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A Distance Based Ranking Methods For Type-1 Fuzzy Numbers and Interval Type-2 Fuzzy Numbers

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***Abstract*—Although there are many methods for ranking type-1 fuzzy numbers, most of which exist some limitations. In this paper, we firstly propose a new distance measure based method to rank type-1 fuzzy numbers, which defines two formats of possibility mean and variation coefficient. It not only clearly discriminates the ranking of the type-1 fuzzy numbers especially for the reasonable ranking of symmetric type-1 fuzzy numbers, but also satisfies the consistence of the ranking with their images. Then, we extend such ranking method to interval type-2 fuzzy numbers. For ranking both the type-1 fuzzy numbers and interval type-2 fuzzy numbers, the proposed methods are easy to understand and their computations are simple. Several typical examples are used to illustrate our new ranking methods for type-1 fuzzy numbers and interval type-2 fuzzy numbers.**

I. INTRODUCTION

Ranking fuzzy numbers is one of the most important step for decision making [31], [21], [15]. So far, there are many ranking methods for type-1 fuzzy numbers. According to the computation principles, we divide the ranking methods into four categories: the centroid point method [33], [23], [34], [19], the minimizing and maximizing sets method [7], [11], [28], the left-right deviation degree (L-R deviation degree) method [30], [24], [35] and the distance method [13], [5], [1], [4].

For the centroid methods, Yagger [33] proposed the centroid index point. Lee and Li [17] proposed the mean and standard deviation values, but the comparison criteria is not clear. Cheng [10] proposed centroid ranking approach to improve Yagers [33] and Lee and Li [17] approaches, but the ranking is inconsistent with peoples' intuition. Chu and Tsao [12] defined the area between the centroid and original point. Wang and Lee [29] used the importance degrees to revise Chu and Tsao method [12], but the ranking result is inconsistent with peoples' intuition, either. For the maximizing and minimizing set methods, Chen [7] introduced the maximizing and minimizing sets, but the ranking is a relatively order. For the L-R deviation degree methods, Wang et al. [30] defined the L-R deviation degree. Asady [3] used the transfer coefficient to revise Wang et al. [30] method, but the ranking of their images is illogical. Nejad and Mashinchi [24] redefined the L-R deviation degree, but the ranking of the fuzzy numbers' images is illogical, either. Yu et al. [35] introduced the epsilon deviation degree to rank the symmetric type-1 fuzzy numbers, but the computation is too complexity. For the distance methods, Diamond and Kloeden [13] proposed the Euclidean distance. Asady and Zendehnam [5] introduced the minimization distance, but it could not discriminate the symmetric type-1 fuzzy numbers with different spread in the bottom. Abbasbandy and Hajjari [2] proposed the magnitude method to revise Asady and Zendehnam [5] method, but it still could not solve the problem. Ezzati et al. [14] revised the magnitude method, but the ranking of type-1 fuzzy numbers' images is not logical.

Compared with the ranking methods for type-1 fuzzy numbers, the ranking methods for interval type-2 fuzzy numbers are far less. Mitchell [22] proposed a method for ranking the interval type-2 fuzzy numbers. Lee and Chen [18], [8] employed mean value and deviation of the vertex points. Wu and Mendel [32] introduced a centroid method.

In this paper, we propose a distance based method to rank fuzzy numbers, that is the combination of the possibility mean and variation coefficient. When used to rank type-1 fuzzy numbers, it avoids the problems that the most existing ranking methods have. Firstly, the ranking is consistent with people's intuition; secondly, the ranking of the type-1 fuzzy numbers' images is also reasonable; thirdly, the symmetric type-1 fuzzy numbers having different spread in the bottom can also be discriminated. Meanwhile, the proposed method also satisfies the general ranking principles of the type-1 fuzzy numbers. Then, we extend the concept to interval type-2 fuzzy numbers environment, and present a new ranking method, which discriminates the interval type-2 fuzzy numbers especially for the symmetric interval type-2 fuzzy numbers with different spread in the bottom, and the ranking with their images' is reasonable as well.

This paper is organized as follows: Section II introduces the concepts of type-1 fuzzy numbers, interval type-2 fuzzy numbers and the related distance based ranking methods for type-1 fuzzy numbers. Section III introduces the main problems of existing ranking methods for type-1 fuzzy numbers, and proposes a new ranking method. Section IV proposes a ranking method for interval type-2 fuzzy numbers. Section V illustrates some examples to rank type-1 fuzzy numbers and interval type-2 fuzzy numbers, and compares the results with those of some existing ranking methods. Section VI summarizes the main results and draws conclusions.

II. PRELIMINARIES

In this section, we introduce the concepts of type-1 fuzzy numbers, interval type-2 fuzzy numbers and the related distance based ranking methods for type-1 fuzzy numbers.

A. The concepts of type-1 fuzzy numbers and interval type-2 fuzzy numbers

Definition 1. [28] A fuzzy number $\tilde{A}$ is a pair $(\underline{A}, \overline{A})$ of functions $\underline{A}(r)$, $\overline{A}(r)$, which satisfies the following conditions:

1) $\underline{A}(r)$ is a bounded increasing continuous function,
2) $\overline{A}(r)$ is a bounded decreasing continuous function,
3) $\underline{A}(r) \leq \overline{A}(r)$, $0 \leq r \leq 1$

The type-1 fuzzy number is a trapezoidal fuzzy number $\tilde{A} = (x_0, y_0, \alpha, \beta)_{(L,R)}$ with two defuzzifiers x_0, y_0, the left fuzziness α and the right fuzziness β, with which the membership function is defined as follows.

$$\underline{A}(x) = x_0 - \sigma + \sigma r, \quad \overline{A}(r) = y_0 + \beta - \beta r.$$

For the support set of $\tilde{A}(S(\tilde{A}))$ is defined as:

$$S(\tilde{A}) = \{x | \tilde{A}(x) > 0\} = [\underline{A}(r), \overline{A}(r)].$$

Definition 2. [20] An interval type-2 fuzzy number $\tilde{\tilde{A}}$ is an objective, which has the parametric form as:

$$\tilde{\tilde{A}} = \int_{x \in X} \int_{u \in J_x} 1/(x,u) = \int_{x \in X} \left[\int_{u \in J_x} 1/(x,u) \right] /x, \quad (1)$$

where x is the primary variable, $J_x \in [0,1]$ is the primary membership of x, u is the secondary variable, and $\int_{u \in J_x} 1/(x,u)$ is the secondary membership function at x.

The uncertainty footprint of $\tilde{\tilde{A}}(FOU(\tilde{\tilde{A}}))$ is defined as:

$$FOU(\tilde{\tilde{A}}) = \bigcup_{x \in X} J_x, \\ = \{(x,y) : y \in J_x = [\tilde{A}^U(x), \tilde{A}^L(x)]\},$$

where FOU is shown as the shaded region. It is bounded by an upper membership function (UMF) $\tilde{A}^U(x)$ and a lower membership function (LMF) $\tilde{A}^L(x)$, both of which are type-1 fuzzy numbers. An example of interval type-2 fuzzy number is shown in Fig. 1.

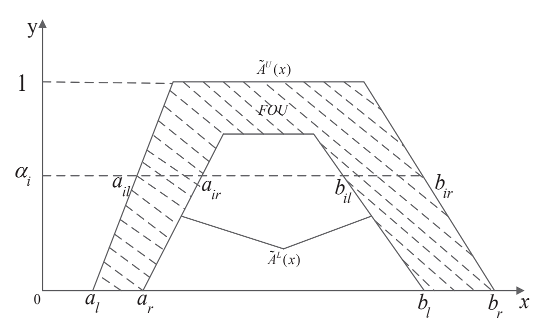


Fig. 1. The sample of interval type-2 fuzzy number

IT2 FSs is an useful tool to deal with vagueness and uncertainty in decision problems, which has been successfully used in many applications [25], [31], [16], [6], [9].

Wang and Kerre [26], [27] proposed some reasonable properties for the ordering of fuzzy quantities. Wu and Mendel [32] discussed the properties of the ordering methods for interval type-2 fuzzy numbers, which are denoted as follows.

Property 1. *For an arbitrary finite subset Γ of set E.*

1) *If $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$ and $\tilde{\tilde{B}} \succeq \tilde{\tilde{A}}$, then $\tilde{\tilde{A}} \sim \tilde{\tilde{B}}$.*
2) *If $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$ and $\tilde{\tilde{B}} \succeq \tilde{\tilde{C}}$, then $\tilde{\tilde{A}} \succeq \tilde{\tilde{C}}$.*
3) *If $\tilde{\tilde{A}} \cap \tilde{\tilde{B}} = \emptyset$ and $\tilde{\tilde{A}}$ is on the right of $\tilde{\tilde{B}}$, then $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$.*
4) *The order of $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is not affected by other interval type-2 fuzzy numbers under comparison.*
5) *If $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$, then $\tilde{\tilde{A}} + \tilde{\tilde{C}} \succeq \tilde{\tilde{B}} + \tilde{\tilde{C}}$.*
6) *If $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$, then $\tilde{\tilde{A}}\tilde{\tilde{C}} \succeq \tilde{\tilde{B}}\tilde{\tilde{C}}$.*

Where $\succeq$ means "larger than or equal to" in the sense of ranking, $\sim$ means "the same rank", $\cap$ means the overlap of two fuzzy sets.

Remark 1. The properties are also applied to type-1 fuzzy numbers, but not all the existing ranking methods satisfy the above properties at the same time.

B. The related distance based ranking methods for type-1 fuzzy numbers

Definition 3. [5] For a trapezoidal fuzzy number $\tilde{A} = (x_0, y_0, \alpha, \beta)$ with parametric form $\tilde{A} = (\underline{A}(r), \overline{A}(r))$, the magnitude of which is defined as:

$$Mag(A) = \frac{1}{2} \left(\int_0^1 (\underline{A}(r) + \overline{A}(r)) dr \right), \quad (2)$$

where $\underline{A}(r) = x_0 - \alpha + \alpha r, \quad \overline{A}(r) = y_0 + \beta - \beta r.$

Definition 4. [2] For a trapezoidal fuzzy number $\tilde{A} = (x_0, y_0, \alpha, \beta)$ with parametric form $\tilde{A} = (\underline{A}(r), \overline{A}(r))$, the revised magnitude of which is defined as:

$$M(A) = \frac{1}{2} \left(\int_0^1 (\underline{A}(r) + \overline{A}(r) + x_0 + y_0) f(r) dr \right), \quad (3)$$

where $f(r)$ is an increasing function satisfying $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(r) dr = \frac{1}{2}$,

$$\underline{A}(r) = x_0 - \alpha + \alpha r, \quad \overline{A}(r) = y_0 + \beta - \beta r.$$

Definition 5. [4] For a trapezoidal fuzzy number $\tilde{A} = (x_0, y_0, \alpha, \beta)_{(L,R)}$ with parametric form $\tilde{A} = (\underline{A}_\epsilon(r), \overline{A}_\epsilon(r))$, the magnitude of which is redefined as:

$$Mag(\tilde{A}_\epsilon) \\ = \frac{1}{2} \int_0^1 \left(\underline{A}_\epsilon(r) + \overline{A}_\epsilon(r) \right) dr, \\ = \frac{(x_0 + y_0) + (\beta - \alpha) + \frac{\sigma}{1+L} \left(L + \epsilon^{\frac{1+L}{L}} \right) - \frac{\beta}{1+R} \left(R + \epsilon^{\frac{1+R}{R}} \right)}{2},$$

where $A_\epsilon = (\underline{A}_\epsilon(r), \overline{A}_\epsilon(r))$ is the best approximate epsilon-neighborhood of fuzzy number $\tilde{A}$.

For two type-1 fuzzy numbers $\tilde{A}$ and $\tilde{B}$, the ranking criteria based on the Asady method [4] is defined as follows.

1) If $Mag(\tilde{A}) > Mag(\tilde{B})$, then $\tilde{A} \succ \tilde{B}$;
2) If $Mag(\tilde{A}) < Mag(\tilde{B})$, then $\tilde{A} \prec \tilde{B}$;

3) If $Mag(\tilde{A}) = Mag(\tilde{B})$, then
   a) if $Mag(\tilde{A}_\epsilon) > Mag(\tilde{B}_\epsilon)$, then $\tilde{A} \succ \tilde{B}$;
   b) if $Mag(\tilde{A}_\epsilon) < Mag(\tilde{B}_\epsilon)$, then $\tilde{A} \prec \tilde{B}$;
   c) else $\tilde{A} \sim \tilde{B}$.

III. The New Ranking Methods for Type-1 Fuzzy Numbers

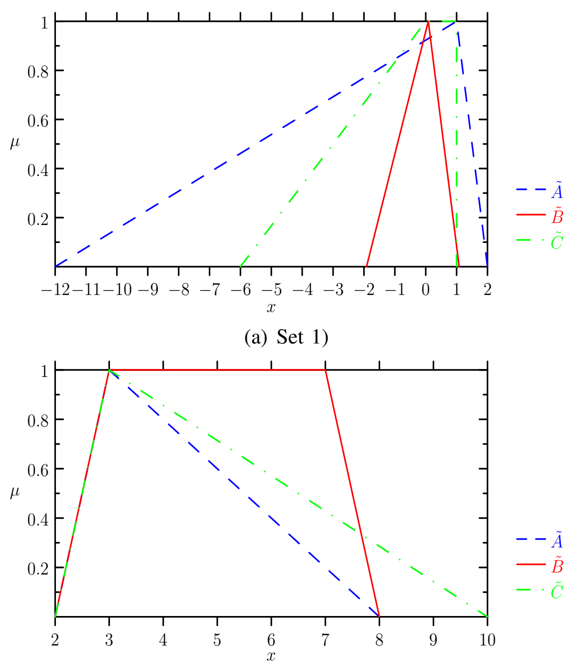
A. The main problems of existing ranking methods for type-1 fuzzy numbers

We give several examples to illustrate the main problems of existing ranking methods for type-1 fuzzy numbers. Here, we just list several methods to present the problems, more methods will be shown in the comparison of ranking results with the proposed method.

Problem 1. The ranking results are not consistent with peoples' intuition, which is shown in Example 1.

Example 1. Consider the following sets of type-1 fuzzy numbers, which are shown in Fig. 2.

1) $\tilde{A} = (1, 13, 1)$, $\tilde{B} = (\frac{1}{12}, 2, 1)$ and $\tilde{C} = (0, 1, 6, 0)$ in Ref. [4].
2) $\tilde{A} = (3, 1, 5)$, $\tilde{B} = (3, 7, 1, 1)$ and $\tilde{C} = (3, 1, 7)$ in Ref. [24].



(a) Set 1)

(b) Set 2)

Fig. 2. The type-1 fuzzy numbers of Example 1

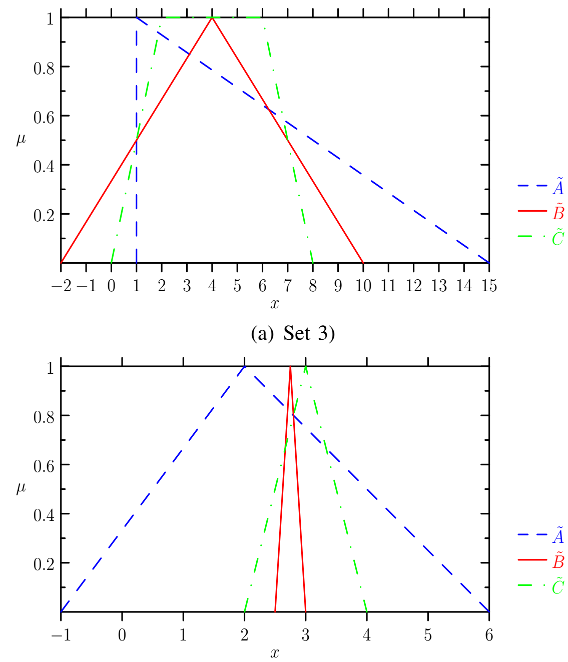
In Set 1), which is shown in Fig. 2(a), the ranking order for Wang and Lee [29] is $\tilde{C} \succ \tilde{B} \succ \tilde{A}$, which is not consistent with our intuition.

In Set 2), which is shown in Fig. 2(b), the ranking order for Wang et al. [30] method is worthless. As the transfer coefficient $\lambda_{\tilde{B}} = \lambda_{\tilde{C}} = 0$, which leads to the left deviation degree $d^L_{\tilde{B}}$ and $d^L_{\tilde{C}}$ are worthless.

Problem 2. The ranking results of the type-1 fuzzy numbers' images are not logical, which is shown in Example 2.

Example 2. Consider the following sets of type-1 fuzzy numbers, which are shown in Fig. 3.

3) $\tilde{A} = (1, 0, 14)$, $\tilde{B} = (4, 6, 6)$ and $\tilde{C} = (2, 6, 2, 2)$ in Ref. [14].
4) $\tilde{A} = (2, 1, 4)$, $\tilde{B} = (2.75, 0.25, 0.25)$ and $\tilde{C} = (3, 1, 1)$ in Ref. [24].



(a) Set 3)

(b) Set 4)

Fig. 3. The type-1 fuzzy numbers of Example 2

In Set 3), the ranking order for Ezzati et al. method [14] is $\tilde{A} \succ \tilde{C} \succ \tilde{B}$, but the ranking of their images is $-\tilde{C} \succ -\tilde{B} \succ -\tilde{A}$, which is illogical.

In Set 4), the ranking order for Nejad and Mashinchi method [24] is $\tilde{C} \succ \tilde{B} \succ \tilde{A}$, but the ranking of their images is $-\tilde{B} \succ -\tilde{A} \succ -\tilde{C}$, which is illogical either.

Problem 3. The ranking result of symmetric type-1 fuzzy numbers having different spread in the bottom is not reasonable, which is shown in Example 3.

Example 3. Consider the two type-1 fuzzy numbers $\tilde{A} = (0.5, 0.3, 0.3)$, $\tilde{B} = (0.5, 0.1, 0.1)$, which are shown in Fig. 4. The ranking order for Abbasbandy and Hajjari method [2] and the Asady method [4] is $\tilde{A} \sim \tilde{B}$, which is not reasonable, bacause they actually present different fuzzy information.

B. The proposed ranking method for type-1 fuzzy numbers

To overcome the problems shown above, we introduce a new concept to rank type-1 fuzzy numbers, that is the combination of possibility mean and variation coefficient. In the proposed ranking method, the variation coefficient is a new format, and the possibility mean is originated from Ref [2], which are defined as follows.

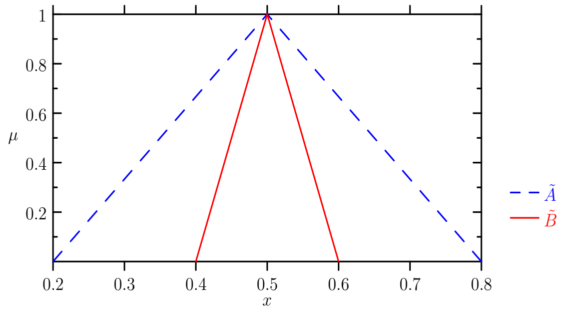


Fig. 4. The type-1 fuzzy numbers of Example 3

Definition 6. [2] For any type-1 fuzzy number $\tilde{A} = (x_0, y_0, \alpha, \beta)$ with parametric form $\tilde{A} = (\underline{A}(r), \overline{A}(r))$, the possibility mean of which is defined as:

$$M(A) = \frac{1}{2}\int_0^1 \left(\underline{A}(r) + \overline{A}(r) + x_0 + y_0\right) f(r)dr, \tag{4}$$

where $f(r)$ is an increasing function satisfying $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(r)dr = \frac{1}{2}$,

$$\underline{A}(r) = x_0 - \alpha + \alpha r, \quad \overline{A}(r) = y_0 + \beta - \beta r. \tag{5}$$

Definition 7. For any arbitrary type-1 fuzzy number $\tilde{A} = (x_0, y_0, \alpha, \beta)$, the variation coefficient of which is defined as:

$$VC(\tilde{A}) = \begin{cases} \frac{D(\tilde{A})}{M(\tilde{A})}, & \text{if} \quad M(\tilde{A}) \neq 0, \\ \frac{D(\tilde{A})}{\epsilon}, & \text{if} \quad M(\tilde{A}) = 0. \end{cases} \tag{6}$$

where ϵ is an extremely small value to present the approximate $M(\tilde{A})$, $D(\tilde{A})$ is the deviation value of the type-1 fuzzy number, and the expression is defined as:

$$D(\tilde{A}) = \frac{1}{4}\int_0^1 \left(\overline{A}(r) + y_0 - \underline{A}(r) - x_0\right)^2 f(r)dr, \tag{7}$$

where $f(r)$ is an increasing function satisfying $f(0) = 0$, $f(1) = 1$ and $\int_0^1 f(r)dr = \frac{1}{2}$.

It is denoted that ϵ can be seen as positive and negative number. For example, if the possibility mean value of type-1 fuzzy number $\tilde{A}$ is approximated to zero from the positive number and the corresponding approximating is ϵ, then the image of $\tilde{A}$ $(-\tilde{A})$ must be approximated to zero from the negative number, and the corresponding approximating is $-\epsilon$.

Regarding the proposed ranking method for type-1 fuzzy numbers, the possibility mean value represents information from the membership degree, variation coefficient reflects the change rate of span length from the right side to the left side. The combination of which not only compares the information of the type-1 fuzzy numbers, but also discriminates the type-1 fuzzy numbers having the same possibility mean value from their variation rate. Hence, we use the combination of both to rank the type-1 fuzzy numbers.

Next, we introduce new ordering rules for the proposed ranking method. For any two type-1 fuzzy numbers $\tilde{A}$ and $\tilde{B}$, the comparison criteria is carried out as follows.

Definition 8. Let $\tilde{A}$ and $\tilde{B}$ be two type-1 fuzzy numbers,

1) If $M(\tilde{A}) > M(\tilde{B})$, then $\tilde{A} \succ \tilde{B}$;
2) If $M(\tilde{A}) < M(\tilde{B})$, then $\tilde{A} \prec \tilde{B}$;
3) If $M(\tilde{A}) = M(\tilde{B})$, then
   a) if $VC(\tilde{A}) > VC(\tilde{B})$, then $\tilde{A} \succ \tilde{B}$;
   b) if $VC(\tilde{A}) < VC(\tilde{B})$, then $\tilde{A} \prec \tilde{B}$;
   c) else $\tilde{A} \sim \tilde{B}$.

Hence, we rank $\tilde{A}$ and $\tilde{B}$ based on their possibility mean if the two values are different. Otherwise, we further compare the variation coefficient to identify their rankings.

In the following, we study some properties of the proposed ranking method for type-1 fuzzy numbers.

Theorem 1. *For the type-1 fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$,*

1) *If $\tilde{A} \succeq \tilde{B}$ and $\tilde{B} \succeq \tilde{A}$, then $\tilde{A} \sim \tilde{B}$.*
2) *If $\tilde{A} \succeq \tilde{B}$ and $\tilde{B} \succeq \tilde{C}$, then $\tilde{A} \succeq \tilde{C}$.*
3) *If $\tilde{A} \cap \tilde{B} = \emptyset$ and $\tilde{A}$ is on the right of $\tilde{B}$, then $\tilde{A} \succeq \tilde{B}$.*
4) *The order of $\tilde{A}$ and $\tilde{B}$ is not affected by other type-1 fuzzy numbers under comparison.*
5) *If $\tilde{A} \succeq \tilde{B}$, then $\tilde{A} + \tilde{C} \succeq \tilde{B} + \tilde{C}$.*

Where $\succeq$ means "larger than or equal to" in the sense of ranking, $\sim$ means "the same rank", $\cap$ means the overlap of the two fuzzy sets.

Proof: The proof of Theorem 1. 1) - 4) are easy, we here give the proof process of Theorem 1. 5) in detail.

Suppose $\tilde{A} = (x_a, y_a, \sigma_a, \beta_a)$, $\tilde{B} = (x_b, y_b, \sigma_b, \beta_b)$ and $\tilde{C} = (x_c, y_c, \sigma_c, \beta_c)$.

As the combination of possibility mean and variation coefficient is used to rank the type-1 fuzzy numbers, there are two cases to rank type-1 fuzzy numbers $\tilde{A}$ and $\tilde{B}$, which can be shown as follows.

1) $\tilde{A} \succeq \tilde{B}$ if and only if $M(\tilde{A}) \succeq M(\tilde{B})$. From Eq. (4), the possibility mean value of $\tilde{A} + \tilde{C}$ and $\tilde{B} + \tilde{C}$ can be written as:

$$M(\tilde{A} + \tilde{C}) = \frac{1}{2}\int_0^1 \big(\underline{A}(r) + \underline{C}(r) + \overline{A}(r) + \overline{C}(r) + x_a + x_c + y_a + y_c\big) f(r)dr,$$

and

$$M(\tilde{B} + \tilde{C}) = \frac{1}{2}\int_0^1 \big(\underline{B}(r) + \underline{C}(r) + \overline{B}(r) + \overline{C}(r) + x_b + x_c + y_b + y_c\big) f(r)dr.$$

So

$$\begin{aligned} &M(\tilde{A} + \tilde{C}) - M(\tilde{B} + \tilde{C}) \\ &= \frac{1}{2}\int_0^1 \big(\underline{A}(r) + \overline{A}(r) + x_a + y_a - \\ &\underline{B}(r) - \overline{B}(r) - x_b - y_b\big) f(r)dr. \end{aligned} \tag{8}$$

Because of $M(\tilde{A}) \succeq M(\tilde{B})$, it is right that

$$\frac{1}{2}\int_0^1 \big(\underline{A}(r) + \overline{A}(r) + x_a + y_a - \underline{B}(r) - \overline{B}(r) - x_b - y_b\big) f(r)dr \geq 0, \tag{9}$$

Hence, from Eqs. (8),(9), it is concluded that $M(A+C)-M(B+C)\geq 0$. That is

$$\tilde{A}+\tilde{C}\succeq\tilde{B}+\tilde{C}.$$

2) $\tilde{A}\succeq\tilde{B}$ if and only if $M(\tilde{A})=M(\tilde{B})$ and $VC(\tilde{A})\succeq VC(\tilde{B})$.
Because of $M(\tilde{A})=M(\tilde{B})$, it is right that $M(\tilde{A}+\tilde{C})=M(\tilde{B}+\tilde{C})$.
From Eq. (7), it is obvious that

$$D(\tilde{A}+\tilde{C})=\frac{1}{4}\int_0^1\big(\overline{A}(r)+y_a-\underline{A}(r)-x_a+\overline{C}(r)+y_c-\underline{C}(r)-x_c\big)^2 f(r)dr,$$

and

$$D(\tilde{B}+\tilde{C})=\frac{1}{4}\int_0^1\big(\overline{B}(r)+y_b-\underline{B}(r)-x_b+\overline{C}(r)+y_c-\underline{C}(r)-x_c\big)^2 f(r)dr.$$

Then

$$D(\tilde{A}+\tilde{C})-D(\tilde{B}+\tilde{C})=\frac{1}{4}\int_0^1\big(\overline{A}(r)+y_a-\underline{A}(r)-x_a+\overline{B}(r)+y_b-\underline{B}(r)-x_b+2\big(\overline{C}(r)+y_c-\underline{C}(r)-x_c\big)\big)\big(\overline{A}(r)+y_a-\underline{A}(r)-x_a-\big(\overline{B}(r)+y_b-\underline{B}(r)-x_b\big)\big)f(r)dr. \tag{10}$$

As $M(\tilde{A})=M(\tilde{B})$ and $VC(\tilde{A})\succeq VC(\tilde{B})$, from Eq. (6), it is concluded that $D(\tilde{A})\geq D(\tilde{B})$.
That is

$$D(\tilde{A})-D(\tilde{B})=\frac{1}{4}\int_0^1\bigg(\overline{A}(r)+y_a-\underline{A}(r)-x_a+\overline{B}(r)+y_b-\underline{B}(r)-x_b\bigg)\bigg(\overline{A}(r)+y_a-\underline{A}(r)-x_a-\big(\overline{B}(r)+y_b-\underline{B}(r)-x_b\big)\bigg)f(r)dr\geq 0.$$

Since $\overline{A}(r)+y_a-\underline{A}(r)-x_a+\overline{B}(r)+y_b-\underline{B}(r)-x_b\geq 0$, it is concluded that

$$\overline{A}(r)+y_a-\underline{A}(r)-x_a-(\overline{B}(r)+y_b-\underline{B}(r)-x_b)\geq 0 \tag{11}$$

In Eq. (10), as $(\overline{A}(r)+y_a-\underline{A}(r)-x_a+\overline{B}(r)+y_b-\underline{B}(r)-x_b+2(\overline{C}(r)+y_c-\underline{C}(r)-x_c))\geq 0$.
Combined with the conclusion of Eq. (11), it is right that $D(\tilde{A}+\tilde{C})-D(\tilde{B}+\tilde{C})\geq 0$. That is $\tilde{A}+\tilde{C}\succeq\tilde{B}+\tilde{C}$.

■

IV. The Ranking Method for Interval Type-2 Fuzzy Numbers

In this section, we extend the concepts of possibility mean and variation coefficient to interval type-2 fuzzy numbers. Detailed definitions and expressions can be found in the following.

Definition 9. For an arbitrary interval type-2 fuzzy number $\tilde{\tilde{A}}=((\alpha_1,a,b,\beta_1),(\alpha_2,c,d,\beta_2);H_1(x_1),H_2(x_2))$, the possibility mean of which is defined as:

$$M(\tilde{\tilde{A}})=\frac{M(\tilde{A}^U)+M(\tilde{A}^L)}{2}, \tag{12}$$

where the possibility mean values of the LMF and UMF are written as:

$$M(\tilde{A}^U)=\frac{1}{2}\int_0^{H_2(x_2)}(\underline{u}^U(r)+\overline{u}^U(r)+c+d)f(r)dr, \tag{13}$$

$$M(\tilde{A}^L)=\frac{1}{2}\int_0^{H_1(x_1)}(\underline{u}^L(r)+\overline{u}^L(r)+a+b)f(r)dr, \tag{14}$$

$f(r)$ is an increasing function satisfying $f(0)=0$, $f(1)=1$ and $\int_0^1 f(r)dr=\frac{1}{2}$.

Definition 10. For any interval type-2 fuzzy number $\tilde{\tilde{A}}=((\alpha_1,a,b,\beta_1),(\alpha_2,c,d,\beta_2);H_1(x_1),H_2(x_2))$, the variation coefficient of which is defined as:

$$VC(\tilde{\tilde{A}})=\begin{cases}\frac{D(\tilde{\tilde{A}})}{M(\tilde{\tilde{A}})}, & \text{if} \quad M(\tilde{\tilde{A}})\neq 0,\\ \frac{D(\tilde{\tilde{A}})}{\epsilon}, & \text{if} \quad M(\tilde{\tilde{A}})=0.\end{cases} \tag{15}$$

where ϵ is an extremely small value to present the approximate $M(\tilde{\tilde{A}})$, $D(\tilde{\tilde{A}})$ is the variation value, and the expression is defined as:

$$D(\tilde{\tilde{A}})=\sqrt{D(\tilde{A}^U)D(\tilde{A}^L)}, \tag{16}$$

and

$$D(\tilde{A}^U)=\frac{1}{4}\int_0^{H_1(x_1)}(\overline{u}^U(r)+d-\underline{u}^U(r)-c)^2 f(r)dr, \tag{17}$$

$$D(\tilde{A}^L)=\frac{1}{4}\int_0^{H_2(x_1)}(\overline{u}^L(r)+b-\underline{u}^L(r)-a)^2 f(r)dr, \tag{18}$$

$f(r)$ is an increasing function satisfying $f(0)=0$, $f(1)=1$ and $\int_0^1 f(r)dr=\frac{1}{2}$.

It is denoted that ϵ can be denoted as positive and negative number, for example, if the possibility mean value of interval type-2 fuzzy number $\tilde{\tilde{A}}$ is approximated to zero from the positive number and the corresponding approximating is ϵ, then the image of $\tilde{\tilde{A}}$ $(-\tilde{\tilde{A}})$ must be approximated to zero from the negative number, and the corresponding approximating is $-\epsilon$.

Regarding the proposed ranking method for interval type-2 fuzzy numbers, the possibility mean represents the information from the LMF to UMF, variation coefficient reflects the change rate of span length from the right side to the left side. The combination of which not only compares the information of the interval type-2 fuzzy numbers, but also discriminates the interval type-2 fuzzy numbers having the same possibility mean from their variation rate. Hence, we also use the combination of both to rank interval type-2 fuzzy numbers.

Next, we introduce new ordering rules for the proposed ranking method. For any two interval type-2 fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$, the comparison criteria is carried out as follows.

Definition 11. Let $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ be two interval type-2 fuzzy numbers.

1) If $M(\tilde{\tilde{A}}) > M(\tilde{\tilde{B}})$, then $\tilde{\tilde{A}} \succ \tilde{\tilde{B}}$;
2) If $M(\tilde{\tilde{A}}) < M(\tilde{\tilde{B}})$, then $\tilde{\tilde{A}} \prec \tilde{\tilde{B}}$;
3) If $M(\tilde{\tilde{A}}) = M(\tilde{\tilde{B}})$, then (a)
    a) if $VC(\tilde{\tilde{A}}) > VC(\tilde{\tilde{B}})$, then $\tilde{\tilde{A}} \succ \tilde{\tilde{B}}$;
    b) if $VC(\tilde{\tilde{A}}) < VC(\tilde{\tilde{B}})$, then $\tilde{\tilde{A}} \prec \tilde{\tilde{B}}$;
    c) else $\tilde{\tilde{A}} \sim \tilde{\tilde{B}}$.

Regarding the interval type-2 fuzzy numbers, we rank $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ based on their possibility mean values $M(\tilde{\tilde{A}})$ and $M(\tilde{\tilde{B}})$ if the two values are different. Otherwise, we further compare the variation coefficients $VC(\tilde{\tilde{A}})$ and $VC(\tilde{\tilde{B}})$ to identify their rankings.

In the following, we study some properties of the proposed ranking method for interval type-2 fuzzy numbers.

Theorem 2. *For the interval type-2 fuzzy numbers $\tilde{\tilde{A}}$, $\tilde{\tilde{B}}$ and $\tilde{\tilde{C}}$, there are some properties as follows.*

1) *If $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$ and $\tilde{\tilde{B}} \succeq \tilde{\tilde{A}}$, then $\tilde{\tilde{A}} \sim \tilde{\tilde{B}}$.*
2) *If $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$ and $\tilde{\tilde{B}} \succeq \tilde{\tilde{C}}$, then $\tilde{\tilde{A}} \succeq \tilde{\tilde{C}}$.*
3) *If $\tilde{\tilde{A}} \cap \tilde{\tilde{B}} = \emptyset$ and $\tilde{\tilde{A}}$ is on the right of $\tilde{\tilde{B}}$, then $\tilde{\tilde{A}} \succeq \tilde{\tilde{B}}$.*
4) *The order of $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$ is not affected by other interval type-2 fuzzy numbers under comparison.*

Where $\succeq$ means "larger than or equal to" in the sense of ranking, $\sim$ means "the same rank", $\cap$ means the overlap of the two fuzzy sets.

Proof: The proof of Theorem 2 1) - 4) are easy, we omit here. ■

Remark 2. In view of the computing complexity of the existing ranking methods for interval type-2 fuzzy numbers, the proposed ranking method just satisfies the conclusions of Theorem 1 1) - 4).

V. EXAMPLES

A. The ranking for type-1 fuzzy numbers

To present the meaning of the proposed method with possibility mean and variation coefficient, it is used to solve the three kinds of problems in Section III-A. An additional example is listed to present the property of Theorem 1 5).

Example 1. Consider the three type-1 fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ of Set 1) shown in Figure 2(a). Using the proposed method, the ranking of the three type-1 fuzzy numbers and their images is $\tilde{B} \succ \tilde{C} \succ \tilde{A}$ and $-\tilde{A} \succ -\tilde{C} \succ -\tilde{B}$, respectively.

Using the same method, the three type-1 fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ of Set 2) shown in Fig. 2(b) and their images is ranked as $\tilde{B} \succ \tilde{C} \succ \tilde{A}$ and $-\tilde{A} \succ -\tilde{C} \succ -\tilde{B}$, respectively.

Table I lists the ranking results with some existing methods and the proposed method. From which it is concluded that the proposed method is able to overcome the condition that

TABLE I. THE COMPARATIVE RESULTS OF EXAMPLE 1

Authors	Set	A	B	C	Ranking
Wang and Lee	1)	-3	-0.25	-1.375	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$
(2008) [29]	2)	4.33	$(5, 0.5)$	$(5, 0.96)$	$\tilde{C} \succ \tilde{B} \succ \tilde{A}$
Wang et al.	1)	0	0.05	8.4	$\tilde{C} \succ \tilde{B} \succ \tilde{A}$
(2009) [30]	2)	0	0.44	0.44	$\tilde{C} \sim \tilde{B} \succ \tilde{A}$
Asady	1)	2.22	4.44	3.14	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$
(2010) [3]	2)	0.31	0.55	0.42	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$
Asady	1)	-2	-0.17	-1.33	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$
(2011) [4]	2)	4	5	4.5	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$
The proposed	1)	$(0, \frac{-4.08}{\epsilon})$	$(0, \frac{-0.19}{\epsilon})$	$(0, \frac{-2.25}{\epsilon})$	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$
method	2)	3.33	5	3.5	$\tilde{B} \succ \tilde{C} \succ \tilde{A}$

the ranking is not consistent with peoples' intuition and the limitation of Wang et al. [30] method.

Example 2. Consider the three type-1 fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ of Set 3) shown in Fig. 3(a). Using the proposed method, the ranking of the three type-1 fuzzy numbers and their images is $\tilde{C} \succ \tilde{B} \succ \tilde{A}$ and $-\tilde{A} \succ -\tilde{B} \succ -\tilde{C}$, respectively.

Using the same method, the three type-1 fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ of Set 4) shown in Fig. 3(b) and their images is ranked as $\tilde{C} \succ \tilde{B} \succ \tilde{A}$ and $-\tilde{A} \succ -\tilde{B} \succ -\tilde{C}$, respectively.

Table II lists the ranking results with some existing methods and the proposed method. From the comparative ranking results, it is clear that the proposed method is able to overcome the defect that the ranking of their images' is not logical.

TABLE II. THE COMPARATIVE RESULTS OF EXAMPLE 2

Authors	Set	$\tilde{A}$	$\tilde{B}$	$\tilde{C}$	$-\tilde{A}$	$-\tilde{B}$	$-\tilde{C}$
Wang et al.	3)	2.17	4	3.33	-2.17	-4	-3.33
(2009) [30]		$\tilde{B} \succ \tilde{C} \succ \tilde{A}$			$-\tilde{A} \succ -\tilde{C} \succ -\tilde{B}$		
	4)	0.42	0	1.17	0	2.12	0
		$\tilde{C} \succ \tilde{A} \succ \tilde{B}$			$-\tilde{B} \succ -\tilde{A} \sim -\tilde{C}$		
Asady	3)	12	0	0	0	22	4.75
(2010) [3]		$\tilde{A} \succ \tilde{B} \sim \tilde{C}$			$-\tilde{B} \succ -\tilde{C} \succ -\tilde{A}$		
	4)	0.47	0	0.57	0	1.44	1.2
		$\tilde{C} \succ \tilde{A} \succ \tilde{B}$			$-\tilde{B} \succ -\tilde{C} \succ -\tilde{A}$		
Nejad and	3)	0.18	0.22	0.28	1.41	1.65	1.28
Mashinchi		$\tilde{C} \succ \tilde{B} \succ \tilde{A}$			$-\tilde{B} \succ -\tilde{A} \succ -\tilde{C}$		
(2011) [24]	4)	2	0	1.5	0	3.13	0
		$\tilde{A} \succ \tilde{C} \succ \tilde{B}$			$-\tilde{B} \succ -\tilde{A} \sim -\tilde{C}$		
Ezzati et al.	3)	2.17	$4 + 6\sigma$	$4 + 4\sigma$	-2.17	$-4 + 6\sigma$	$-4 + 4\sigma$
(2012) [14]		$\tilde{B} \succ \tilde{C} \succ \tilde{A}$			$-\tilde{A} \succ -\tilde{B} \succ -\tilde{C}$		
	4)	2.25	2.75	3	-2.25	-2.75	-3
		$\tilde{C} \succ \tilde{B} \succ \tilde{A}$			$-\tilde{A} \succ -\tilde{B} \succ -\tilde{C}$		
The proposed	3)	2.17	4	3.33	-2.17	-4	-3.33
method		$\tilde{B} \succ \tilde{C} \succ \tilde{A}$			$-\tilde{A} \succ -\tilde{C} \succ -\tilde{B}$		
	4)	2.25	2.75	3	-2.25	-2.75	-3
		$\tilde{C} \succ \tilde{B} \succ \tilde{A}$			$-\tilde{A} \succ -\tilde{B} \succ -\tilde{C}$		

Example 3. Consider the two type-1 fuzzy numbers $\tilde{A}$ and $\tilde{B}$ shown in Fig. 4. Using the proposed method, the ranking of the two type-1 fuzzy numbers and their images is $\tilde{A} \succ \tilde{B}$ and $-\tilde{B} \succ -\tilde{A}$, respectively.

Table III lists the ranking results with the proposed method. From the comparative ranking results, it is obvious that the proposed method is able to overcome the defeat that the ranking of the symmetric type-1 fuzzy numbers is not reasonable.

Example 4 demonstrates the property of Theorem 1 5).

Example 4. Consider the two type-1 fuzzy numbers $\tilde{A}$ and $\tilde{B}$, $\tilde{C}$ and $\tilde{C}'$ are two arbitrary type-1 fuzzy numbers shown

TABLE III. THE COMPARATIVE RESULTS OF EXAMPLE 3

Authors	$\tilde{A}$	$\tilde{B}$	$-\tilde{A}$	$-\tilde{B}$
Abbasbandy and Hajjari (2009) [2]	0.5 $\tilde{A} \sim \tilde{B}$	0.5	-0.5 $-\tilde{A} \sim -\tilde{B}$	-0.5
Asady (2011) [4]	0.5 $\tilde{A} \sim \tilde{B}$	0.5	-0.5 $-\tilde{A} \sim -\tilde{B}$	-0.5
Ezzati et al. (2012) [14]	0.65 $\tilde{A} \succ \tilde{B}$	0.55	-0.35 $-\tilde{A} \succ -\tilde{B}$	-0.45
The proposed method	$(0.5, 0.015)$ $\tilde{A} \succ \tilde{B}$	$(0.5, 0.002)$	$(-0.5, -0.015)$ $-\tilde{B} \succ -\tilde{A}$	$(-0.5, -0.002)$

in Fig. 5. Let $\tilde{A}' = \tilde{A} + \tilde{C}$, $\tilde{B}' = \tilde{B} + \tilde{C}$, $\tilde{A}'' = \tilde{A} + \tilde{C}'$, $\tilde{B}'' = \tilde{B} + \tilde{C}'$.

Using the proposed method, the ranking of the two type-1 fuzzy numbers is $\tilde{A} \succ \tilde{B}$, $\tilde{A}' \succ \tilde{B}'$ and $\tilde{A}'' \succ \tilde{B}''$, respectively.

Table IV lists the ranking results with the proposed method. From the comparative ranking results, it is right that the proposed method satisfies the property that the ranking order of $\tilde{A}$ and $\tilde{B}$ does not changed when adding any type-1 fuzzy numbers to them.

TABLE IV. THE COMPARATIVE RESULTS OF EXAMPLE 4

Fuzzy numbers	$\tilde{A}$	$\tilde{B}$	$\tilde{A}'$	$\tilde{B}'$	$\tilde{A}''$	$\tilde{B}''$
Values	1.66	1.33	7.13	7.08	-0.16	-0.21
Ranking	$\tilde{A} \succ \tilde{B}$		$\tilde{A}' \succ \tilde{B}'$		$\tilde{A}'' \succ \tilde{B}''$	

B. The ranking for interval type-2 fuzzy numbers

Example 5 illustrate the effectiveness of the proposed ranking method for interval type-2 fuzzy numbers.

Example 5. Consider the following interval type-2 fuzzy numbers, $\tilde{\tilde{A}} = ((0,4,8),(2,4,6);1,0.9)$, $\tilde{\tilde{B}} = ((1,4,7),(3,4,5);1,0.9)$, and $\tilde{\tilde{C}} = ((-3,-1/2,2,5),(-2,0,1,4);1,0.9)$, which are shown in Fig. 6. According to Eqs. (12),(14),(13), Table V shows the possibility mean values and variation coefficient of these interval type-2 fuzzy numbers.

TABLE V. THE COMPARATIVE RESULTS OF EXAMPLE 5

	$\tilde{\tilde{A}}$	$\tilde{\tilde{B}}$	$\tilde{\tilde{C}}$	-$\tilde{\tilde{A}}$	-$\tilde{\tilde{B}}$	-$\tilde{\tilde{C}}$
Values	$(3.62, 0.18)$	$(3.62, 0.07)$	0.7	$(-3.62, -0.18)$	$(-3.62, -0.07)$	-0.7
Ranking	$\tilde{\tilde{B}} \succ \tilde{\tilde{A}} \succ \tilde{\tilde{C}}$			$-\tilde{\tilde{C}} \succ -\tilde{\tilde{A}} \succ -\tilde{\tilde{B}}$		

From Table V, it is concluded that the ranking result for interval type-2 fuzzy numbers $\tilde{\tilde{A}}$, $\tilde{\tilde{B}}$ and $\tilde{\tilde{C}}$ is consistent with peoples' intuition. the ranking of symmetric interval type-2 fuzzy numbers $\tilde{\tilde{A}}$ and $\tilde{\tilde{B}}$. Moreover, the ranking result of their images' is also reasonable.

VI. CONCLUSION

In this paper, we have presented a new distance based ranking method for fuzzy numbers. Compared with the currently main ranking methods for type-1 fuzzy numbers, the proposed method can avoid the defects that most ranking methods have, that is it not only correctly ranks the type-1 fuzzy numbers especially for the symmetric type-1 fuzzy numbers having different spread in the bottom, but also reasonably rank their images. Then, we extend the concept to interval type-2 fuzzy numbers environment, and introduced a new ranking method with possibility mean and variation coefficient, which can discriminate the ranking result of any interval type-2 fuzzy numbers, and the ranking of their images' is also logical. Finally, several examples are illustrated to compare the ranking results with the existing main ranking methods.

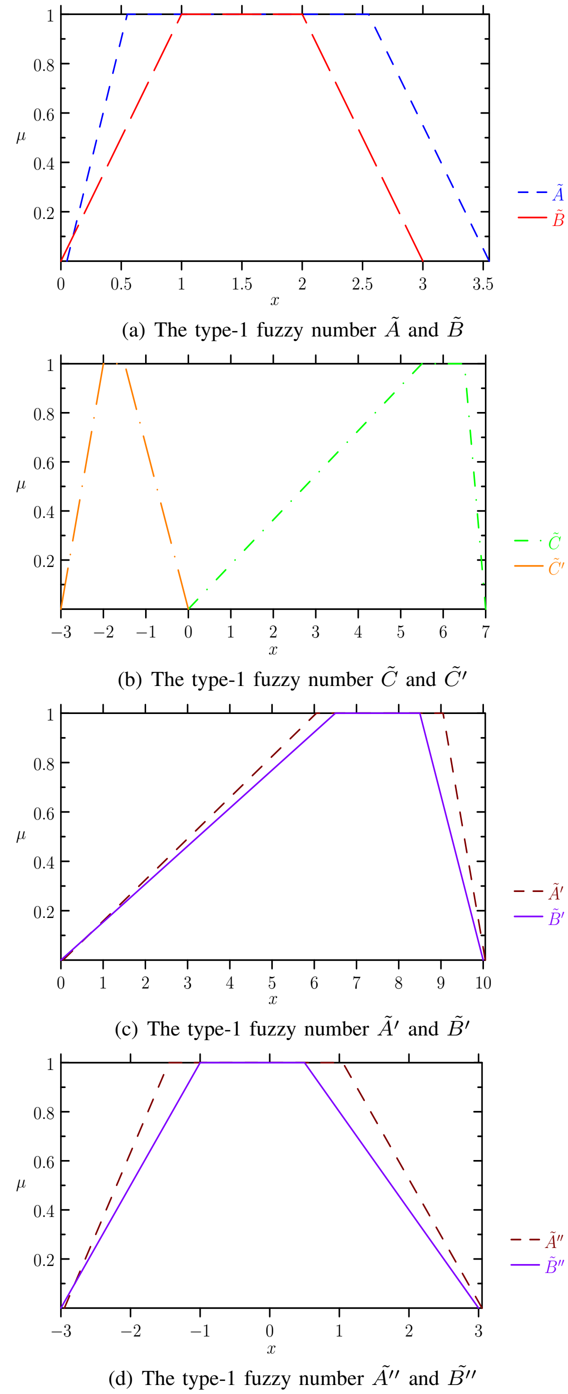


Fig. 5. The type-1 fuzzy numbers of Example 4

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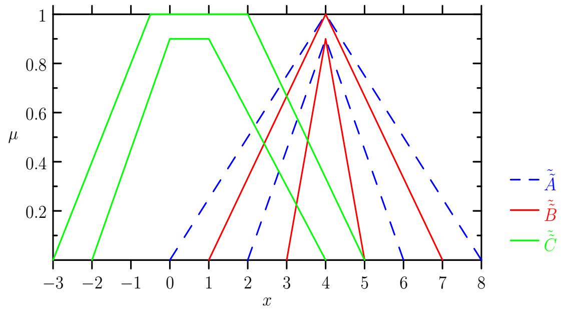

Fig. 6. The interval type-2 fuzzy numbers for Example 5

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