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A New Interval-based Method for Handling Non-Monotonic Information

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***Abstract*—The focus of this paper is on handling non-monotone information in the modelling process of a single-input target monotone system. On one hand, the monotonicity property is a piece of useful prior (or additional) information which can be exploited for modelling of a monotone target system. On the other hand, it is difficult to model a monotone system if the available information is not monotonically-ordered. In this paper, an interval-based method for analysing non-monotonically ordered information is proposed. The applicability of the proposed method to handling a non-monotone function, a non-monotone data set, and an incomplete and/or non-monotone fuzzy rule base is presented. The upper and lower bounds of the interval are firstly defined. The region governed by the interval is explained as a coverage measure. The coverage size represents uncertainty pertaining to the available information. The proposed approach constitutes a new method to transform non-monotonic information to interval-valued monotone system. The proposed interval-based method to handle an incomplete and/or non-monotone fuzzy rule base constitutes a new fuzzy reasoning approach.**

Keywords*: *Fuzzy ordering, monotonicity property, interval-valued, coverage measure, fuzzy sets, fuzzy reasoning

I. INTRODUCTION

A common problem in modelling a target monotone system is to approximate an unknown monotone function based on the available information. Examples of information include a set of experimental data, a mathematical model, or a set of fuzzy rules for a Fuzzy Inference System (FIS). It should be noted that the available information may not always satisfy the monotonicity property. Examples include non-monotonically ordered experimental data, non-monotone mathematical model, and non-monotonically ordered fuzzy rules for a target monotone system. The monotonicity property has been shown to be useful prior (or additional) information which can be exploited for modelling a monotone target system. However, exploitation of the monotonicity property in modelling is difficult, if the provided information does not conform to the monotonicity property. Therefore, it is important to represent non-monotone information as a monotone system to ensure the usefulness of the resulting model.

A search in the literature reveals that techniques to handle such information are available. As an example, noise was defined in the form of non-monotonicity, and a method to relabel non-monotone data set was proposed [1]. In [2], an original estimate (in the form of a non-monotone mathematical function) was considered. A method to re-arrange and transform the original estimates into a monotone form of estimates was proposed. In [3-4], methods to re-label non-monotone fuzzy rules were also proposed.

Three potential options for handling non-monotone experimental data have been discussed in [1]: (i) keep the data samples as they are; (2) identify noisy data samples and remove them; (3) identify noisy data samples and re-label them. In this paper, we argue that these options may also be applicable to non-monotone functions and non-monotone fuzzy rules. As a result, three potential options for handling non-monotone information are: (1) keep the information as it is; (2) identify and remove noisy information; and (3) identify and modify noisy information. Examples of the third option include the re-labeling techniques for non-monotone experimental data [1], the re-arrangement techniques to monotonize a non-monotone original function [2], and the development of fuzzy rule re-labeling techniques to re-label non-monotone fuzzy rules of an FIS [3-4].

The focus of this paper is on the first option, i.e., keep the information as it is, and the focus is on fuzzy modeling. While this option has been discussed [1], it is not clear how it can be implemented practically. In this paper, non-monotone information that describes a target monotone system is defined as a form of noise. The monotonicity property is exploited as a piece of useful prior (or additional) information for modelling a monotone system. An interval-based method to analyze noisy information is further suggested. The idea is to keep the original noisy information as it is, and represent the noisy information as an interval. As such, the upper and lower bounds of the interval are firstly defined. The region governed by the interval is explained as a coverage measure, and the coverage size represents the degree of uncertainty of the information. It is important to represent non-monotone information as an interval because it provides the lower and upper limits of the output for a particular input point of a monotone system.

In this paper, the use of the proposed interval-based method for handling a non-monotone function, a non-monotone data set, and an incomplete and/or non-monotone fuzzy rule base is examined. The interval-based method for handling a non-monotone function can be perceived as an alternative to the re-arrangement technique [2] in processing a non-monotone function. The non-monotone function can be a non-monotone FIS model. The non-monotone function is represented as a set of lower and upper bounds, instead of being modified by the re-arrangement technique. The region governed by the interval is explained as a coverage measure, and the coverage size represents the degree of uncertainty of the information.

For the case of non-monotone data, the original non-monotone experimental data are considered. The intervals of the dependent variables of a data sample in the independent variable space are obtained. Using such method, none of the data samples need to be removed or re-labeled. Such method offers an alternative to the relabeling technique for processing a non-monotone function.

The proposed interval-based method for handling a non-monotone data set is further extended to fuzzy modeling. The idea of fuzzy ordering [5] is adopted. Instead of interpolating fuzzy rules (i.e., fuzzy rule interpolation (FRI) [6]), the lower of upper bounds of a conclusion are obtained. We argue that if fuzzy ordering exists, and a set of monotonically-ordered fuzzy rules is necessary, it is possible to deduce the lower and upper bounds of the conclusion; therefore an alternative fuzzy reasoning approach. If the original fuzzy rule base is non-monotone, it is possible to represent the fuzzy rule consequence as an interval too, instead of re-labelling them [3-4]. This extension serves as an alternative solution to our previous works on fuzzy rule re-labeling and optimization-based similarity reasoning methods [3].

This paper is organized as follows. In Section II, the proposed interval-based method for handling non-monotone original estimates is presented. In Section III, an interval-based method for handling a set of non-monotone data is presented. In Section IV, FRI is firstly presented, which is followed by our proposed method. A method for handling non-monotone fuzzy rule is further presented. Finally, concluding remarks are presented in Section V.

II. A NON-MONOTONE ESTIMATE

A. A non-monotone estimate

The definitions of the input and output spaces of a target monotone function are presented as Definitions 1 and 2, as follows.

Definition 1: Consider an input space, X, and an output space, Y. Variables x and y are the elements of X and Y, respectively, i.e., $x \in X$, and $y \in Y$. The lower and upper bounds of X are represented by $\underline{x}$ and $\overline{x}$, respectively. Similarly, the lower and upper bounds of Y are represented by $\underline{y}$ and $\overline{y}$, respectively.

Definition 2: Consider a target monotone system $y = f(x)$, $f: X \to Y$. For all x and y, $f(x_2) \geq f(x_1)$, where $x_2 > x_1 \in X$, is always true.

A mathematical model, $y = g(x)$, i.e., $g: X \to Y$, which attempts to approximate the target monotone system, is considered. Note that g is obtained through an identification, regression, or interpolation process. As the monotonicity relationship is not adopted as the prior information during modelling, $y = g(x)$ may not be monotone. As a result, $y = g(x)$ is considered as the available information pertaining to the target monotone system. The uncertainty of $g(x)$ can be represented as an interval-valued mathematical model (i.e., $\widehat{y_g} = \hat{g}(x)$), as defined in Definition 3. The lower and upper bounds of $\hat{g}(x_1)$, where $x_1 \in X$, are obtained using Equations 1 and 2, respectively.

Definition 3: An interval-valued mathematical model, $\widehat{y_g} = \hat{g}(x)$, is considered. For all x, $\widehat{y_g}$ exists, where $\widehat{y_g(x)} = \left[\underline{y_g(x)}, \overline{y_g(x)}\right]$ and $\underline{y_g(x)} \leq \overline{y_g(x)}$.

$$\underline{y_g(x_1)} = min(g(x \geq x_1)). \tag{1}$$

$$\overline{y_g(x_1)} = max(g(x \leq x_1)) \tag{2}$$

The size of $\hat{g}(x)$ and its coverage are obtained using Equations 3 and 4, respectively.

$$size(x_1) = \overline{y_g(x_1)} - \underline{y_g(x_1)}\, ,\; x_1 \in X \tag{3}$$

$$coverage = \int_{\underline{x}}^{\overline{x}} \left(\overline{y_g(x)} - \underline{y_g(x)}\right) dx \tag{4}$$

Theorem 1 indicates a number of properties of $g(x)$ that satisfy the monotonicity property.

Theorem 1: If $g(x)$ satisfies the monotonicity property, i.e., $g(x_2) \geq g(x_1)$, where $x_2 > x_1 \in X$, is always true, then,

1.1 $\underline{y_g(x)} = g(x), \forall x$

1.2 $\overline{y_g(x)} = g(x), \forall x$

1.3 $size(x) = 0, \forall x$

1.4 $\overline{coverage} = 0$

As an example, consider domain, X, where $\underline{x} = 0$ and $\overline{x} = 12$. An original estimate, $g(x) = (x-1) \times (x-7) \times (x-9)$ is available. Figure 1 illustrates $g(x)$, its $\widehat{y_g}$ and size. Its coverage is shaded, and the area is estimated to be 448 in unit of $X \times Y$.

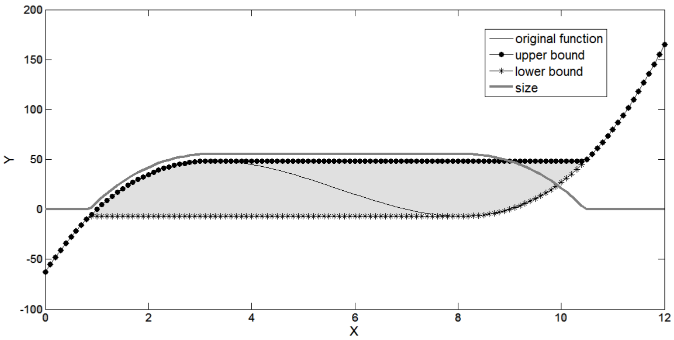

Figure 1. An illustrative example

III. DATA-DRIVEN MODELLING

A. A data set

Definitions 1 and 2 are considered. A data set in the input and output spaces is further defined, as follows.

Definition 4: A data set with m data samples is considered. Each data sample is denoted as $\overline{d_k} = [x_k, y_k]$ in the X and Y spaces (Definition 1), $k = 1,2,3,\ldots,m$. To simplify the notation, y_k of x_k is denoted as $y_k(x_k)$.

Definition 5 is considered. Note that $\underline{y_l}$ and $\overline{y_l}$ are obtained using Equations (5) and (6), respectively.

Definition 5: A list of identical x_k samples is written x_l, where $l = 1,2,\ldots,n \le m$. Each x_l is associated with an interval in space Y (Definition 1), i.e., $\left[\underline{y_l}, \overline{y_l}\right]$. To simplify the notation, the lower and upper bounds of the interval at x_l are denoted as $\underline{y_l}(x_l)$ and $\overline{y_l}(x_l)$, respectively.

$$\underline{y_l(x_l)} = min(y_k(x_k \ge x_l) \quad (5)$$

$$\overline{y_l(x_l)} = max(y_k(x_k \le x_l) \quad (6)$$

B. Simulated examples

As an example, consider two data sets with $k = 7$, and $l = 6$, as shown in Table 1. Data set 1 satisfies the monotonicity property, while data set 2 does not satisfy the property. Based on Equations (5) and (6), the results shown in Table 2 can be obtained. With data set 1, there are two data samples for x_k=5, i.e., $y_k = 5,6$, as shaded in Table 1. With the proposed method, x_l=5 is mapped to an interval, i.e.. [5,6], as listed in Table 2. With data set 2, there are two non-monotone data samples for x_k=4, x_k=5, as shaded in Table 1. With the proposed method, x_l=4 and x_l=5 are mapped to [4,5] and [4,6], respectively, as in Table 2.

Table 1 Data set for experiments

k	x_k	y_k	y_k
		Data set 1	Data set 2
1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	5
5	5	5	4
6	5	6	6
7	6	7	7

Table 2 Experimental results

l	x_l	$y_l = \left[\underline{y_l}, \overline{y_l}\right]$	$y_l = \left[\underline{y_l}, \overline{y_l}\right]$
		Data set 1	Data set 2
1	1	[1,1]	[1,1]
2	2	[2,2]	[2,2]
3	3	[3,3]	[3,3]
4	4	[4,4]	[4,5]
5	5	[5,6]	[4,6]
6	6	[7,7]	[7,7]

IV. AN EXTENSION TO FUZZY REASONING

A. The zero-order Sugeno Fuzzy Inference System

The focus of this study is on the zero-order Sugeno FIS model. Definitions 1 and 2 are considered. The fuzzy If-Then rules for a single input zero-order Sugeno FIS model (hereafter refer to as the FIS model) with M fuzzy rules, is represented as follows.

R^j : $If(X \ is \ A_j), THEN(Y \ is \ b_j), 1 \le j \le M$

The fuzzy membership functions (MFs) are written as $\mu_j(x)$ for linguistic term A_j. The output is obtained by using the weighted average of the representative value, b_j, with respect to its compatibility grade, i.e., $\mu_j(x)$, as shown in Equation (7).

$$y = \frac{\sum_{j=1}^{j=M}(\mu_j(x) \times b_j)}{\sum_{j=1}^{j=M}(\mu_j(x))} \quad (7)$$

Using the findings in [7-9], the following theorem is formulated for the FIS model.

Theorem 2. The FIS model (i.e., Equation (7)) fulfills the monotonicity property between y and x, if the following conditions are satisfied.

2.1 At the rule consequent part, $b_{p+1} \ge b_p$, where $p \in [1,2,3,\ldots,M]$.

2.2 At the rule antecedent part, $(d\mu_{p+1}(x)/dx)/\mu_{p+1}(x) \ge (d\mu_p(x)/dx)/\mu_p(x)$, where $p \in [1,2,3,\ldots,M]$.

Note that $(d\mu(x)/dx)/\mu(x)$ is the ratio between the rate of change of the membership degree and the membership degree itself. A Gaussian MF is written as $\mu_G(x\colon c,\sigma) = e^{-[x-c]^2/2\sigma^2}$, where c is the center of the MF, and σ parameterizes the width of the MF. Note that $(d\mu_G(x)/dx)/\mu_G(x)$ for the Gaussian MF return a linear function, i.e, $E(x) = (d\mu_G(x)/dx)/\mu_G(x) = -(1/\sigma^2)x + (c/\sigma^2)$ [9].

B. Monotone Fuzzy Rule Interpolation

Suppose a set of fuzzy variables and fuzzy rules is given, as defined in Definitions 6 and 7, respectively.

Definition 6*:* Fuzzy variables of the input space, i.e., X, are denoted as A, and partial ordering among fuzzy variables at X exists, i.e.. $A_{r+1} \succcurlyeq A_r$. In this study, $A_{r+1} \succcurlyeq A_r$ is true, if Theorem 2.2 is satisfied. Fuzzy singletons of the output space, i.e., Y, are denoted as B, and ordering among fuzzy singletons at Y exists, i.e.. $B_{r+1} \geq B_r$, respectively.

Definition 7. A set of original fuzzy rules, i.e., R_r: $A_r \rightarrow B_{r,k}$, is provided. Note that $B_{r,k}$ is fuzzy singletons of the output space, i.e., Y. To simplify the notation, R_r: $A_r \rightarrow B_{r,k}$ is denoted as $B_{r,k}(A_r)$. If the rule base is incomplete, some $B_{r,k}$ are unknown, and fuzzy rules with unknown $B_{r,k}$ are labelled as R^* : $A_r^* \rightarrow B_{r,k}^*$.

The general idea of FRI is explained in Fig. 2. As can be seen in Fig. 2, partial ordering among fuzzy variables in space X exists, i.e., $A_6 \succcurlyeq A_5^* \succcurlyeq A_4 \succcurlyeq A_3^* \succcurlyeq A_2^* \succcurlyeq A_1$. Fuzzy variables in space X satisfy Theorem 2.2. Ordering among fuzzy singletons in space Y also exists, i.e., $B_6 \geq B_5^* \geq B_4 \geq B_3^* \geq B_2^* \geq B_1$. Three fuzzy rules are provided, i.e., R_1: $A_1 \rightarrow B_1$, R_4: $A_4 \rightarrow B_4$, and R_6: $A_6 \rightarrow B_6$. Interpolation between two fuzzy rules is possible [6]. As an example, from R_1 and R_4, R_2^* and R_3^* can be interpolated with linear interpolation. Other polynomial functions can also be used for estimating R_2^* and R_3^* [6]. Such method is called fuzzy interpolative reasoning [6].

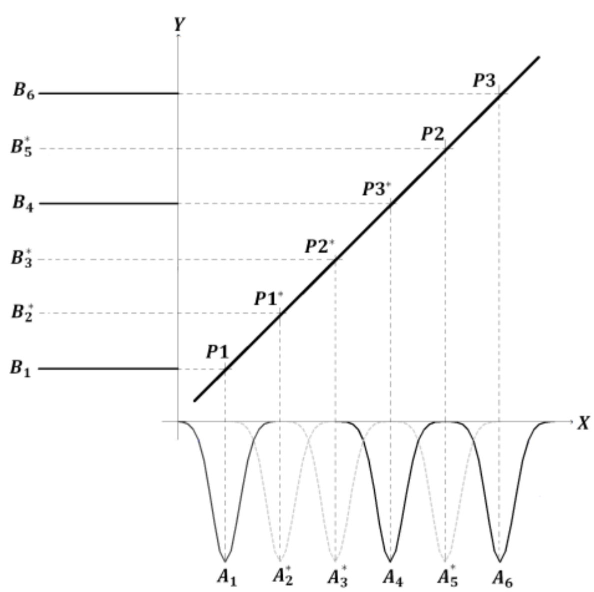

Figure 2. Interpolation of fuzzy rules

C. An alternative approach to fuzzy reasoning

Definitions 1, 6, and 7, are considered, and additional information is available, i.e., the monotonicity property (Definition 2). Based on the monotone ordering conditions of fuzzy rules (i.e., Theorem 2), it is known that $B_{r+1,k} \geq B_{r,k}$. Some of $B_{r,k}$ (Definition 7) are unknown, and those known $B_{r,k}$ satisfy the monotone ordering conditions of fuzzy rules. Instead of interpolating the fuzzy rules [6], it is possible to represent $B_{r,k}^*$ as an interval. To simplify the notation, R^* : $A_r^* \rightarrow B_{r,k}^*$ is denoted as $B_{r,k}^*(A_r^*)$, and the interval of $B_{r,k}^*(A_r^*)$ is denoted as $\left[\underline{B_{r,k}^*(A_r^*)}, \overline{B_{r,k}^*(A_r^*)}\right]$. The lower and upper bounds of $B_{r,k}^*(A_r^*)$ can be obtained using Equations 8 and 9, respectively.

$$\underline{B_{r,k}^*(A_r^*)} = max(B_{r,k}(A \preccurlyeq A_r^*)) \tag{8}$$

$$\overline{B_{r,k}^*(A_r^*)} = min\ (B_{r,k}(A \succcurlyeq A_r^*)) \tag{9}$$

The lower and uppers bounds result in an interval-valued FIS model. The size and coverage of the resulting interval-valued FIS model are obtained using Equations 3 and 4, respectively. Based on the example in Figure 2, $\underline{B_{2,k}^*(A_2^*)} = max(B_{1,k}(A_1))$ and $\overline{B_{2,k}^*(A_2^*)} = min\ (B_{4,k}(A_4), B_{6,k}(A_6))$.

An illustrative example with six fuzzy rules is further exemplified. Note that A_r is a Gaussian MF, which is specified by two parameters, i.e., c_r and σ_r, where $y = e^{-\frac{1}{2}((x-c_r)/\sigma_r)^2}$. Referring to Rule Set 1 in Table 3, a monotone but incomplete fuzzy rule base is formed, whereby $B_{2,k}(A_2)$, $B_{3,k}(A_3)$, and $B_{5,k}(A_5)$ are unknown. With FRI, $B_{2,k}(A_2)$=2. Using the proposed interval-based method, the results in Table 4 are obtained. As an example, with $r = 2$, the interval obtained is $B_{2,k}(A_2) = [1,4]$.

Table 3 Fuzzy rules set for experiments

r	A_r	$B_{r,k}(A_r)$	$B_{r,k}(A_r)$	$B_{r,k}(A_r)$
	$[c, \sigma]$	Rules set 1 (monotone but incomplete)	Rules set 2 (non-monotone)	Rules set 3 (incomplete and non-monotone)
1	[1,0.3]	1	1	1
2	[2,0.3]	2*	3	
3	[3,0.3]	3*	2	4
4	[4,0.3]	4	4	3
5	[5,0.3]	5*	2	
6	[6,0.3]	6	6	6

Table 4 Experimental results

r	$\underline{B_{r,k}(A_r)}$	$B_{r,k}(A_r)$	$\overline{B_{r,k}(A_r)}$
	Rules set 1 (monotone but incomplete)	Rules set 2 (non-monotone)	Rules set 3 (incomplete and non-monotone)
1	[1,1]	[1,1]	[1,1]
2	[1,4]*	[2,3]	[1,4]*
3	[1,4]*	[2,3]	[3,4]
4	[4,4]	[2,4]	[3, 4]
5	[4,6] *	[2,4]	[3,6]*
6	[6,6]	[6,6]	[6, 6]

Figure 3 shows the resulting FIS models from FRI as well as our proposed method. With $\underline{x} = 1$ and $\overline{x} = 6$, the estimated coverage of the our proposed interval-valued FIS model is 7.99.

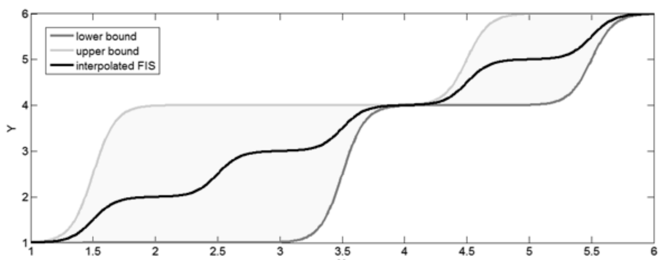

Figure 3. The FIS models from FRI and our proposed method

D. Handling of a non-monotone fuzzy rule base

In this section, Definitions 1, 6, and 7 are considered, and additional information is available, i.e., the monotonicity property (Definition 2) is satisfied. It is known that $B_{r+1,k} \geq B_{r,k}$. All $B_{r,k}$ (Definition 7) are known, but, they do not follow a monotone order.

Again, each $B_{r,k}$ is denoted as an interval, i.e., $\left[\underline{B_{r,k}(A_r)}, \overline{B_{r,k}(A_r)}\right]$. Following Equations 5 and 6, the lower and upper bounds of $B_{r,k}(A_r)$ can be obtained using Equations (10) and (11), respectively.

$$\underline{B_{r,k}(A_r)} = min(B_{r,k}(A \succcurlyeq A_r)) \tag{10}$$

$$\overline{B_{r,k}(A_r)} = max\,(B_{r,k}(A \preccurlyeq A_r)) \tag{11}$$

Again, the lower and uppers bounds result in an interval-valued FIS model. The size and coverage of the resulting interval-valued FIS model are obtained using Equations (3) and (4), respectively.

An illustrative example with r=6 is examined, and Rule set 2 in Table 3 is considered. It is a non-monotone and complete fuzzy rule base. With the proposed method, the results in Table 4 are obtained. Figure 4 shows the FIS models. The resulting FIS model with the non-monotone and complete fuzzy rule base does not satisfy Definition 2. The interval-valued FIS model from our proposed method is also plotted, and its coverage is 5.99.

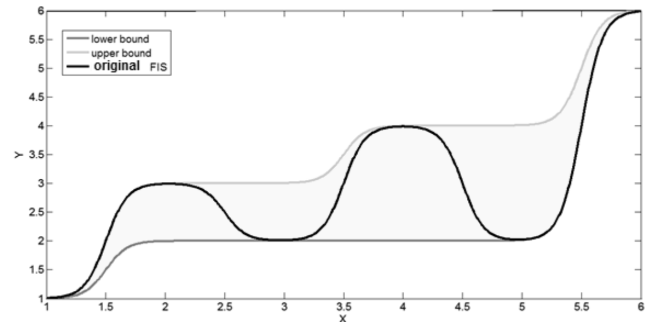

Figure 4. The FIS models from FRI and our proposed method

E. Handling of a non-monotone and incomplete fuzzy rule base

The findings in Sections IV(C) and IV(D) can be combined to handle a non-monotone and incomplete fuzzy rule base. The original fuzzy rules from experts are incomplete and non-monotone. As such, the method in Section IV(D) is used to identify the interval of $B_{r,k}(A_r)$. Then, the interval of $B^*_{r,k}(A^*_r)$ is obtained with Equations (12) and (13), respectively.

$$\underline{B^*_{r,k}(A^*_r)} = max(\underline{B_{r,k}(A \preccurlyeq A^*_r)}) \tag{12}$$

$$\overline{B^*_{r,k}(A^*_r)} = min\,(\overline{B_{r,k}(A \succcurlyeq A^*_r)}) \tag{13}$$

Again, an illustrative example with r=6 is studied, and Rule set 3 in Table 3 is considered. It is a non-monotone and incomplete fuzzy rule base. With the proposed method, the results in Table 4 are obtained. Figure 5 shows the resulting interval-valued FIS model with our proposed method. Its coverage is 7.99.

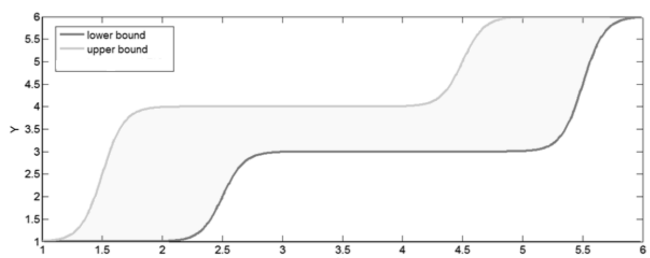

Figure 5. The resulting interval-valued FIS model

F. Remarks

The proposed interval approach to handling non-monotone information in the modelling process of single-input target monotone systems can be extended to multi-input target monotone systems. Besides, the proposed approach in Section IV can be extended to Mamdani [10] and SIRM [11] FIS models, in which the monotonicity property is of importance. In practice, the proposed approach is useful for assessment and decision making problems [12-13], in which the available fuzzy rules can be incomplete and/or non-monotone.

V. Conclusions

In this paper, a new direction to solve problems relating to modelling of monotonicity-preserving models is described. The rationale is to keep the original information as it is, and to represent the information as an interval. The upper and lower bounds of the interval are defined. The region governed by the interval is explained as a coverage, and the size of the coverage is a measure of uncertainty. The usefulness of the proposed method for handling a non-monotone original estimate (i.e., a mathematical function) and a non-monotone data set has been evaluated using simulated examples.

The proposed interval-based method for handling a non-monotone data set is further extended to fuzzy modeling, for solving problems related to incomplete and non-monotone fuzzy rules. Simulation examples have been further demonstrated. The outcomes indicate that the proposed method constitutes an alternative to monotone fuzzy reasoning. For future research, the proposed method for undertaking interval-valued functions and fuzzy functions will be studied.

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