that is to be minimized. Let Y be a dual feasible solution of the dual problem $DLP(w)$, and let $\psi(Y)$ the corresponding value of the dual function that is to be maximized. Then the objective value $\phi(X)$ for a feasible solution to the primal problem $LP(w)$ will always be less than or equal to the objective value $\psi(Y)$ for a feasible solution to the dual problem $DLP(w)$. That is, $\phi(X) \geq \psi(Y)$.

Theorem 2 (Optimality Criterion)[8] If both the primal problem $LP(w)$ and the dual problem $DLP(w)$ have optimal feasible solutions, then the two optimal objective values are equal. That is, $Max\ \psi = Min\ \phi$.

Theorem 3 (Sufficient Optimality Criterion)[8] If $\bar{X}^*$ and $\bar{Y}^*$ are feasible solutions to the primal problem $LP(w)$ and its dual problem $DLP(w)$, respectively, and if the primal objective function $\sum_{j=1}^{m} w_j(u_j + v_j)$ and the dual objective function $\sum_{i=1}^{m} o_i y_i^*$ satisfy $\sum_{j=1}^{m} w_j(u_j^* + v_j^*) = \sum_{i=1}^{m} o_i y_i^*$, then $\bar{X}^*$ and $\bar{Y}^*$ are the optimal solutions to $LP(w)$ and $DLP(w)$, respectively.

Theorem 4 (Strong Duality)[8] If either the primal problem $LP(w)$ or the dual problem $DLP(w)$ have an optimal feasible solution, then so does the other problem and the two optimal objective values are equal. That is, $Max\ \psi = Min\ \phi$.

V. A CONSENSUS MODEL BASED ON THE MINIMUM COST OF $k-th$ INDIVIDUAL

In this section, we discuss a consensus model based on the minimum cost of the $k-th$ individual DM.

A. The ε Restriction Problem $P_k(\varepsilon)$ Based on the Minimum Cost of $k-th$ Individual and Its Dual Problem $DP_k(\varepsilon)$

For the sake of convenience of communication, all the individual DMs' opinions are acceptable except for the individual DM d_k. From the viewpoint of the moderator, on the one hand, he/she hopes that the smaller the value of $f_k(o') = |o_k - o'|$ is, the closer the deviation between o' and o_k, and the lower the total compensation $r_k f_k(o')$ that needs to paid according to the value of the unit cost r_k and the value of deviation $f_k(o')$. On the other hand, the moderator does not need to pay any compensations if the moderator's opinion o' is allowed in the deviation limits ε_j of the most individual DMs' opinions. I.e., the restriction conditions $f_j(o') \leq \varepsilon_j$ hold for all $j \in M, j \neq k$.

Thus, a nonlinear programming model $P_k(\varepsilon)$ is constructed as follows, where its objective function satisfies that the total compensation $r_k f_k(o')$ is as small as possible, and the restriction condition $f_j(o'),\ j \in M,\ i \neq k$ is limited within the allowed deviation ε_j.

$$
P_k(\varepsilon): \qquad \begin{array}{l} min\ Z = r_k f_k(o') \\ s.t. \begin{cases} o' \in O \\ f_j(o') \leq \varepsilon_j,\ j \in M,\ j \neq k \end{cases} \end{array} \tag{4}
$$

Let $O(\varepsilon)$ be the feasible field of $P_k(\varepsilon)$. By letting $u_k = [|o'-o_k|+(o'-o_k)]/2$, $v_k = [|o'-o_k|-(o'-o_k)]/2$, $P_k(\varepsilon)$ is equivalent to the following linear programming model $LP_k(\varepsilon)$:

$$
LP_k(\varepsilon): \qquad \begin{array}{l} min\ Z = r_k u_k + r_k v_k \\ s.t. \begin{cases} o' \in O \\ o' \leq o_j + \varepsilon_j,\ j \in M,\ j \neq k \\ o' \geq o_j - \varepsilon_j,\ j \in M,\ j \neq k \\ o' - u_k + v_k = o_k \\ o' \geq 0,\ u_k \geq 0,\ v_k \geq 0 \end{cases} \end{array} \tag{5}
$$

Let's reconsider the feasible field of model (5). For all $j \in M,\ j \neq k$, there must exist $s,\ t \in M, s \neq k,\ t \neq k$, such that the set

$$\{o' | o' \leq o_j + \varepsilon_j,\ o' \geq o_j - \varepsilon_j,\ j \in M,\ j \neq k\}$$

is equal to the set

$$\{o' | o' \leq o_t + \varepsilon_t,\ o' \geq o_s - \varepsilon_s, s,\ t \in M,\ s \neq k,\ t \neq k\},$$

where $o_t + \varepsilon_t = min\{o_j + \varepsilon_j | j \in M,\ j \neq k\}$, $o_s - \varepsilon_s = max\{o_j - \varepsilon_j | j \in M,\ j \neq k\}$.

Therefore, model (5) can be further simplified into

$$
LP_k(\varepsilon): \qquad \begin{array}{l} min\ Z = r_k u_k + r_k v_k \\ s.t. \begin{cases} o' \in O \\ o' \leq o_t + \varepsilon_t,\ t \in M, t \neq k \\ o' \geq o_s - \varepsilon_s,\ s \in M,\ s \neq k \\ o' - u_k + v_k = o_k \\ o' \geq 0,\ u_k \geq 0,\ v_k \geq 0 \end{cases} \end{array} \tag{6}
$$

According to the theory of linear programming, the dual problem of Model (6) is as follows:

$$
\begin{array}{l} DLP_k(\varepsilon): \\ max\ \ S = (o_s - \varepsilon_s)x_s - (o_t + \varepsilon_t)x_t + o_k x_k \\ s.t. \begin{cases} x_s - x_t + x_k \leq 0 \\ -x_k \leq r_k \\ x_k \leq r_k \\ x_s, x_t, x_k \geq 0 \end{cases} \end{array} \tag{7}
$$

According to the principle of complementary slackness of the primal-dual linear programming, Model (7) is equivalent to

$$
DLP_k(\varepsilon): \qquad \begin{array}{l} max\ \ S = (o_k - o')x_k \\ s.t. \begin{cases} x_s - x_t + x_k = 0 \\ -x_k \leq r_k \\ x_k \leq r_k \\ x_s, x_t, x_k \geq 0 \end{cases} \end{array} \tag{8}
$$

B. The relation between the primal problem $LP_k(\varepsilon)$and its Dual problem $DLP_k(\varepsilon)$.

Theorem 5 (Weak Duality) [8] Let X be a primal feasible solution of the primal problem $LP_k(\varepsilon)$, and let $Z(X)$ be the corresponding value of the primal function that is to be minimized. Let Y be a dual feasible solution of the dual problem $DLP_k(\varepsilon)$, and let $S(Y)$ be the corresponding value of the dual function that is to be maximized. Then the objective value $Z(X)$ for a feasible solution to the primal problem $LP_k(\varepsilon)$ will always be less than or equal to the objective value $S(Y)$ for a feasible solution to the dual problem $DLP_k(\varepsilon)$. That is, $S(Y) \leq Z(X)$.

Theorem 6 (Optimality Criterion)[8] If both the primal problem $LP_k(\varepsilon)$ and the dual problem $DLP_k(\varepsilon)$ have optimal feasible solutions, then the two optimal objective values are equal. That is, $Max\ S = Min\ Z$.

Theorem 7 (Sufficient Optimality Criterion)[8] If $\hat{X}^* = (o^*, u_k^*, v_k^*)^T$ and $\hat{Y}^* = (X_s^*, x_t^*, x_k^*)^T$ are feasible solutions to the primal problem $LP_k(\varepsilon)$ and its dual problem $DLP_k(\varepsilon)$, respectively, and if the primal objective function$r_k u_k + r_k v_k$ and the dual objective function $(o_k - o')r_k$ satisfy $r_k u_k + r_k v_k$ and the dual objective function $((o_k - o')r_k)$ satisfy $r_k u_k^* + r_k v_k^* = (o_k - o^*)r_k^*$, then $\hat{X}^*$ and $\hat{Y}^*$ are the optimal solutions to $LP_k(\varepsilon)$ and $DLP_k(\varepsilon)$, respectively.

Theorem 8 (Strong Duality)[8] If either the primal problem $LP_k(\varepsilon)$ or the dual problem $DLP_k(\varepsilon)$ have an optimal feasible solution, then other problem also does and the two optimal objective values are equal. That is, $Max\ S = Min\ Z$.

VI. THE LINK BETWEEN $P(w)$ AND $P_k(\varepsilon)$

Both $P(w)$ model and $P_k(\varepsilon)$ model may solve different optimal consensus opinions. Next, we explore the conditions that the optimal consensus opinions solved by these two model are the same.

In $P_k(\varepsilon)$ Model (7), for a given point o^*, we use the symbol $P_k(\varepsilon^*)$ to represent the problem $P_k(\varepsilon)$, where $\varepsilon_j = \varepsilon_j^* = f_j(o^*), j \in M, j \neq k$.

Theorem 9 [4] For any given k, o^* is the optimal consensus opinion solved by $P_k(\varepsilon^*)$: then there exist $w^* \in W; w^* \geq 0$, such that o^* is also the optimal consensus opinion solved by $P(w^*)$.

Theorem 10 If there exists $w \in W$ such that o^* is the optimization solution to $P(w)$, then either
(1) if $w_k > 0$, then o^* is also the optimization solution to $P_k(\varepsilon^*)$; or
(2)if o^* is the unique optimization solution to $P(w)$, then o^* is also the optimization solution to $P_k(\varepsilon^*)$ for all $k, k \in M$.

Theorem 11 Assume that there exists the optimal solution o^* to $LP(w)$, and the restriction conditions of $LP_k(\varepsilon)$ attain the upper limitations

$$\varepsilon_j = |o^* - o_j|, j \in M, j \neq k$$

Then for all $j \in M, o^*$ must be the unique optimal solution to $LP_k(\varepsilon)$.

Obviously, the condition of Theorem 11 is more relax than that of Theorem 10.

VII. CONCLUSIONS AND FUTURE RESEARCH

A kind of consensus model regarding with all individuals and a kind of consensus model regarding with only one individual have been investigated in this paper: A minimum cost primal model and its dual model - a maximum return model for reaching greatest consensus have been developed from the standpoint of all individuals. A ε restriction problem $P_k(\varepsilon)$ based on the minimum cost with a particular individual and its dual problem also have been explored from the standpoint of a particular individual. The close interrelation between theses two kind of consensus model is built, showing that when some conditions met, the optimal consensus opinion derived by theses two model are identical.

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