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Interval Type-2 Relational Analysis and Its Application to Multiple Attribute Decision Making

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***Abstract*—In this paper, we present the interval type-2 fuzzy rational degree to measure the similarity of the interval type-2 fuzzy sets and apply it to multiple attribute decision making with interval type-2 fuzzy information. First, we introduce the concept of the interval type-2 fuzzy metric spaces, which include interval type-2 fuzzy distance space and interval type-2 fuzzy inner product space, respectively. Based on which, we derive the distance measure and inner product of interval type-2 fuzzy sets, respectively. Then, we introduce the axiomatic definition of the interval type-2 fuzzy rational degree and propose an interval type-2 fuzzy rational degree formula. Moreover, we construct the mathematical optimal model based on two types of interval type-2 fuzzy measures (center of gravity and fuzziness) to determine the optimal attribute weights. Based on the interval type-2 fuzzy rational degree and the optimal weights solution model we proposed, an approach to multiple attribute decision making under interval type-2 fuzzy environment is developed. Finally, an illustrative example is given to demonstrate the practicality and effectiveness of our method.**

I. Introduction

Type-2 fuzzy sets (T2FSs) theory initially introduced by Zadeh in 1975 [1], which can be viewed as an effective extension of tradition type-1 fuzzy sets (T1FSs), it is more capable for handling imprecision and imperfect information in real-world application. In recent years, the type-2 fuzzy sets theory [2] has widely used in computing with words [3], information fusion [4], pattern recognition [5] and other domains [6]–[8].

Type-2 fuzzy multiple attribute decision making (MADM) is the most hotly topic in the research of type-2 fuzzy theory, which aims to find the best solution from a finite number of feasible alternatives assessed on multiple attributes. To date, many useful decision making techniques have been presented. Chen et al. [9] developed an extended QUALIFLEX approach with interval type-2 fuzzy information to solve MADM problems. Chen and Li [10] developed extended TOPSIS method to handle MADM with interval type-2 information. Wang and Liu [11] presented some optimization models to determine the attribute weights and developed its application to interval type-2 fuzzy decision making. Celik et al. [12] developed an integrated novel interval type-2 fuzzy MADM method to improve customer satisfaction in public transportation. Naim and Hagras [13] proposed a hybrid approach for multi-attribute group decision making (MAGDM) based on interval type-2 fuzzy logic and intuitionistic fuzzy information. In additional, some studies have focus on other interval type-2 fuzzy decision making methods based on a variety of classical decision making techniques such as AHP [14], TOPSIS [15], PROMETHEE [16] etc. Recently, the research of general type-2 fuzzy MADM are receiving more attention from scholars, some valuable researches have been presented [17][18].

Type-2 fuzzy measure is an interesting research direction of the T2FSs theory. As the most important fuzzy measure, the similarity of the T2FSs are receiving more and more attention from scholars in different domains. Many valuable research works of type-2 fuzzy similarity measure have been published during the last several years. For example, Wu and Mendel [19] studied the uncertainty fuzzy similarity measure for interval type-2 fuzzy sets (IT2FSs) and made a comparative study of ranking methods, similarity measures and other uncertainty measures of IT2FSs. Zeng and Li [20] introduced the axiomatic definition of the interval type-2 fuzzy similarity measure and discussed the relationship between the similarity measure and entropy of IT2FSs. Hwang et al. [21] proposed a similarity measure based on Sugeno integral. Recently, Zhai and Mendel [22] investigated the general type-2 fuzzy similarity measure. Zhao et al. [23] defined two new similarity measures based on $\alpha-$ plane representation theory, the main characteristic of these similarity measures are expressed as T1FSs.

However, the current researches of interval type-2 fuzzy similarity measure can only measure the similarity of two IT2FSs. In real practical situations, we usually should consider all the IT2FSs as a whole. That means we should find a new aggregation tool to measure the relationship among the multiple interval type-2 fuzzy sets vector sequence. To fill this gap, we focus our attention on this issue and introduce a new concept of interval type-2 fuzzy measure: interval type-2 fuzzy relational degree and develop a new method based on the proposed interval type-2 fuzzy relational degree to handle multiple attribute decision making problems.

The remainder of this paper is organized as follows. In Section II, we briefly introduce some basic concepts of T2FSs and IT2FSs. In Section III, we investigate the interval type-2 fuzzy metric spaces and derive the distance measure and inner product of the IT2FSs, respectively. In Section IV, we propose an interval type-2 fuzzy relational degree formula and give a strictly proof process. In Section V, we develop an approach based on the proposed relational degree to solve multiple attribute decision making. Furthermore, a optimization model in accordance with center of gravity and fuzziness is provided to determine the attribute weights. An illustrative example is given to demonstrate the practicality and effectiveness of our method in Section VI. Section VII concludes the paper.

II. Preliminaries

In this section, we briefly introduce some basic concepts related to T2FSs and IT2FSs, which will be used in the next sections.

Definition 1: [2] Let X be a universe of discourse, a type-2 fuzzy sets A can be represented by a membership function $\mu_A(x,u)$, shown as follows:

$$A = \{((x,u), \mu_A(x,u)) | \forall x \in X, \forall u \in J_x \subseteq [0,1]\} \quad (1)$$

where $0 \leq \mu_A(x,u) \leq 1$. A can also be expressed as the following form:

$$\begin{aligned} A &= \int_{x\in X}\int_{u\in J_x} \mu_A(x,u)/(x,u) \\ &= \int_{x\in X} (\int_{u\in J_x} \mu_A(x,u)/u) \Big/ x \end{aligned} \quad (2)$$

where $J_x \subseteq [0,1]$ is the primary membership at x, and $\int_{u\in J_x} \mu_A(x,u)/u$ indicates the second membership at x. For discreet situations, $\int$ is replaced by $\sum$.

Definition 2: [24] Let A be a type-2 fuzzy sets, if all $\mu_A(x,u) = 1$, that is

$$A = \int_{x\in X}\int_{u\in J_x} 1/(x,u) = \int_{x\in X} \left(\int_{u\in J_x} 1/u\right) \Big/ x \quad (3)$$

then A is called an interval type-2 fuzzy sets (IT2FSs).

It is obvious that the IT2FSs A is completely determined by the primary membership which is called the footprint of uncertainty (FOU), the FOU can be expressed as follows:

$$\mathrm{FOU(A)} = \bigcup_{\forall x\in X} J_x = \{(x,u) | u \in J_x \subseteq [0,1]\} \quad (4)$$

Let $\overline{\mu}_A(x)$ and $\underline{\mu}_A(x)$ be the upper membership function (UMF) and lower membership function (LMF), respectively. Based on the definition of FOU, for any $x \in X$, we have:

$$\begin{aligned} \overline{\mu}_A(x) &= \sup(\mathrm{FOU(A)}) \\ \underline{\mu}_A(x) &= \inf(\mathrm{FOU(A)}) \end{aligned} \quad (5)$$

Therefore, the FOU of IT2FSs A can be expressed as:

$$\mathrm{FOU(A)} = \bigcup_{\forall x\in X} [\underline{\mu}_A(x), \overline{\mu}_A(x)] \quad (6)$$

For further explanation, we let $A = (\overline{\mu}_A(x), \underline{\mu}_A(x)) = ((a_{11}^+, a_{12}^+, a_{13}^+, a_{14}^+; h_A^+), (a_{11}^-, a_{12}^-, a_{13}^-, a_{14}^-; h_A^-))$ be a trapezoidal interval type-2 fuzzy sets (TIT2FSs), as shown in Fig.1, where h_A^+ denotes the upper membership value of a_{12}^+ and a_{13}^+ ; h_A^- denotes the upper membership value of a_{12}^- and a_{13}^-; and $0 \leq h_A^- \leq h_A^+ \leq 1$. Especially, if $a_{12}^+ = a_{13}^+$ and $a_{12}^- = a_{13}^-$, then the trapezoidal interval type-2 sets A reduces to a triangular interval type-2 fuzzy sets.

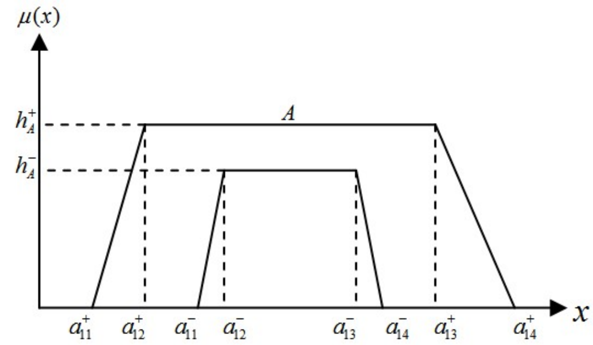

Fig. 1. The ten reference points to determine an FOU. $(a_{11}^+, a_{12}^+, a_{13}^+, a_{14}^+)$ determines a trapezoidal UMF A with the height h_A^+, and $(a_{11}^-, a_{12}^-, a_{13}^-, a_{14}^-)$ determines a trapezoidal LMF A with the height h_A^-.

Definition 3: [25] Let A be an interval type-2 fuzzy sets, then the ranking value of A is defined as:

$$\begin{aligned} &Rank(A) = \sum_{i=1}^{3} (M_i(A^+) + M_i(A^-)) \\ &-\tfrac{1}{4}\sum_{i=1}^{4} (S_i(A^+) + S_i(A^-)) \; + \sum_{i=1}^{2} (H_i(A^+) + H_i(A^-)) \end{aligned} \quad (7)$$

where $M_i(A^j)$ denotes the average value of the elements a_{1p}^j and a_{1p+1}^j,$M_i(A^j) = (a_{1p}^j + a_{1p+1}^j)/2 (i = 1,2,3)$, $S_i(A^j)$ denotes the standard deviation of the elements a_{1p}^j and a_{1p+1}^j, $S_i(A^j) = \sqrt{\frac{1}{2}\sum_{k=i}^{i+1}(a_{1i}^j - \frac{1}{2}\sum_{k=i}^{i+1} a_{1i}^j)^2} (i = 1,2,3)$, $H_i(A^j)$ denotes the membership value of the element a_{1p+1}^j in the interval type-2 fuzzy sets A, $1 \leq i \leq 2, j \in \{+,-\}$.

III. The Metric space of trapezoidal interval type-2 fuzzy sets

A. Distance space of trapezoidal interval type-2 fuzzy sets

Definition 4: Let A and B be two TIT2FSs on X, there exists a real number $d(A,B)$, which satisfies the following properties:

1) $d(A,B) \geq 0$, when $d(A,B) = 0$, if and only if $A = B = ((0,0,0,0;0),(0,0,0,0;0))$;
2) $d(A,B) = d(B,A)$;
3) $d(A,B) \leq d(A,C) + d(C,B)$ for any $C \in X$.

then (X,d) is called a trapezoidal interval type-2 fuzzy distance space (TIT2FDS).

Based on functional analysis theory [30], we know that the distance measure usually derived by norm operations directly. Therefore, we first define some trapezoidal interval type-2 fuzzy norms as follows.

Definition 5: Let $A = ((a_{11}^+, a_{12}^+, a_{13}^+, a_{14}^+; h_A^+), (a_{11}^-, a_{12}^-, a_{13}^-, a_{14}^-; h_A^-))$ be a TIT2FSs on X, then the commonly norms of A (denote by $\| A \|$) are defined as follows:

1) 1-norm $\| A \|_1 = \sum_{i=1}^{4} \left(|a_{1i}^+| + |a_{1i}^-|\right) + |h_A^+| + |h_A^-|$
2) 2-norm
$$\| A \|_2 = \sqrt{\sum_{i=1}^{4}\left(|a_{1i}^+|^2 + |a_{1i}^-|^2\right) + |h_A^+|^2 + |h_A^-|^2}$$

3) p-norm
$\| A \|_p = \left(\sum_{i=1}^{4} \left(\left|a_{1i}^{+}\right|^p + \left|a_{1i}^{-}\right|^p \right) + \left|h_A^{+}\right|^p + \left|h_A^{-}\right|^p \right)^{1/p}$

4) ∞-norm
$\| A \|_\infty = \max_{1\le i\le 4} \left\{ \left|a_{1i}^{+}\right|, \left|a_{1i}^{-}\right|, \left|h_A^{+}\right|, \left|h_A^{-}\right| \right\}$

Definition 6: Let A and B be two TIT2FSs on X, then the distance measure between A and B is defined as

$$d(A,B) = \| A - B \| \tag{8}$$

Without loss of generality, we take 2-norm to derive the distance measure, which is defined as:

$$d(A,B) = \left(\left(\sum_{i=1}^{4} (a_{1i}^{+} - b_{1i}^{+})^2 + \sum_{i=1}^{4} (a_{1i}^{-} - b_{1i}^{-})^2 + (h_A^{+} - h_B^{+})^2 + (h_A^{-} - h_B^{-})^2 \right) / 10 \right)^{1/2} \tag{9}$$

where $0 \le d(A,B) \le 1$.

B. Inner product space of trapezoidal interval type-2 fuzzy sets

Definition 7: Let A and B be two TIT2FSs on X, there exists a real number $\langle A, B\rangle$, which satisfies the following properties:

1) $\langle A, B\rangle = \langle B, A\rangle$
2) $\langle k_1A + k_2B, C\rangle = k_1\langle A, C\rangle + k_2\langle B, C\rangle$, for any $C \in X$, $k_1, k_2 \in Z$.
3) $\langle A, A\rangle \ge 0$, if and only if $A = ((0,0,0,0;0),(0,0,0,0;0))$, $\langle A, A\rangle = 0$

then $\langle A, B\rangle$ is called an inner product of A and B, and $(X, \langle\rangle)$ is called trapezoidal interval type-2 fuzzy inner product space (TIT2FIPS).

Based on Definition 7, we propose a inner product formula of TIT2FSs as follows:

$$< A, B >= \sum_{i=1}^{4} a_{1i}^{+} b_{1i}^{+} + \sum_{i=1}^{4} a_{1i}^{-} b_{1i}^{-} + h_A^{+} h_B^{+} + h_A^{-} h_B^{-} \tag{10}$$

It can be easily proved that Eq.(10) satisfying the axiom of TIT2FIPS described in Definition 7.

IV. Interval type-2 fuzzy relational degree

In this section, we shall give the definition of interval type-2 fuzzy relational degree and propose an interval type-2 fuzzy relational degree formula.

Definition 8: Let $A_0 = (A_{01}, A_{02}, \cdots, A_{0n})$ be an interval type-2 fuzzy reference sequence, and $B_i = (B_{i1}, B_{i2}, \cdots, B_{in})(i = 1, 2, \cdots, m)$ be an interval type-2 fuzzy sets sequences in the universe of discourse X. For given real numbers $R_j(A_{0j}, B_{ij})(j = 1, 2, \cdots, n)$, such that

$$R(A_0, B_i) = \frac{1}{n} \sum_{j=1}^{n} R_j(A_{0j}, B_{ij}) \tag{11}$$

which satisfying the following conditions:

1) Normality
$0 \le R(A_0, B_i) \le 1$, when $R(A_0, B_i) = 1$ if and only if $A_0 = B_i$;
2) Symmetry
For any $A_0, B_i \in X$, we have $R(A_0, B_i) = R(B_i, A_0)$;
3) Closeness
If $|A_{0j} - B_{ij}| \to 0$, then $R_j(A_{0j}, B_{ij}) \to 0$.

then $R(A_0, B_i)$ is referred to as a interval type-2 fuzzy relational degree between A_0 and B_i, where $R_j(A_{0j}, B_{ij})$ is the relational coefficient of A_0 and B_i at point j.

Based on the concept of interval type-2 fuzzy relational degree proposed above, we can derive the following Theorem 1.

Theorem 1: Let $A_0 = (A_{01}, A_{02}, \cdots, A_{0n})$ be an interval type-2 fuzzy sets reference sequence and $B_i = (B_{i1}, B_{i2}, \cdots, B_{in})(i = 1, 2, \cdots, m)$ be a common interval type-2 fuzzy sets sequence, then

$$R_j(A_{0j}, B_{ij}) = \frac{1-\sin\Delta_{ij}}{\cos(\min_i\{\Delta_{ij}\})} \frac{\sum_{j=1}^{n} <A_{0j}, B_{ij}>}{\sum_{j=1}^{n} \|A_{0j}\|^2 + \sum_{j=1}^{n} \|B_{ij}\|^2 - \sum_{j=1}^{n} <A_{0j}, B_{ij}>} \tag{12}$$

and

$$R(A_0, B_i) = \frac{1}{n} \sum_{j=1}^{n} \frac{1-\sin\Delta_{ij}}{\cos(\min_i\{\Delta_{ij}\})} \frac{\sum_{j=1}^{n} <A_{0j}, B_{ij}>}{\sum_{j=1}^{n} \|A_{0j}\|^2 + \sum_{j=1}^{n} \|B_{ij}\|^2 - \sum_{j=1}^{n} <A_{0j}, B_{ij}>} \tag{13}$$

then $R(A_0, B_i)$ is called an interval type-2 fuzzy relational degree between A_0 and B_i.

where $\Delta_{ij} = \frac{\sum_{p=\{+,-\}} \sum_{k=1}^{4} \left(\left| a_{1k(0j)}^{p} - b_{1k(ij)}^{p} \right| + \left| h_{A_{0j}}^{p} - h_{B_{ij}}^{p} \right| \right)}{10}$.

Proof:

1) Normality
Based on Cauchy inequality, we can easy to obtain $\sum_{j=1}^{n} \|A_{0j}\|^2 + \sum_{j=1}^{n} \|B_{ij}\|^2 \ge 2 \sum_{j=1}^{n} < A_{0j}, B_{ij} > \Rightarrow \frac{\sum_{j=1}^{n} <A_{0j}, B_{ij}>}{\sum_{j=1}^{n} \|A_{0j}\|^2 + \sum_{j=1}^{n} \|B_{ij}\|^2 - \sum_{j=1}^{n} <A_{0j}, B_{ij}>} \le 1$. Also since $0 \le \Delta_{ij} \le 1$, so we have $\min_i\{\Delta_{ij}\} \le \Delta_{ij} \Rightarrow \cos(\min_i\{\Delta_{ij}\}) \ge \cos(\Delta_{ij})$, then it follows that $1 - \sin\Delta_{ij} - \cos(\min_i\{\Delta_{ij}\}) \le 1 - \sin\Delta_{ij} - \cos\Delta_{ij}$. Since $0 \le \Delta_{ij} \le 1 \Rightarrow 1 - \sin\Delta_{ij} - \cos\Delta_{ij} \le 0 \Rightarrow 1 - \sin\Delta_{ij} - \cos(\min_i\{\Delta_{ij}\}) \le 0 \Rightarrow \frac{1-\sin\Delta_{ij}}{\cos(\min_i\{\Delta_{ij}\})} \le 1$.
Therefore, we can deduce that $0 \le R_j(A_{0j}, B_{ij}) \le 1$ for all $j = 1, 2, \ldots, n$, then $0 \le R(A_0, B_i) = \frac{1}{n} \sum_{j=1}^{n} R_j(A_{0j}, B_{ij}) \le \frac{n}{n} = 1$. When $R(A_0, B_i) = 1$, if and only if $\Delta_{ij} = 0$ for all j, thus, we can easy to obtain $A_0 = B_i$.
2) Symmetry
The symmetry is obvious, omitted in here.
3) Closeness
If $|A_{0j} - B_{ij}| \to 0$ for all j, then we have $\Delta_{ij} \to 0$,

we can further deduce that $\frac{1-\sin\Delta_{ij}}{\cos(\min_i\{\Delta_{ij}\})} \to 0$ and $\frac{\sum_{j=1}^{n} <A_{0j},B_{ij}>}{\sum_{j=1}^{n}\|A_{0j}\|^2+\sum_{j=1}^{n}\|B_{ij}\|^2-\sum_{j=1}^{n}<A_{0j},B_{ij}>} \to 0$, then it follows that $R_j(A_{0j}, B_{ij}) \to 1$ for all $j = 1, 2, \ldots, n$. Therefore, $R(A_0, B_i) \to 0$.

which completes the proof of Theorem 1. ■

Example 1: Let $A_0 = \{A_{01}, A_{02}\} = (((0, 0, 0, 0.1; 1), (0, 0, 0, 0.05; 0.9)), ((0, 0.1, 0.1, 0.3; 1), (0.05, 0.1, 0.1, 0.2; 0.9))$ be an interval type-2 fuzzy sets reference sequence , $B_1 = \{B_{11}, B_{12}\} = (((0.1, 0.3, 0.3, 0.5; 1), (0.2, 0.3, 0.3, 0.4; 0.9)), ((0.3, 0.5, 0.5, 0.7; 1), (0.4, 0.5, 0.5, 0.6; 0.9)))$, and $B_2 = \{B_{21}, B_{22}\} = (((0.5, 0.7, 0.7, 0.9; 1), (0.6, 0.7, 0.7, 0.8; 0.9)), ((0.7, 0.9, 0.9, 1; 1), (0.8, 0.9, 0.9, 0.95; 0.9)))$ be the two interval type-2 fuzzy sets sequences, then based on Eqs.(10) and (13), we have

$$\begin{aligned} R_1(A_{01}, B_{11}) &= \frac{1-\sin 0.225}{\cos(\min(0.225, 0.545))}\frac{4.24}{3.805+6.54-4.24} \\ &= 0.5535 \\ R_2(A_{02}, B_{12}) &= \frac{1-\sin 0.305}{\cos(\min(0.305, 0.61))}\frac{4.24}{3.805+6.54-4.24} \\ &= 0.5094 \end{aligned} \tag{14}$$

then

$$R(A_0, B_1) = \frac{R_1(A_{01}, B_{11}) + R_2(A_{02}, B_{12})}{2} = 0.5314$$

Similarly, we can obtain the $R(A_0, B_2)$ as follows:

$$R(A_0, B_2) = \frac{R_1(A_{01}, B_{21}) + R_2(A_{02}, B_{22})}{2} = 0.2597$$

Since $R(A_0, B_1) > R(A_0, B_2)$, therefore, B_1 is closeness to A_0 than B_2.

In Eq.(10), we assume that the importance of each element in interval type-2 fuzzy sets sequence is equal. However, in some practical situations, especially in MADM problems, we usually should consider the weight of each element. In this case, we can introduce the weighted interval type-2 fuzzy relational degree as follows:

$$\begin{aligned} &R_w(A_0, B_i) \\ &= \sum_{j=1}^{n} \omega_j \frac{1-\sin\Delta_{ij}}{\cos(\min_i\{\Delta_{ij}\})} \frac{\sum_{j=1}^{n} <A_{0j},B_{ij}>}{\sum_{j=1}^{n}\|A_{0j}\|^2+\sum_{j=1}^{n}\|B_{ij}\|^2-\sum_{j=1}^{n}<A_{0j},B_{ij}>} \end{aligned} \tag{15}$$

Similarly to Theorem 1, we can easily to prove that the weighted interval type-2 fuzzy relational degree we proposed is satisfying axiom rules described in Definition 8.

V. An approach to multiple attribute decision making with interval type-2 relational degree

A. The description of MADM problems

For an interval type-2 fuzzy MADM problem, let $A = \{A_1, A_2, \cdots, A_m\}$ be a discrete set of alternatives and $C = \{C_1, C_2, \cdots, C_n\}$ be a set of attributes. $W = (< w_1, [w_1^L, w_1^R] >, < w_2, [w_2^L, w_2^R] >, \cdots, < w_n, [w_n^L, w_n^R] >)^T$ is the weighting vector of attributes, where $w_j (j = 1, 2, \cdots, n)$ is in the form of IT2FSs, w_j^L and w_j^R be the lower extreme and the upper extreme of the IT2FSs w_j, and $\sum_{j=1}^{n} COG(w_j) = 1$, where $COG(w_j)$ indicate the center of gravity of w_j. Assume that decision matrix $R = (A_{ij})_{m\times n}$, where the attribute value A_{ij} takes the form of the IT2FSs provided by decision maker (DM) for alternative A_i with respect to attribute C_j. Based on these necessary conditions, the ranking of alternatives is required.

B. Weight solution method

Solving attribute weight is the most important part in decision making procedure. In the following, we shall propose a new method to determine the traditional trapezoidal interval type-2 fuzzy attribute weights based on trapezoidal type-1 fuzzy COG and the fuzziness of trapezoidal type-1 fuzzy sets (TT1FSs).

For any TT1FSs $A = (a_1, a_2, a_3, a_4; h_A)$, Chen and Chen [26] introduced a simple formula to calculate the center of gravity of A, denote by (x_A^*, y_A^*), which is defined as follows:

$$y_A^* = \begin{cases} \frac{h_A(\frac{a_3-a_2}{a_4-a_1}+2)}{6} & \text{if } a_1 \neq a_4 \text{ and } 0 < x \leq 1 \\ \frac{h_A}{2} & \text{if } a_1 = a_4 \text{ and } 0 < x \leq 1 \end{cases} \tag{16}$$

$$x_A^* = \frac{y_A^*(a_2 + a_3 + (a_1 + a_4)(h_A - y_A^*))}{2h_A} \tag{17}$$

In [27], Li propose the measure of fuzziness of TT1FSs A, denote by $F(A)$, which can be calculated as:

$$F(A) = \frac{(a_4 - 2a_2 + 2a_3 - a_1)h_A^2}{6} \tag{18}$$

In order to obtain the related feature information, we should maximize the COG of $W = (w_1, w_2, \cdots, w_n)^T$, where $w_j = [w_j^L, w_j^R](j = 1, 2, \cdots, n)$ be an interval type-2 fuzzy weight, and minimize the weighted fuzziness measure, respectively. The ranges of attribute weights (incomplete known attribute weights) are provided by decision maker as: $\Omega = \{w_j | w_j^- \leq w_j \leq w_j^+, \sum_{j=1}^{n} w_j = 1\}$. Based on the above analysis, we can establish the maximize x-COG optimal linear programming model first as follows:

(**Mod 1**)

$$\begin{aligned} &\max \sum_{j=1}^{n}\sum_{i=1}^{m}(x^*_{a^L_{ij}} x^*_{w^L_j} + x^*_{a^R_{ij}} x^*_{w^R_j}) \\ &s.t. \begin{cases} x^*_{w^L_j} \leq x^*_{w^R_j} (j = 1, 2, \cdots, n) \\ x^*_{w^L_j} \geq w_j^- (j = 1, 2, \cdots, n) \\ x^*_{w^R_j} \leq w_j^+ (j = 1, 2, \cdots, n) \\ \left|x^*_{w^L_j} - x^*_{w^R_j}\right| \leq L\left|w_j^L - w_j^R\right| \\ \sum_{j=1}^{n} \frac{x^*_{w_j^+} + x^*_{w_j^-}}{2} = 1 \\ 0 \leq x^*_{w^L_j}, x^*_{w^R_j} \leq 1 (j = 1, 2, \cdots, n) \end{cases} \end{aligned} \tag{19}$$

It is noted that the fourth constrain condition is a Lipschitz condition, where L is a Lipschitz constant and $0 < L < 1$. From contraction mapping principle in convex functional analysis [29], we can get the globe optimal solution of this model. In general, we take $L = 0.5$ to use. Similarly, we can construct the minimize fuzziness measure optimal linear programming model and the maximize y-COG optimal linear programming model, respectively.

(**Mod 2**)

$$\max \sum_{j=1}^{n}\sum_{i=1}^{m}\left(F(a_{ij}^{L})F(w_j^{L})+F(a_{ij}^{R})F(w_j^{R})\right)$$
$$s.t.\begin{cases} F(w_j^L)\le F(w_j^R)(j=1,2,\cdots,n)\\ F(w_j^L)\ge \frac{5}{18}(x^*_{w_j^R}-x^*_{w_j^L})(j=1,2,\cdots,n)\\ F(w_j^R)\le \frac{5}{18}(w_j^+-w_j^-)(j=1,2,\cdots,n)\\ \frac{\sum_{j=1}^{n}F(w_j^L)+F(w_j^R)}{2n}=\frac{5}{18}\frac{w_j^+-w_j^-+x^*_{w_j^R}-x^*_{w_j^L}}{2}\\ 0\le F(w_j^L),F(w_j^R)\le 1(j=1,2,\cdots,n)\end{cases} \tag{20}$$

(**Mod 3**)

$$\max \sum_{j=1}^{n}\sum_{i=1}^{m}\left(y^*_{a_{ij}^L}y^*_{w_j^L}+y^*_{a_{ij}^R}y^*_{w_j^R}\right)$$
$$s.t.\begin{cases} y^*_{w_j^L}\le y^*_{w_j^R}(j=1,2,\cdots,n)\\ y^*_{w_j^L}\ge \frac{7}{18}(1-F(w_j^R))(j=1,2,\cdots,n)\\ y^*_{w_j^R}\le \frac{7}{18}(1-F(w_j^L))(j=1,2,\cdots,n)\\ \frac{\sum_{j=1}^{n}y^*_{w_j^L}+y^*_{w_j^R}}{2n}=\frac{7}{18}(1-\frac{\sum_{j=1}^{n}F(w_j^R)+F(w_j^R)}{2})\\ 0\le y^*_{w_j^L},y^*_{w_j^R}\le 1(j=1,2,\cdots,n)\end{cases} \tag{21}$$

Based on the models above, we can construct the algebraic equations to solve the interval type-2 fuzzy attribute weight $w_j=[w_j^L,w_j^R]$, where $w_j^L=(a_{1j}^L,a_{2j}^L,a_{3j}^L,a_{4j}^L;h_{wj}^L)$, and $w_j^R=(a_{1j}^R,a_{2j}^R,a_{3j}^R,a_{4j}^R;h_{wj}^R)$. For computational convenience, we consider a special case that the w_j is a arithmetic symmetry interval type-2 fuzzy sets, which means the reference point of w_j^L and w_j^R are satisfying $a_{2j}^L-a_{1j}^L=a_{3j}^L-a_{2j}^L=a_{4j}^L-a_{3j}^L=d_j^L$ and $a_{2j}^R-a_{1j}^R=a_{3j}^R-a_{2j}^R=a_{4j}^R-a_{3j}^R=d_j^R$, respectively. Therefore, we have $a_{nj}^{L(R)}=a_{1j}^{L(R)}+(n-1)d_j^{L(R)}(n=1,2,3,4)$. Based on this assumption and related mathematical relationship, we can establish the linear equations for solving the IT2FSs attribute weights as follows:

$$\begin{cases} \frac{y^*_{w_j^L}(2a_{1j}^L+3d_j^L)(h_{w_j}^L-y^*_{w_j^L}+1)}{2h_{w_j}^L}=x^*_{w_j^L}\\ \frac{h_{w_j}^L(\frac{d_j^L}{3d_j^L}+2)}{6}=y^*_{w_j^L}\\ \frac{5d_j^L(h_{w_j}^L)^2}{6}=F(w_j^L)\end{cases} \tag{22}$$

and

$$\begin{cases} \frac{y^*_{w_j^R}(2a_{1j}^R+3d_j^R)(h_{w_j}^R-y^*_{w_j^R}+1)}{2h_{w_j}^L}=x^*_{w_j^R}\\ \frac{h_{w_j}^R(\frac{d_j^R}{3d_j^R}+2)}{6}=y^*_{w_j^R}\\ \frac{5d_j^R(h_{w_j}^R)^2}{6}=F(w_j^R)\end{cases} \tag{23}$$

Solving these two algebraic equations, we can obtain the optimal attribute weights under interval type-2 fuzzy environment.

Remark 1: In **Mod-2** and **Mod-3**, we take Lipschitz constant $L=\frac{7}{18}$ and $\frac{5}{18}$, respectively. Where $\frac{7}{18}$ is the maximum value of F and $\frac{5}{18}$ is the maximum value of y^*. In what follows, we shall give a strictly proof.

Theorem 2: Let A be a type-1 fuzzy sets, then max $y_A^*=\frac{7}{18}$ and $F(A)=\frac{5}{18}$.

Proof: Based on Eq.(16), we have $y_A^*=\frac{h_A(\frac{a_3-a_2}{a_4-a_1}+2)}{6}$, also since $a_3-a_2=\frac{1}{3}(a_4-a_1)$, then it follows that $y_A^*=\frac{h_A(\frac{1}{3}+2)}{6}=\frac{7}{18}h_A\le\frac{7}{18}$. Similarly, from Eq.(18), we have $F(A)=\frac{(a_4-2a_2+2a_3-a_1)h_A^2}{6}=\frac{5}{6}d_Ah_A^2$. Since $0\le h_A\le 1$ and $0\le 3d=a_4-a_1\le 1\Longrightarrow 0\le d\le\frac{1}{3}$, then we have $F(A)=\frac{5}{6}d_Ah_A^2\le\frac{5}{6}\times\frac{1}{3}\times 1^2=\frac{5}{18}$. Thus, the proof of Theorem 2 is completed. ∎

C. Decision making steps

Step 1: Calculate the IT2FSs attribute weights based on Eqs.(20-23).

Step 2: Define the positive ideal solution (PIS) and the negative ideal solution (NIS) based on interval type-2 fuzzy numbers, respectively:

$$A^+=(A_1^+,A_2^+,\cdots,A_n^+) \tag{24}$$

$$A^-=(A_1^-,A_2^-,\cdots,A_n^-) \tag{25}$$

where

$$A_j^+=\max_i Rank(A_{ij}),j=1,2,\cdots,n \tag{26}$$

$$A_j^-=\min_i Rank(A_{ij}),j=1,2,\cdots,n \tag{27}$$

Step 3: Calculate the interval type-2 fuzzy relational coefficient of each alternative from the PIS and NIS by using the Eq.(12), respectively:

The interval type-2 fuzzy relational coefficient of each alternative from the PIS is defined as:

$$R_{ij}^+=\frac{1-\sin d(A_{ij},A_j^+)}{\cos(\min_{i=1}^{m}d(A_{ij},A_j^+))}\frac{\sum_{j=1}^{n}<A_{ij},A_j^+>}{\sum_{j=1}^{n}\|A_{ij}\|^2+\sum_{j=1}^{n}\left\|A_j^+\right\|^2-\sum_{j=1}^{n}<A_{ij},A_j^+>} \tag{28}$$

Similarly, the interval type-2 fuzzy relational coefficient of each alternative from the NIS is defined as:

$$R_{ij}^-=\frac{1-\sin d(A_{ij},A_j^-)}{\cos(\min_{i=1}^{m}d(A_{ij},A_j^-))}\frac{\sum_{j=1}^{n}<A_{ij},A_j^->}{\sum_{j=1}^{n}\|A_{ij}\|^2+\sum_{j=1}^{n}\left\|A_j^-\right\|^2-\sum_{j=1}^{n}<A_{ij},A_j^->} \tag{29}$$

Step 4: Calculate the weighted interval type-2 fuzzy relational degree of each alternative based on the PIS and NIS, we utilize the following two formulas, respectively:

$$R_i^+=\sum_{j=1}^{n}\frac{Rank(w_j)}{\sum_{j=1}^{n}Rank(w_j)}R_{ij}^+\quad(i=1,2,\cdots,m) \tag{30}$$

$$R_i^-=\sum_{j=1}^{n}\frac{Rank(w_j)}{\sum_{j=1}^{n}Rank(w_j)}R_{ij}^-\quad(i=1,2,\cdots,m) \tag{31}$$

Step 5: Calculate the interval type-2 fuzzy relative relational degree of each alternative as follows:

$$R_i = \frac{R_i^+}{R_i^+ + R_i^-} \quad (i = 1, 2, \cdots, m) \tag{32}$$

Step 6: Rank the alternatives $A_i (i = 1, 2, \cdots, m)$ based on the value of relative relational degree R_i. Obviously, the greater the value R_i, the better the alternative A_i will be.

Step 7: End.

VI. Numerical example

In this section, we consider an investment problem of the transport facilities, which aims to search the best investment alternative (adapted from [28]). There are seven potential investment alternatives $A_i (i = 1, 2, \cdots, 7)$ to be evaluated and four critical attributes should be considerate: (1) Resources(C_1); (2) Politics and Policy(C_2); (3) Economy(C_3); (4) Infrastructure(C_4). Assume that the decision maker (DM) use the linguistic terms from Table I to express the characteristics of the potential investment alternatives $A_i (i = 1, 2, \cdots, 7)$ with respect to four attributes $C_j (j = 1, 2, 3, 4)$ under interval type-2 fuzzy environment. The decision making information provide by DM as shown in Table II. The incomplete known attribute weights provided by DM is: $\Omega = \{0.25 \le w_1 \le 0.4, 0.3 \le w_2 \le 0.4, 0.2 \le w_3 \le 0.4, 0.2 \le w_4 \le 0.35\}$. Now, we apply our proposed method to the ranking and selection of the best investment alternative, which involves the following steps:

TABLE I. Linguistic terms and their corresponding interval type-2 fuzzy sets

Linguistic terms	Interval type-2 fuzzy sets
Very low(VL)	((0,0,0,0.1;1),(0,0,0,0.05;0.9))
Low(L)	((0,0.1,0.1,0.3;1),(0.05,0.1,0.1,0.2;0.9))
Medium low(ML)	((0.1,0.3,0.3,0.5;1),(0.2,0.3,0.3,0.4;0.9))
Medium(M)	((0.3,0.5,0.5,0.7;1),(0.4,0.5,0.5,0.6;0.9))
Medium high(MH)	((0.5,0.7,0.7,0.9;1),(0.6,0.7,0.7,0.8;0.9))
High(H)	((0.7,0.9,0.9,1;1),(0.8,0.9,0.9,0.95;0.9))
Very high(VH)	((0.9,1,1,1;1),(0.95,1,1,1;0.9))

TABLE II. Decision making information provided by decision maker

	Resources	Politics and Policy	Economy	Infrastructure
A_1	VH	M	H	MH
A_2	H	VH	ML	M
A_3	H	H	MH	H
A_4	VH	H	H	M
A_5	VH	H	H	MH
A_6	H	M	H	VH
A_7	VH	VH	ML	MH

A. Decision making procedure based on interval type-2 fuzzy relational degree

Step 1: Utilize Eqs. (20-23) to calculate the IT2FSs attribute weights.

Based on **Mod (1)-(3)**, we can obtain the optimal models from x-COG, y-COG, and the minimize fuzziness with $w_j (j = 1, 2, 3, 4)$, the results are shown as follows:

$$\begin{aligned}
x_{w_1}^* &= [x_{w_1^L}^*, x_{w_1^R}^*] = [0.25, 0.32] \\
x_{w_2}^* &= [x_{w_2^L}^*, x_{w_2^R}^*] = [0.30, 0.34] \\
x_{w_3}^* &= [x_{w_3^L}^*, x_{w_3^R}^*] = [0.20, 0.22] \\
x_{w_4}^* &= [x_{w_4^L}^*, x_{w_4^R}^*] = [0.20, 0.21] \\
y_{w_1}^* &= [y_{w_1^L}^*, y_{w_1^R}^*] = [0.2908, 0.3781] \\
y_{w_2}^* &= [y_{w_2^L}^*, y_{w_2^R}^*] = [0.2462, 0.3841] \\
y_{w_3}^* &= [y_{w_3^L}^*, y_{w_3^R}^*] = [0.2462, 0.3506] \\
y_{w_4}^* &= [y_{w_4^L}^*, y_{w_4^R}^*] = [0.2443, 0.2593] \\
F(w_1^*) &= [F(w_1^{L^*}), F(w_1^{R^*})] = [0.023, 0.038] \\
F(w_2^*) &= [F(w_2^{L^*}), F(w_2^{R^*})] = [0.010, 0.020] \\
F(w_3^*) &= [F(w_3^{L^*}), F(w_3^{R^*})] = [0.010, 0.020] \\
F(w_4^*) &= [F(w_4^{L^*}), F(w_4^{R^*})] = [0.015, 0.030]
\end{aligned}$$

Take w_1 as an example, we can obtain the parameter value of w_1 based on Eqs.(22-23) as follows:

$$\begin{cases} \frac{y_{w_1^L}^* (2a_{11}^L + 3d_1^L)(h_{w_1}^L - y_{w_1^L}^* + 1)}{2h_{w_1}^L} = 0.25 \\ \frac{h_{w_1}^L (\frac{d_1^L}{3d_1^L} + 2)}{6} = 0.2908 \\ \frac{5d_1^L (h_{w_1}^L)^2}{6} = 0.023 \end{cases} \tag{33}$$

and

$$\begin{cases} \frac{y_{w_1^R}^* (2a_{11}^R + 3d_1^R)(h_{w_1}^R - y_{w_1^R}^* + 1)}{2h_{w_1}^L} = 0.32 \\ \frac{h_{w_1}^R (\frac{d_1^R}{3d_1^R} + 2)}{6} = 0.3781 \\ \frac{5d_1^R (h_{w_1}^R)^2}{6} = 0.038 \end{cases} \tag{34}$$

By solving the linear equations above, we can obtain the interval type-2 fuzzy attribute weight w_1 as follows:

$$\begin{aligned} w_1 = ((&0.367, 0.416, 0.465, 0.514; 0.747), \\ &(0.438, 0.488, 0.538, 0.588; 0.956)) \end{aligned}$$

Similarly, we can solve other attribute weights $w_j (j = 2, 3, 4)$ respectively, the results are shown as follows:

$$\begin{aligned}
w_2 = ((&0.511, 0.540, 0.569, 0.598; 0.633), \\
&(0.461, 0.485, 0.509, 0.533; 0.987)) \\
w_3 = ((&0.325, 0.355, 0.385, 0.415; 0.633), \\
&(0.305, 0.334, 0.363, 0.392; 0.902)) \\
w_4 = ((&0.304, 0.349, 0.394, 0.439; 0.628), \\
&(0.262, 0.343, 0.424, 0.505; 0.667))
\end{aligned} \tag{35}$$

Step 2: Define the positive ideal solution (PIS) and the negative ideal solution (NIS) based on interval type-2 fuzzy numbers, respectively:

Based on Eq.(7), we can obtain the positive ideal solution (PIS) and the negative ideal solution (NIS) as follows:

1) Positive ideal solution (PIS)

$$
\begin{aligned}
A^{+} = \{&((0.9,1,1,1;1,1),(0.95,1,1,1;0.9,0.9)),\\
&((0.9,1,1,1;1,1),(0.95,1,1,1;0.9,0.9)),\\
&((0.7,0.9,0.9,1;1,1),(0.8,0.9,0.9,0.95;0.9,0.9)),\\
&((0.9,1,1,1;1,1),(0.95,1,1,1;0.9,0.9))\}
\end{aligned}
$$

2) Negative ideal solution (NIS)

$$
\begin{aligned}
A^{-} = \{&((0.7,0.9,0.9,1;1,1),(0.8,0.9,0.9,0.95;0.9,0.9)),\\
&((0.3,0.5,0.5,0.7;1,1),(0.4,0.5,0.5,0.6;0.9,0.9)),\\
&((0.1,0.3,0.3,0.5;1,1),(0.2,0.3,0.3,0.4;0.9,0.9)),\\
&((0.1,0.3,0.3,0.5;1,1),(0.2,0.3,0.3,0.4;0.9,0.9))\}
\end{aligned}
$$

Step 3: Calculate the interval type-2 fuzzy relational coefficient of each alternative from the PIS and NIS by using the Eq.(22), respectively.

Based on Eqs.(12-13) and Eqs.(28-29), we can calculate the relational coefficient of each alternative. Take R_{11}^{+} as an example, we have the following calculation process:

$$
\begin{aligned}
R_{11}^{+} &= \frac{1-\sin d(A_{11},A_1^{+})}{\cos(\min_{i=1}^{7} d(A_{i1},A_1^{+}))} \frac{\sum_{j=1}^{4} < A_{1j}, A_j^{+} >}{\sum_{j=1}^{4} \|A_{1j}\|^2 + \sum_{j=1}^{4} \left\|A_j^{+}\right\|^2 - \sum_{j=1}^{4} < A_{1j}, A_1^{+} >}\\
&= \frac{1-0}{\cos 0} \frac{37.935}{34.585+43.89-37.935} = 0.936
\end{aligned}
$$

Similarly, we can calculate the other relational coefficients of each alternative from the PIS and NIS, the results are shown as the following matrix form:

$$
R^{+} = \begin{bmatrix}
0.936 & 0.539 & 0.936 & 0.693\\
0.791 & 0.882 & 0.443 & 0.508\\
0.884 & 0.884 & 0.823 & 0.508\\
0.951 & 0.854 & 0.951 & 0.548\\
0.981 & 0.881 & 0.981 & 0.725\\
0.854 & 0.548 & 0.951 & 0.951\\
0.846 & 0.846 & 0.425 & 0.626
\end{bmatrix}
$$

$$
R^{-} = \begin{bmatrix}
0.794 & 0.884 & 0.444 & 0.734\\
0.929 & 0.536 & 0.929 & 0.929\\
0.886 & 0.589 & 0.576 & 0.589\\
0.786 & 0.582 & 0.439 & 0.875\\
0.781 & 0.578 & 0.437 & 0.723\\
0.858 & 0.858 & 0.431 & 0.494\\
0.829 & 0.533 & 0.924 & 0.767
\end{bmatrix}
$$

Step 4: Utilize Eq.(7) to calculate the ranking value of attribute weight $w_j (j = 1, 2, 3, 4)$, the results are shown as follows:

$$
\begin{aligned}
&R_1^{+} = 0.7581, R_2^{+} = 0.7036, R_3^{+} = 0.8043, R_4^{+} = 0.8454\\
&R_5^{+} = 0.9009, R_6^{+} = 0.7869, R_7^{+} = 0.7253, R_1^{-} = 0.7463\\
&R_2^{-} = 0.7964, R_3^{-} = 0.6725, R_4^{-} = 0.6666, R_5^{-} = 0.6359\\
&R_6^{-} = 0.7101, R_7^{-} = 0.7362.
\end{aligned}
$$

Step 5: Calculate the interval type-2 fuzzy relative relational degrees of each alternative based on Eq.(32), the results are shown as follows:

$$
\begin{aligned}
&R_1 = 0.5039, R_2 = 0.4691, R_3 = 0.5446, R_4 = 0.5591\\
&R_5 = 0.5862, R_6 = 0.5256, R_7 = 0.4962
\end{aligned}
$$

Since

$$R_5 > R_4 > R_3 > R_6 > R_1 > R_7 > R_2$$

Therefore, we have

$$A_5 \succ A_4 \succ A_3 \succ A_6 \succ A_1 \succ A_7 \succ A_2$$

Thus, the best investment alternative is A_5.

B. Comparisons of our proposed method with other existing methods

In what follows, we compare our proposed method with other previous methods including similarity measure method [20], possibility degree method [28], and ranking value method [10]. The results are shown in Table III.

TABLE III. COMPARISONS OF FOUR METHODS

Methods	Order of alternatives
Similarity method [20]	$A_5 \succ A_4 \succ A_3 \succ A_6 \succ A_1 \succ A_7 \succ A_2$
Possibility degree method [28]	$A_5 \succ A_4 \succ A_3 \succ A_6 \succ A_1 \succ A_7 \succ A_2$
Ranking value method [10]	$A_5 \succ A_4 \succ A_3 \succ A_6 \succ A_1 \succ A_7 \succ A_2$
The proposed method	$A_5 \succ A_4 \succ A_3 \succ A_6 \succ A_1 \succ A_7 \succ A_2$

From Table III, it is clear that four methods have the same ranking results, this verifies the method we proposed is reasonable and validity in this paper.

(1) Compared with Zeng and Li's similarity measure method in [20], the main advantage of our method is that we consider all the IT2FSs as a whole, not individual. Furthermore, the similarity measure proposed by Zeng and Li [20] can only reflect the relationship between two IT2FSs, which means that our method in this paper is more generalized and versatility. In addition, the proposed method only use the discreet reference points of the membership to calculate the relational degree whereas Zeng and Li's method must use the continues membership function to obtain the value of similarity, so the computational complexity of our method is much lower than Zeng and Li's method.

(2) Compared with possibility degree method in [28], the proposed method is based on interval type-2 fuzzy relational degree, and the method by Hu et al. [28] is based on possibility degree. In real decision making, the decision maker usually difficult to give a exact value of attribute weight under various uncertain factors, it is usually converted to IT2FSs information. In our method, we construct a optimization model to solve interval type-2 fuzzy attribute weights easily, so the method in this paper is more suitable for MADM problems under interval type-2 fuzzy environment.

(3) Compared with ranking value method in [10], the method in this paper propose a new uncertainty measure for IT2FSs, and extend to MADM problems. The ranking value method by Chen et al. [10] can only rank the order of any two IT2FSs, however, it is not reflect the correlation or similarity of IT2FSs. Moreover, the ranking value method include some complex operational laws and its geometric meaning is not clear, therefore, it is difficult to use in real-life decision making. Our method can overcome these drawbacks, because we give a simple relational degree measure to rank the IT2FSs and apply to MADM problem, the prominent characteristic is that the geometric meaning is clear and easy to use in practice.

According to the comparisons and analysis above, our proposed relational degree of IT2FSs method is better than the other three methods.

VII. Conclusion

Due to the powerful ability of IT2FSs in describing the uncertainty and fuzzy information, and motivated by the similarity measure of IT2FSs, we have first investigated the interval type-2 fuzzy metric spaces and propose the distance measure and inner product measure in this paper. Then, we introduce the concept of interval type-2 fuzzy relational degree and derive to a simple relational degree formula which satisfies the axiomatic definition we proposed. On the basis of our theoretical analysis, we construct a group of mathematical optimization models based on the center of gravity and fuzziness of IT2FSs to solve the optimal attribute weights. Furthermore, we have developed the procedure of interval type-2 fuzzy multiple attribute decision making problems. To illustrate the interval type-2 fuzzy relational analysis decision making method we proposed, an practical example is given. The result has shown that our method is effective and validity for handling MADM problems with interval type-2 fuzzy information.

For further research, we shall extend the proposed method to general type-2 fuzzy sets, and apply it to decision making, cluster analysis, computing with words and data mining.

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References

[1] L. A. Zadeh, “Concept of a linguistic variable and its application to approximate reasoning .1,” *Information Sciences*, vol. 8, no. 3, pp. 199-249, 1975.

[2] J. M. Mendel and R. I. John, “Type-2 fuzzy sets made simple,” *IEEE Transactions on Fuzzy Systems*, vol. 10, no. 2, pp. 117-127, 2002.

[3] D. Wu and J. M. Mendel, “Computing With Words for Hierarchical Decision Making Applied to Evaluating a Weapon System,” *IEEE Transactions on Fuzzy Systems*, vol. 18, no. 3, pp. 441-460, 2010.

[4] S.-M. Zhou, R. I. John, F. Chiclana, and J. M. Garibaldi, “On Aggregating Uncertain Information by Type-2 OWA Operators for Soft Decision Making,” *International Journal of Intelligent Systems*, vol. 25, no. 6, pp. 540-558, 2010.

[5] H. B. Mitchell, “Pattern recognition using type-II fuzzy sets,” *Information Sciences*, vol. 170, no. 2-4, pp. 409-418, 2005.

[6] T. Dereli and K. Altun, “Technology evaluation through the use of interval type-2 fuzzy sets and systems,” *Computers & Industrial Engineering*, vol. 65, no. 4, pp. 624-633, 2013.

[7] M. Ha, Y. Yang, and C. Wang, “A new support vector machine based on type-2 fuzzy samples,” *Soft Computing*, vol. 17, no. 11, pp. 2065-2074, 2013.

[8] C. Li, G. Zhang, M. Wang, and J. Yi, “Data-driven modeling and optimization of thermal comfort and energy consumption using type-2 fuzzy method,” *Soft Computing*, vol. 17, no. 11, pp. 2075-2088, 2013.

[9] T.-Y. Chen, C.-H. Chang, and J.-f. R. Lu, “The extended QUALIFLEX method for multiple criteria decision analysis based on interval type-2 fuzzy sets and applications to medical decision making,” *European Journal of Operational Research*, vol. 226, no. 3, pp. 615-625, 2013.

[10] S.-M. Chen, M.-W. Yang, L.-W. Lee, and S.-W. Yang, “Fuzzy multiple attributes group decision-making based on ranking interval type-2 fuzzy sets,” *Expert Systems with Applications*, vol. 39, no. 5, pp. 5295-5308, 2012.

[11] W. Wang, X. Liu, and Y. Qin, “Multi-attribute group decision making models under interval type-2 fuzzy environment,” *Knowledge-Based Systems*, vol. 30, pp. 121-128, 2012.

[12] E. Celik, O. N. Bilisik, M. Erdogan, A. T. Gumus, and H. Baracli, “An integrated novel interval type-2 fuzzy MCDM method to improve customer satisfaction in public transportation for Istanbul,”*Transportation Research Part E-Logistics and Transportation Review*, vol. 58, pp. 28-51, 2013.

[13] S. Naim, H. Hagras, “A Hybrid Approach for Multi-Criteria Group Decision Making Based on Interval Type-2 Fuzzy Logic and Intuitionistic Fuzzy Evaluation,” *IEEE International Conference on Fuzzy Systems*, 2012.

[14] K.-P. Chiao, “Trapezoidal Interval Type-2 Fuzzy Set Extension of Analytic Hierarchy Process,” *IEEE International Conference on Fuzzy Systems*, 2012.

[15] M. Z. A. Langroudi, A. Emrouznejad, A. Mustafa, and J. Ignatius, “Type-2 TOPSIS: A Group Decision Problem When Ideal Values are not Extreme Endpoints,” *Group Decision and Negotiation*, vol. 22, no. 5, pp. 851-866, 2013.

[16] T.-Y. Chen, “A PROMETHEE-based outranking method for multiple criteria decision analysis with interval type-2 fuzzy sets,” *Soft Computing*, in press, 2013.

[17] S. Naim, H. Hagras, “A general type-2 fuzzy logic based multi-criteria group decision making for lighting level selection in an intelligent environment,” *The 5th Computer Science and Electronic Engineering Conference*, 2013.

[18] S. Naim, H. Hagras, “A general type-2 fuzzy logic based approach for multi-criteria group decision making,” *IEEE International Conference on Fuzzy Systems*, pp. 1-8, 2013.

[19] D. Wu and J. M. Mendel, “Uncertainty measures for interval type-2 fuzzy sets,” *Information Sciences*, vol. 177, no. 23, pp. 5378-5393, 2007.

[20] W. Y. Zeng and H. X. Li, “Relationship between similarity measure and entropy of interval valued fuzzy sets,” *Fuzzy Sets and Systems*, vol. 157, no. 11, pp. 1477-1484, 2006.

[21] C.-M. Hwang, M.-S. Yang, W.-L. Hung, and E. S. Lee, “Similarity, inclusion and entropy measures between type-2 fuzzy sets based on the Sugeno integral,” *Mathematical and Computer Modelling*, vol. 53, no. 9-10, pp. 1788-1797, 2011.

[22] D. Zhai and J. M. Mendel, “Uncertainty measures for general Type-2 fuzzy sets,” *Information Sciences*, vol. 181, no. 3, pp. 503-518, 2011.

[23] T. Zhao, J. Xiao, Y.X. li and X.S. Deng, “A new approach to similarity and inclusion measures between general type-2 fuzzy sets,” *Soft Computing*, In press, 2013.

[24] J. M. Mendel, R. I. John, and F. Liu, “Interval type-2 fuzzy logic systems made simple,” *IEEE Transactions on Fuzzy Systems*, vol. 14, no. 6, pp. 808-821, 2006.

[25] L.-W. Lee and S.-M. Chen, “Fuzzy multiple attributes group decision-making based on the extension of TOPSIS method and interval type-2 fuzzy sets,” *Proceedings of 2008 International Conference on Machine Learning and Cybernetics*, vol. 1-7, pp. 3260-3265, 2008.

[26] S. J. Chen and S. M. Chen, “Fuzzy risk analysis based on similarity measures of generalized fuzzy numbers,” *IEEE Transactions on Fuzzy Systems*, vol. 11, no. 1, pp. 45-56, 2003.

[27] D.-F. Li, “A ratio ranking method of triangular intuitionistic fuzzy numbers and its application to MADM problems,” *Computers & Mathematics with Applications*, vol. 60, no. 6, pp. 1557-1570, 2010.

[28] J. Hu, Y. Zhang, X. Chen, and Y. Liu, “Multi-criteria decision making method based on possibility degree of interval type-2 fuzzy number,” *Knowledge-Based Systems*, vol. 43, pp. 21-29, 2013.

[29] S. Boyd and L. Vandenberghe, *Convex Optimization.* Cambridge University Press, Cambridge, UK. 2004.

[30] W. Rudin, *Functional analysis.* Mcgraw-Hill, New York, USA. 1991.