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Consistency based estimation of fuzzy linguistic preferences. The case of reciprocal intuitionistic fuzzy preference relations

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***Abstract*—The decision-making assumption of all experts being able to express their preferences on all available alternatives of a decision-making problem might be considered unrealistic. This is specially true when the number of alternatives is considerable high and/or when sources of information are conflicting and dynamic. Thus, the presence of incomplete information, which is not equivalent to low quality information, is worth investigation and its processing within decision-making processes desirable. A consistency based approach to deal with incomplete fuzzy linguistic preferences is the focus of this contribution. Consistency is considered here as linked to the transitivity of preferences, and in particular to Tanino's multiplicative transitivity property of reciprocal fuzzy preference relations. The first result presented is the formal modelling and representation of Tanino's multiplicative transitivity property to the case of fuzzy linguistic preference relations. This is done via Zadeh's extension principle and the representation theorem of fuzzy sets. The second result derives the multiplicative transitivity property of reciprocal intuitionistic fuzzy preference relations, which can be isomorphically mapped to a particular type of linguistic preference relation: reciprocal interval-valued fuzzy preference relations. The third result is the computation of the consistency based estimated reciprocal intuitionistic fuzzy preference values using an indirect chain of alternatives, which can be used to address incomplete information in decision-making problems with this type of preference relations.**

I. Introduction

THE assumption in the majority of decision-making models by which experts express preferences on all aspect of the problem is not realistic and when enforced could lead to the provision by experts of preferences values that might not reflect accurately their knowledge of the problem. Indeed, decision making situations involving a large number of alternatives to choose from and/or conflicting and dynamic sources of information can be too complex for the above assumption to be valid. An empirical study conducted by Deparis et al. [1] corroborates the following hypothesis: "increasing the intensity of conflict in a multicriteria comparison increases the likelihood that DMs consider two alternatives as incomparable." Thus, decision-making problem with conflicting criteria might lead to the presence of incomplete preferences. Their results indicate that, in the scenario of conflicting criteria, if incomparability statements are not allowed, an increase of indifference statements happens. However, it is obvious that incomparability is not equivalent to indifference and consequently it becomes necessary to develop decision models to address the presence of incomplete information, i.e. information with missing data.

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Some methodologies widely adopted in situations with incomplete information discard or rate more negatively those experts that provide preferences with missing values [2], [3]. These methodologies are based on the assumption that a good solution to a decision making problem cannot be achieved from incomplete information, or that the solution would not be as good as the one that would derive using complete information. However, empirical evidence suggests that the incomplete preference relation derived from the random deletion of as much as 50 % of the elements of a complete pairwise preference relation provides good results without compromising accuracy [4]. Therefore, these two groups of methods might end eliminating or undervaluing useful information in the data provided, which could introduce bias and lead to inaccurate results [5]. Indeed, incomplete information is not equivalent to low quality information, and consequently imposing penalties in the decision making processes to experts providing incomplete information could lead to misleading solution, specially when the incomplete information is consistent and the complete information is not. Thus, alternative approaches to manage incomplete information in decision making are desirable. One of these approaches is based on the selection of an appropriate methodology to 'build' the preference relation, and/or to assign importance values to experts based not on the amount of information provided but on how consistent the information provided is [6], [7], [8], [9], [10], [11].

A comparison study between different alternative preference elicitation methods is reported in [12], where it was concluded that pairwise comparison methods are more accurate than non-pairwise methods. The main advantage of pairwise comparison methods is that facilitates individuals expressing their preferences because they focus exclusively on two alternatives at a time.

In classical preference modelling the set of numerical values $\{1, 0.5, 0\}$, or its equivalent $\{1, 0, -1\}$ [13], is used to repre-

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sent when the first alternative is preferred to the second alternative, when both alternatives are considered equally preferred (indifference), and when the second alternative is preferred to the first one, respectively. This classical preference modelling constitutes the simplest numeric discrimination model of preferences, and it proves insufficient in many decision making situations where the implementation of some kind of 'intensity of preference' between alternatives might be necessary [13], [14].

The concept of fuzzy set, which extends the classical concept of set, when applied to a classical relation leads to the concept of a fuzzy relation, which in turn allows the implementation of intensity of preferences [15]. In [16], we can find for the first time the fuzzy interpretation of intensity of preferences via the concept of a reciprocal fuzzy preference relation, which was later reinterpreted by Nurmi in [17]. In this approach, the numeric scale to evaluate intensity of preferences is the whole unit interval $[0,\ 1]$ instead of $\{1, 0.5, 0\}$, which it is argued though to assume unlimited computational abilities and resources from the individuals [14]. However, subjectivity, imprecision and vagueness in the articulation of opinions pervade real world decision applications, and individuals usually find difficult to evaluate their preference using exact numbers. Individuals might feel more comfortable using words by means of linguistic labels or terms to articulate their preferences [18], [19].

The aim of this contribution is to formally model and represent the concept of consistency of linguistic preferences represented as fuzzy subsets of the unit interval [20]. For reciprocal fuzzy preference relations, consistency is based on the concept of transitivity, which is modelled in many different ways (see for example [21]). Tanino [22] proposed the multiplicative transitivity property of reciprocal fuzzy preference relations, which was proved to be the most appropriate one for modelling cardinal consistency of such type of preference relations [14]. The first objective is thus to extend Tanino's multiplicative transitivity property to the case of fuzzy linguistic preference relations. This is done in Section II via Zadeh's extension principle and the representation theorem of fuzzy sets [18]. In Section III the multiplicative transitivity property is derived for reciprocal intuitionistic fuzzy preference relations [23], [24], which can be isomorphically mapped to a particular type of linguistic preference relation: reciprocal interval-valued fuzzy preference relations [25]. The computation of the consistency based estimated values of reciprocal intuitionistic fuzzy preferences using an indirect chain of alternatives, which can be used to address incomplete information in decision-making problems with this type of preference relations is provided in Section IV. Conclusions are drawn in V.

II. Consistency of Fuzzy Linguistic Preference Relations

In decision-making, experts usually need to compare a finite set of alternatives $X = \{x_1, x_2, \ldots, x_n\}$ with respect to a single criterion, and construct preference relations. The preferences of an expert can be represented using a reciprocal matrix, $R = (r_{ij})$, whith element r_{ij} interpreted as the degree or intensity of preference of alternative x_i over x_j (se Section II-B). The elements of R can be numerical values or linguistic labels.

A. Fuzzy Linguistic Preference Relation

There are two main types of numeric preference relations: crisp preference relations [26] and fuzzy preference relations or [0,1]–valued preference relations [16], [17]; with the second one being an extension of the first one, i.e. [0,1]–valued preference relations have crisp relations as a particular case. However, as mentioned before, subjectivity, imprecision and vagueness in the articulation of opinions pervade real world decision applications [27], and individuals might feel more comfortable using words by means of linguistic labels or terms to articulate their preferences [18]. In these cases it is still valid the following quotation by Zadeh [28]: 'Since words, in general, are less precise than numbers, the concept of a linguistic variable serves the purpose of providing a means of approximate characterisation of phenomena which are too complex or too ill-defined to be amenable to description in conventional quantitative terms.'

Let $\mathcal{L} = \{l_0, \ldots, l_s\}$ be a set of linguistic labels ($s \geq 2$), with semantic underlying a ranking relation that can be precisely captured with a linear order, i.e., $l_0 < l_1 < \cdots < l_s$. Assuming that the number of labels is odd and the central label $l_{s/2}$ stands for the indifference state when comparing two alternatives, the remaining labels are usually located symmetrically around that central assessment, which guarantees that a kind of reciprocity property holds as in the case of numerical preferences previously mentioned. Thus, if the linguistic assessment associated to the pair of alternatives (x_i, x_j) is $r_{ij} = l_h \in \mathcal{L}$, then the linguistic assessment corresponding to the pair of alternatives (x_j, x_i) would be $r_{ji} = l_{s-h}$. Therefore, the operator defined as $N(l_h) = l_g$ with $(g+h) = s$ is a negator operator because $N(N(l_h)) = N(l_g) = l_h$ [20].

Convex normal fuzzy subsets of the real line, also known as fuzzy numbers, are commonly used to represent linguistic terms [20], [29]. By doing this, each linguistic assessment is represented using a fuzzy number that is characterised by a membership function, with base variable the unit interval $[0, 1]$, describing its semantic meaning. The membership function maps each value in $[0, 1]$ to a degree of performance representing its compatibility with the linguistic assessment [18].

B. Multiplicative Transitivity Property of a Fuzzy Linguistic Preference Relation

Rationality is related with *consistency*, which is associated with the *transitivity property* [21], [30]. Many properties have been suggested to model transitivity of reciprocal fuzzy preference relations (RFPRs)

$$R = (r_{ij}): \quad r_{ij} \in [0, 1] \quad \wedge \quad r_{ij} + r_{ji} = 1 \ \ \forall i, j. \tag{1}$$

Some of these properties have been proved to be inappropriate in [14]. The assumption of experts being able to quantify their preferences in the domain [0,1], instead of

$\{0,1\}$, underlies unlimited computational abilities and resources from the experts, which was used by Chiclana et al. [14] to propose the modelling of the cardinal consistency of reciprocal fuzzy preference relations via a functional equation, and to subsequently prove that when such a function is almost continuous and monotonic (increasing) then it must be a representable uninorm. Cardinal consistency with the conjunctive representable Cross Ratio uninorm is equivalent to Tanino's multiplicative transitivity property (MTP). Because any two representable uninorms are order isomorphic, then multiplicative transitivity is indeed characterised as the most appropriate property to model consistency of reciprocal preference relations. Notice that multiplicative transitivity property extends weak stochastic transitivity, and therefore extends the classical transitivity property of crisp preference relations. The following definition summarises this result:

Definition 1 (MTP of a RFPR): A reciprocal fuzzy preference relation $R = (r_{ij})$ on a finite set of alternatives X is multiplicative transitive if and only if

$$\forall i,k,j \in \{1,2,\dots n\}: \quad r_{ij} \cdot r_{jk} \cdot r_{ki} = r_{ik} \cdot r_{kj} \cdot r_{ji}. \quad (2)$$

Tanino proposed the above transitivity property when $r_{ij} > 0 \ \ \forall i,j$ [22], and it can be expressed as

$$r_{ij} = \frac{r_{ik} \cdot r_{kj}}{r_{ik} \cdot r_{kj} + (1 - r_{ik}) \cdot (1 - r_{kj})}. \quad (3)$$

In what follows, we will formally generalise the multiplicative transitivity property of a reciprocal fuzzy preference relation to the case of a fuzzy linguistic preference relation (FLPR). We will do this by applying Zadeh's *Extension Principle* [18] to the case when the preference values are fuzzy sets rather than crisp values in $[0,1]$. The *Representation Theorem* [18] will be applied to the corresponding fuzzy sets that are obtained after applying the extension principle, so that we can obtain the corresponding multiplicative transitivity property of fuzzy linguistic preference values.

The extension principle allows the domain of a functional mapping to be extended from crisp elements to fuzzy sets as given below [31]:

Definition 2 (Extension Principle): Let $X_1 \times X_2 \times \ldots \times X_n$ be a universal product set and F a functional mapping of the form

$$F\colon X_1 \times X_2 \times \ldots \times X_n \longrightarrow Y$$

that maps the element $(x_1, x_2, \ldots, x_n) \in X_1 \times X_2 \times \ldots \times X_n$ to the element $y = F(x_1, x_2, \ldots, x_n)$ of the universal set Y. Let A_i be a fuzzy set over the universal set X_i with membership function μ_{A_i} $(i = 1, 2, \ldots, n)$. The membership function μ_B of the fuzzy set $B = F(A_1, ., A_n)$ over the universal set Y is:

- If $\exists\ x_1, \ldots, x_n$ such that $y = F(x_1, \ldots, x_n)$:

$$\mu_B(y) = \sup_{y=F(x_1,x_2,\ldots,x_n)} \Big[\mu_{A_1}(x_1) * \mu_{A_2}(x_2) * \ldots * \mu_{A_n}(x_n)\Big]$$

- Otherwise: $\mu_B(y) = 0$, where $*$ is a t-norm.

In what follows, the minimum t-norm ($\wedge$) is used. Expression (2) involves the comparison of two products of three crisp numbers (preference values) in the unit interval $[0,1]$. In what follows we will extend the function $f\colon [0,1] \times [0,1] \times [0,1] \longrightarrow [0,1]$,

$$f(x_1, x_2, x_3) = x_1 \cdot x_2 \cdot x_3, \quad (4)$$

to $f(A_1, A_2, A_3)$ where A_1, A_2, A_3 are fuzzy sets over the set $[0,1]$ and associated membership functions μ_{A_1}, μ_{A_2}, μ_{A_3}, respectively.

The *extension principle* states that

$$B = f(A_1, A_2, A_3) \quad (5)$$

is a fuzzy set over the set $[0,1]$ with membership function $\mu_B\colon [0,1] \to [0,1]$;

$$\mu_B(y) = \sup_{\substack{x_1 \cdot x_2 \cdot x_3 = y \\ x_1, x_2, x_3 \in [0,1]}} \left[\mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2) \wedge \mu_{A_3}(x_3)\right].$$

The *representation theorem of fuzzy sets* provides an alternative and convenient way to define a fuzzy set via its corresponding family of crisp α-level sets.

The α-level set of a fuzzy set A over the universe Z is defined as

$$A^\alpha = \{z \in Z | \mu_A(z) \geq \alpha\}. \quad (6)$$

The set of crisp sets $\{A^\alpha | 0 < \alpha \leq 1\}$ is said to be a representation of the fuzzy set A. Indeed, the fuzzy set A can be represented as

$$A = \bigcup_{0<\alpha\leq 1} \alpha A^\alpha \quad (7)$$

with membership function

$$\mu_A(z) = \sup_{\alpha\colon z \in A_\alpha} \alpha. \quad (8)$$

Let A_1^α, A_2^α and A_3^α be the α-level sets of fuzzy sets A_1, A_2 and A_3 described above. We have

$$f\left(A_1^\alpha, A_2^\alpha, A_3^\alpha\right) = \Big\{x_1 \cdot x_2 \cdot x_3 | \; x_1 \in A_1^\alpha, x_2 \in A_2^\alpha, x_3 \in A_3^\alpha\Big\}. \quad (9)$$

The following result holds:

Proposition *1:* Let function $f\colon [0,1] \times [0,1] \times \ [0,1] \longrightarrow [0,1]$ be:

$$f(x_1, x_2, x_3) = x_1 \cdot x_2 \cdot x_3.$$

Let A_1, A_2, A_3 be fuzzy sets over the set $[0,1]$ with associated membership functions μ_{A_1}, μ_{A_2}, μ_{A_3}, respectively, and $B = f(A_1, A_2, A_3)$ the fuzzy set over the set $[0,1]$ with membership function $\mu_B\colon [0,1] \to [0,1]$;

$$\mu_B(y) = \sup_{\substack{x_1 \cdot x_2 \cdot x_3 = y \\ x_1, x_2, x_3 \in [0,1]}} \left[\mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2) \wedge \mu_{A_3}(x_3)\right].$$

Then the following equality holds:

$$B^\alpha = f\left(A_1^\alpha, A_2^\alpha, A_3^\alpha\right). \quad (10)$$

Proof: Notice that that both B^α and $f(A_1^\alpha, A_2^\alpha, A_3^\alpha)$ are crisp sets.

I. Let $y \in B^\alpha$. By definition, we have $\mu_B(y) \geq \alpha$ and there exists at least three values $x_1, x_2, x_3 \in [0,1]$ such that $x_1 \cdot x_2 \cdot x_3 = y$ and $[\mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2) \wedge \mu_{A_3}(x_3)] \geq \alpha$. Therefore, it is true that $\mu_{A_1}(x_1) \geq \alpha$, $\mu_{A_2}(x_2) \geq \alpha$ and $\mu_{A_3}(x_3) \geq \alpha$, which means that $x_1 \in A_1^\alpha$, $x_2 \in A_2^\alpha$ and $x_3 \in A_3^\alpha$. Consequently, $y \in f(A_1^\alpha, A_2^\alpha, A_3^\alpha)$, i.e. $B^\alpha \subseteq f(A_1^\alpha, A_2^\alpha, A_3^\alpha)$.

II. Let $y \in f(A_1^\alpha, A_2^\alpha, A_3^\alpha)$. There exist $x_1 \in A_1^\alpha$, $x_2 \in A_2^\alpha$ and $x_3 \in A_3^\alpha$ such that $x_1 \cdot x_2 \cdot x_3 = y$. We have that $\mu_{A_1}(x_1) \geq \alpha$, $\mu_{A_2}(x_2) \geq \alpha$ and $\mu_{A_3}(x_3) \geq \alpha$ and therefore it is:

$$\sup_{\substack{x_1 \cdot x_2 \cdot x_3 = y \\ x_1 \in A_1^\alpha, x_2 \in A_2^\alpha, x_3 \in A_3^\alpha}} [\mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2) \wedge \mu_{A_3}(x_3)] \geq \alpha.$$

Because $A_1^\alpha, A_2^\alpha, A_3^\alpha \subseteq [0,1]$, then we have:

$$\sup_{\substack{x_1 \cdot x_2 \cdot x_3 = y \\ x_1, x_2, x_3 \in [0,1]}} [\mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2) \wedge \mu_{A_3}(x_3)] \geq \sup_{\substack{x_1 \cdot x_2 \cdot x_3 = y \\ x_1 \in A_1^\alpha, x_2 \in A_2^\alpha, x_3 \in A_3^\alpha}} [\mu_{A_1}(x_1) \wedge \mu_{A_2}(x_2) \wedge \mu_{A_3}(x_3)].$$

We conclude that $y \in B^\alpha$, i.e. $f(A_1^\alpha, A_2^\alpha, A_3^\alpha) \subseteq B^\alpha$.

∎

Therefore, the following definition is justified:

Definition 3 (MTP of a FLPR): A fuzzy linguistic preference relation $R = (r_{ij})$ on a finite set of alternatives X is multiplicative transitive if and only if

$$\forall \alpha \in (0,1] : f(r_{ij}^\alpha, r_{jk}^\alpha, r_{ki}^\alpha) = f(r_{ik}^\alpha, r_{kj}^\alpha, r_{ji}^\alpha) \quad \forall i,k,j. \tag{11}$$

Notice that under the assumption of being the linguistic labels characterised by fuzzy numbers in the unit interval, we would have that the alpha level set of linguistic label r_{ij} is the following closed interval: $r_{ij}^\alpha = [r_{ij}^{\alpha-}, r_{ij}^{\alpha+}]$. On the other hand, interval arithmetic yields:

$$f(r_{ij}^\alpha, r_{jk}^\alpha, r_{ki}^\alpha) = [r_{ij}^{\alpha-} \cdot r_{jk}^{\alpha-} \cdot r_{ki}^{\alpha-}, r_{ij}^{\alpha+} \cdot r_{jk}^{\alpha+} \cdot r_{ki}^{\alpha+}]. \tag{12}$$

Thus, the above definition can be reformulated as follows:

Definition 4 (MTP of a FLPR): A fuzzy linguistic preference relation $R = (r_{ij})$ on a finite set of alternatives X is multiplicative transitive if and only if

$$\forall \alpha \in (0,1] \wedge \forall i,k,j : \left.\begin{array}{l} r_{ij}^{\alpha-} \cdot r_{jk}^{\alpha-} \cdot r_{ki}^{\alpha-} = r_{ik}^{\alpha-} \cdot r_{kj}^{\alpha-} \cdot r_{ji}^{\alpha-} \\ r_{ij}^{\alpha+} \cdot r_{jk}^{\alpha+} \cdot r_{ki}^{\alpha+} = r_{ik}^{\alpha+} \cdot r_{kj}^{\alpha+} \cdot r_{ji}^{\alpha+} \end{array}\right\} \tag{13}$$

III. Consistency of Reciprocal Intuitionistic Fuzzy Preference Relations

Xu in [24] introduced the reciprocal intuitionistic fuzzy preference relation (RIFPR), which generalises the concept of reciprocal fuzzy preference relation, as follows:

Definition 5 (Reciprocal Intuitionistic FPR (RIFPR)): A reciprocal intuitionistic fuzzy preference relation R on a finite set of alternatives X is characterised by a membership function $\mu_R\colon X \times X \to [0,1]$ and a non-membership function $\nu_R\colon X \times X \to [0,1]$ such that:

$$0 \leq \mu_R(x_i, x_j) + \nu_R(x_i, x_j) \leq 1 \quad \forall (x_i, x_j) \in X \times X.$$

The value $\mu_R(x_i, x_j) = \mu_{ij}$ is interpreted as the certainty degree up to which x_i is preferred to x_j, while the value $\nu_R(x_i, x_j) = \nu_{ij}$ represents the certainty degree up to which x_i is non-preferred to x_j. Additionally, the following conditions are imposed:

$$\forall i,j : \quad \mu_{ii} = \nu_{ii} = 0.5; \quad \mu_{ji} = \nu_{ij}. \tag{14}$$

Using matrix notation, a reciprocal intuitionistic fuzzy preference relation is represented as $R = (r_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$.

Notice that an interval-valued fuzzy preference relation (IVFPR) is a particular type of fuzzy linguistic preference relation. Indeed, and interval-valued fuzzy preference relation $R = (r_{ij})$ is characterised for having elements $r_{ij} = [r_{ij}^-, r_{ij}^+]$, and therefore they can be seen as having associated the following fuzzy membership functions:

$$\mu_{r_{ij}}(x) = \begin{cases} 1 & x \in [r_{ij}^-, r_{ij}^+] \\ 0 & Otherwise. \end{cases} \tag{15}$$

The only non-empty α-level set of $\mu_{r_{ij}}$ is the 1-level set.

It is well known that an intutionistic fuzzy preference relation (IPR) $P = (\langle \mu_{ij}, \nu_{ij} \rangle)$ is isomorphic to the interval-valued preference relation $R = (r_{ij}) = ([\mu_{ij}, 1 - \nu_{ij}])$ [25], [32], [33]. Consequently, the multiplicative transitivity property of a reciprocal intutionistic fuzzy preference relation can be defined as follows:

Definition 6 (Multiplicative Transitivity Property of IRPR): A reciprocal intutionistic fuzzy preference relation $R = (r_{ij}) = (\langle \mu_{ij}, \nu_{ij} \rangle)$ is multiplicative transitive if and only if

$$\forall i,j,k : \left.\begin{array}{l} \mu_{ij} \cdot \mu_{jk} \cdot \mu_{ki} = \mu_{ik} \cdot \mu_{kj} \cdot \mu_{ji} \\ (1-\nu_{ij}) \cdot (1-\nu_{jk}) \cdot (1-\nu_{ki}) = \\ \qquad (1-\nu_{ik}) \cdot (1-\nu_{kj}) \cdot (1-\nu_{ji}) \end{array}\right\} \tag{16}$$

IV. Consistency Based Estimated Reciprocal Intuitionistic Fuzzy Preferences

Notice that reciprocity property (14) when applied to Expression (16) yields:

$$\left.\begin{array}{l} \mu_{ij} = \dfrac{\mu_{ik} \cdot \mu_{kj} \cdot \mu_{ji}}{\mu_{jk} \cdot \mu_{ki}} \\ \nu_{ij} = 1 - \dfrac{(1-\nu_{ik}) \cdot (1-\nu_{kj}) \cdot (1-\nu_{ji})}{(1-\nu_{jk}) \cdot (1-\nu_{ki})} \end{array}\right\} \tag{17}$$

These expressions can be used to estimate the intuitionistic preference value between a pair of alternatives (x_i, x_j) with $(i < j)$ using another different intermediate alternative x_k $(k \neq i,j)$ as follows:

$$\left.\begin{aligned} mr_{ij}^{k-} &= \frac{\mu_{ik}\cdot\mu_{kj}\cdot\mu_{ji}}{\mu_{jk}\cdot\mu_{ki}} \\ mr_{ij}^{k+} &= 1-\frac{(1-\nu_{ik})\cdot(1-\nu_{kj})\cdot(1-\nu_{ji})}{(1-\nu_{jk})\cdot(1-\nu_{ki})} \end{aligned}\right\} \quad (18)$$

as long as the denominators are not zero.

We call $mr_{ij}^{k} = \left\langle mr_{ij}^{k-}, mr_{ij}^{k+}\right\rangle$ the *partially multiplicative transitivity based estimated intuitionistic preference value* of the pair of alternatives (x_i, x_j) obtained using the intermediate alternative x_k.

Notice that both Equations in (16) are always true when two of the three subindexes are equal. Furthermore, although it is possible to obtain the multiplicative transitivity based estimated intuitionistic preference value of the pair of alternatives (x_i, x_j) when $k \in \{i, j\}$ and $(r_{ij}, r_{ji}) \neq \{(\langle 1,0\rangle, \langle 0,1\rangle), (\langle 0,1\rangle, \langle 1,0\rangle)\}$, it is also true that there is no indirect estimation process as described above. Finally, when $i = j$ we have by definition that $r_{ii} = \langle 0.5, 0.5\rangle$ and we would have $mr_{ii}^{k} = r_{ii}$ whenever $r_{ik} \notin (\langle 0,1\rangle, \langle 1,0\rangle)$. Thus, this case will not be relevant, and it is also not assumed from now on.

The average of all possible partially multiplicative transitivity based estimated values of the pair of alternatives (x_i, x_j) can be interpreted as their *global multiplicative transitivity based estimated value*

$$\left.\begin{aligned} mr_{ij}^{-} &= \frac{\sum_{k\in R_{ij}^{01}} mr_{ij}^{k-}}{\#R_{ij}^{01}} \\ mr_{ij}^{+} &= \frac{\sum_{k\in R_{ij}^{01}} mr_{ij}^{k+}}{\#R_{ij}^{01}} \end{aligned}\right\} \quad (19)$$

where $R_{ij}^{01} = \{k \neq i, j | (r_{ik}, r_{kj}) \notin R^{01}\}$, $R^{01} = \{(\langle 1,0\rangle, \langle 0,1\rangle), (\langle 0,1\rangle, \langle 1,0\rangle)\}$, and $\#R_{ij}^{01}$ is the cardinality of R_{ij}^{01}.

Therefore, given a reciprocal intutionistic fuzzy preference relation, $R = (r_{ij}) = (\langle \mu_{ij}, \nu_{ij}\rangle)$, the following *multiplicative transitivity based reciprocal intutionistic fuzzy preference relation*, $MR = \left(\left\langle mr_{ij}^{-}, mr_{ij}^{+}\right\rangle\right)$, can be constructed:

$$mr_{ij}^{-} = \begin{cases} \dfrac{\sum_{k\in R_{ij}^{01}} mr_{ij}^{k-}}{\#R_{ij}^{01}}, & i<j \\ 0.5, & i=j\ ; \\ \dfrac{\sum_{k\in R_{ji}^{01}} mr_{ji}^{k+}}{\#R_{ji}^{01}}, & i>j \end{cases}$$

$$mr_{ij}^{+} = \begin{cases} \dfrac{\sum_{k\in R_{ij}^{01}} mr_{ij}^{k+}}{\#R_{ij}^{01}}, & i<j \\ 0.5, & i=j \\ \dfrac{\sum_{k\in R_{ji}^{01}} mr_{ji}^{k-}}{\#R_{ji}^{01}}, & i>j \end{cases}$$

Example 1. Given the reciprocal intutionistic fuzzy preference relation

$$\mathbf{R} = \begin{pmatrix} \langle 0.5,0.5\rangle & \langle 0.4,0.3\rangle & \langle 0.5,0.4\rangle & \langle 0.4,0.5\rangle \\ \langle 0.3,0.4\rangle & \langle 0.5,0.5\rangle & \langle 0.5,0.4\rangle & \langle 0.3,0.4\rangle \\ \langle 0.4,0.5\rangle & \langle 0.4,0.5\rangle & \langle 0.5,0.5\rangle & \langle 0.3,0.4\rangle \\ \langle 0.5,0.4\rangle & \langle 0.4,0.3\rangle & \langle 0.4,0.3\rangle & \langle 0.5,0.5\rangle \end{pmatrix}$$

The computation of the multiplicative transitivity based estimation of the intuitionistic preference value between alternatives x_1 and x_2 is provided. The value mr_{12} is obtained using the intermediate alternatives x_3 and x_4.

Using (18), we have (rounding to 2 decimal places):

$$mr_{12}^{3-} = \frac{\mu_{13}\cdot\mu_{32}\cdot\mu_{21}}{\mu_{23}\cdot\mu_{31}} = \frac{0.5\cdot 0.4\cdot 0.3}{0.5\cdot 0.4} = 0.3;$$

$$mr_{12}^{4-} = \frac{\mu_{14}\cdot\mu_{42}\cdot\mu_{21}}{\mu_{24}\cdot\mu_{41}} = \frac{0.4\cdot 0.4\cdot 0.3}{0.3\cdot 0.5} = 0.32;$$

$$\begin{aligned} mr_{12}^{3+} =& 1-\frac{(1-\nu_{13})\cdot(1-\nu_{32})\cdot(1-\nu_{21})}{(1-\nu_{23})\cdot(1-\nu_{31})} = \\ & 1-\frac{(1-0.4)\cdot(1-0.5)\cdot(1-0.4)}{(1-0.4)\cdot(1-0.5)} = 0.4; \end{aligned}$$

$$\begin{aligned} mr_{12}^{4+} =& 1-\frac{(1-\nu_{14})\cdot(1-\nu_{42})\cdot(1-\nu_{21})}{(1-\nu_{24})\cdot(1-\nu_{41})} = \\ & 1-\frac{(1-0.5)\cdot(1-0.3)\cdot(1-0.4)}{(1-0.4)\cdot(1-0.4)} = 0.42. \end{aligned}$$

Thus, it is:

$$mr_{12}^{-} = \frac{0.3+0.32}{2} = 0.31; \quad mr_{12}^{+} = \frac{0.4+0.42}{2} = 0.41.$$

We have that $mr_{12} = \langle 0.31, 0.41\rangle$.

The rest of values are obtained following a similar computation process, leading to the following multiplicative transitivity based reciprocal intutionistic fuzzy preference relation:

$$\mathbf{MR} = \begin{pmatrix} \langle 0.50,0.50\rangle & \langle 0.31,0.41\rangle & \langle 0.55,0.41\rangle & \langle 0.48,0.39\rangle \\ \langle 0.41,0.31\rangle & \langle 0.50,0.50\rangle & \langle 0.39,0.49\rangle & \langle 0.31,0.39\rangle \\ \langle 0.41,0.55\rangle & \langle 0.49,0.39\rangle & \langle 0.50,0.50\rangle & \langle 0.25,0.51\rangle \\ \langle 0.39,0.48\rangle & \langle 0.39,0.31\rangle & \langle 0.51,0.25\rangle & \langle 0.50,0.50\rangle \end{pmatrix}$$

V. Conclusions

This contribution presented an application of Zadeh's original extension principle methodology to extend concepts and properties already known and accepted in the context of numerical inputs to the context of fuzzy inputs represented using fuzzy membership functions. In particular, the concept of consistency of preferences has been extended from the case of reciprocal $[0, 1]$-valued fuzzy preference relations to the case of linguistic fuzzy preference relations, with linguistic fuzzy labels are modelled and represented via fuzzy membership functions. Tanino's multiplicative transitivity property is used to model consistency of preferences, and as such it has been obtained its corresponding linguistic representation via Zadeh's representation theorem of fuzzy sets. The particular case of reciprocal intuitionistic fuzzy preference relations was also studied and the concept of consistency based on the multiplicative transitivity property was derived making use of their isomorphism to reciprocal interval-valued fuzzy preference relations, which in turn are a particular type of linguistic fuzzy preference relations. Finally, a procedure to compute consistency based estimated reciprocal intuitionistic fuzzy preference values using an indirect chain of alternatives is pointed out, which can be used to address incomplete information in decision-making problems with this type of preference relations. This last problem is not covered in this contribution and it is left for a different/future research efforts.

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