2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)
July 6-11, 2014, Beijing, China

Hierarchical Fuzzy Sliding-Mode Control for Uncertain Nonlinear Under-Actuated Systems

Chiang-Cheng Chiang
Department of Electrical Engineering
Tatung University
Taipei, Taiwan, R.O.C.
e-mail: ccchiang@ttu.edu.tw

Yao-Wei Yeh
Department of Electrical Engineering
Tatung University
Taipei, Taiwan, R.O.C.

***Abstract*—This paper presents a hierarchical fuzzy sliding mode control scheme for a class of uncertain nonlinear under-actuated systems. First, the sliding surface of one subsystem is selected as the first layer sliding surface. Hence, we further construct a second layer sliding surface from the first layer sliding surface and the sliding surface of another subsystem till all the subsystem sliding surfaces are included. The fuzzy system and some adaptive laws are applied to approximate the unknown nonlinear functions and estimate the upper bounds of the unknown uncertainties, respectively. By means of Lyapunov stability theorem and the theory of sliding mode control, the proposed control scheme ensures the robust stability of the uncertain nonlinear under-actuated systems. Finally, simulation results show the validity of the proposed method.**

Keywords—under-actuated systems; fuzzy systems; hierarchical fuzzy sliding mode control; Lyapunov stability theorem

I. INTRODUCTION

In recent years, there has been growing interest in control problems of under-actuated systems [1-4]. Under-actuated systems are commonly encountered in mechanical systems which are fewer actuators than degrees-of-freedom to be controlled. The applications are used widely in practical systems such as free-flying space robots, underwater robots, manipulators with structural flexibility, overhead crane, etc.

Many papers on the control of under-actuated systems have been proposed in [5-8]. In [5], the hybrid switching control strategy was developed to cope with a class of nonlinear and under-actuated mechanical systems. The optimal control of nonholonomic, under-actuated mechanical systems was studied in [6]. The research [7] introduced a motion planning-based adaptive control method in under-actuated crane systems. In the case of the under-actuated surface ship with rudder actuator dynamics under external disturbances, the adaptive fuzzy controller was developed in [8]. Thus, the analyses and control of nonlinear under-actuated systems have been an important research area.

The sliding mode control (SMC) [9-10] has shown to be one of the effective nonlinear robust control strategies to tackle systemic parameters and external disturbances for nonlinear systems. SMC has some advantages such as insensitivity to system parameter variations, invariance to external disturbances, good transient performance, and fast response, etc. In [11], the sliding-mode controller on the basis of the incremental hierarchical structure and aggregated hierarchical structure were used to treat the under-actuated system. The work [12] was developed a hybrid sliding-mode controller for under-actuated systems. SMC laws usually consist of two parts: switching controller design and equivalent controller design. First, the switching controller design drives the system states toward the sliding surface. Then, when the system states are on the sliding surface, the equivalent controller design guarantees the system states to keep on the sliding surface and converge to zero along the sliding surface.

The fuzzy control methods [13-14] have been adopted widely to treat the problem of under-actuated systems with unknown nonlinear functions. Especially, the decoupled fuzzy adaptive SMC method has been applied to the control of under-actuated systems with mismatched uncertainties [15]. According to the universal approximation theorem, fuzzy systems, which are constructed from a collection of fuzzy If-Then rules, are employed here as a way to approximate the unknown nonlinear functions of the system. Moreover, the adaptive laws are used to adjust the parameters of the fuzzy model. Thus, the adaptive schemes guarantee all signals to be bounded, and the tracking error of the closed-loop system will asymptotically track our desired trajectory and achieve the desired tracking performance.

The main object of this paper is on the design of the robust hierarchical fuzzy sliding mode control for uncertain nonlinear under-actuated systems. The fuzzy system and some adaptive laws are applied to approximate the unknown nonlinear functions and estimate the upper bounds of the unknown uncertainties, respectively. Furthermore, by Lyapunov stability theorem and the theory of sliding mode control, the presented hierarchical fuzzy sliding mode controller can not only guarantee the convergence to zero of each sliding surface, but also ensure the robust stability of the uncertain nonlinear under-actuated system.

This paper is organized as follows. First, the control problems of the uncertain under-actuated system and the concept of fuzzy system are introduced in Section II. The hierarchical fuzzy sliding mode control is proposed to deal with the control problem for uncertain under-actuated systems in Section III. The simulation results are illustrated to show the validity of the proposed control method in Section IV. Finally, a conclusion is given in Section V.

II. PROBLEM STATEMENT AND PRELIMINARIES

A. Problem Statement

Consider a single-input-multi-output uncertain nonlinear underactuated system expressed as follows:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1(\mathbf{x}) + b_1(\mathbf{x})u + d_1(\mathbf{x},t) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2(\mathbf{x}) + b_2(\mathbf{x})u + d_2(\mathbf{x},t) \\ \vdots \\ \dot{x}_{2n-1} = x_{2n} \\ \dot{x}_{2n} = f_n(\mathbf{x}) + b_n(\mathbf{x})u + d_n(\mathbf{x},t) \end{cases} \tag{1}$$

$$\mathbf{y}(t) = \left[x_1, x_3, \cdots, x_{2n-1}\right]^T$$

where $\mathbf{x} = \left[x_1, x_2, \cdots, x_{2n}\right]^T \in R^{2n}$ is the system state vector which is assumed to be available for measurement, $u \in R^1$ is the control input, $\mathbf{y} \in R^n$ is the system output, $f_1(\mathbf{x}), f_2(\mathbf{x}), \cdots, f_n(\mathbf{x})$ and $b_1(\mathbf{x}), b_2(\mathbf{x}), \cdots, b_n(\mathbf{x})$ are unknown real continuous nonlinear functions, and $d_1(\mathbf{x},t), d_2(\mathbf{x},t), \cdots, d_n(\mathbf{x},t)$ are unknown external bound uncertainties. Without loss of generality, the following assumptions are made for the controller design:

Assumption 1 : $0 \le |f_i(\mathbf{x})| \le F_i < \infty,\ 0 < |b_i(\mathbf{x})| \le B_i < \infty$, for $\mathbf{x} \in \Gamma$, $i = 1,2,\cdots,n$,

where F_i and B_i are known positive constants, and Γ is a set given as follows

$$\Gamma = \left\{ \mathbf{x} \middle| \left\| \mathbf{x} - \mathbf{x_0} \right\|_{p,\omega} \le \Delta \right\}. \tag{2}$$

Here ω is a set of weight, and Δ is a positive constant which denotes all state variables' boundary. $\mathbf{x_0} \in \mathbb{R}^{2n}$ is a fixed point, and $\|\mathbf{x}\|_{p,\omega}$ is a weighted p-norm, which is defined as

$$\|\mathbf{x}\|_{p,\omega} = \left[\sum_{i=1}^{2n} \left(\frac{x_i}{\omega_i} \right)^p \right]^{1/p}. \tag{3}$$

If $p = \infty$,

$$\|\mathbf{x}\|_{p,\omega} = \max\left(\frac{|x_1|}{\omega_1}, \frac{|x_2|}{\omega_2}, \cdots, \frac{|x_{2n}|}{\omega_{2n}} \right). \tag{4}$$

If $p = 2$, $\omega = 1$, $\|\mathbf{x}\|_{p,\omega}$ will denote Euclidean norm $\|\mathbf{x}\|$.

Assumption 2 : $0 \le |d_i(\mathbf{x},t)| \le \rho_i(\mathbf{x}) < \infty$, for $\mathbf{x} \in \Gamma$, $i = 1,2,\cdots,n$, where $\rho_i(\mathbf{x})$ are unknown bounded positive smooth continuous functions.

Control Objective: Design a controller for (1) such that the system states $\mathbf{x}(t)$ would converge to zero as $t \to \infty$.

B. Description of Fuzzy Logic Systems

The fuzzy logic system performs a mapping from $U \subset R^n$ to $V \subset R$. Let $U = U_1 \times \cdots \times U_n$ where $U_i \subset R$, $i = 1,2,\cdots,n$. The fuzzy rule base consists of a collection of fuzzy IF-THEN rules:

$$R^{(l)}: \text{IF } x_1 \text{ is } F_1^l, \text{ and } x_2 \text{ is } F_2^l, \text{ and } \cdots \text{ and, } x_n \text{ is } F_n^l \quad \text{THEN } y \text{ is } G^l, \quad \text{for } l = 1,2,\cdots,M. \tag{5}$$

in which $\mathbf{x} = \left[x_1, x_2, \cdots, x_n\right]^T \in U$ and $y \in V \subset R$ are the input and output of the fuzzy logic system, F_i^l and G^l are fuzzy sets in U_i and V, respectively. The fuzzifier maps a crisp point $\mathbf{x} = \left[x_1, x_2, \cdots, x_n\right]^T$ into a fuzzy set in U. The fuzzy inference engine performs a mapping from fuzzy sets in U to fuzzy sets in V, based upon the fuzzy IF-THEN rules in the fuzzy rule base and the compositional rule of inference. The defuzzifier maps a fuzzy set in V to a crisp point in V.

The fuzzy systems with center-average defuzzifier, product inference and singleton fuzzifier are of the following form:

$$y(\mathbf{x}) = \frac{\sum_{l=1}^{M} \theta^l \left(\prod_{i=1}^{n} \mu_{F_i^l}(x_i) \right)}{\sum_{l=1}^{M} \left(\prod_{i=1}^{n} \mu_{F_i^l}(x_i) \right)}, \tag{6}$$

where θ^l is the point at which fuzzy membership function $\mu_{G^l}(\theta^l)$ achieves its maximum value, and we assume that $\mu_{G^l}(\theta^l) = 1$. Eq. (6) can be rewritten as

$$y(\mathbf{x}) = \mathbf{\theta}^T \xi(\mathbf{x}) \tag{7}$$

where $\mathbf{\theta} = \left[\theta^1, \theta^2, \cdots, \theta^l\right]^T$ is a parameter vector, and $\xi(\mathbf{x}) = \left[\xi^1(\mathbf{x}), \cdots, \xi^M(\mathbf{x})\right]^T$ is a regressive vector with the regressor $\xi^l(\mathbf{x})$, which is defined as fuzzy basis function

$$\xi^l(\mathbf{x}) = \frac{\prod_{i=1}^{n} \mu_{F_i^l}(x_i)}{\sum_{l=1}^{M} \left(\prod_{i=1}^{n} \mu_{F_i^l}(x_i) \right)}. \tag{8}$$

III. CONTROLLER DESIGN AND STABILITY ANALYSIS

In view of (1), let the sliding surfaces be defined as follows:

$$s_i = c_i x_{2i-1} + x_{2i}, \quad \text{for } i = 1,2,\ldots,n. \tag{9}$$

where c_i are positive constants. Differentiating s_i with respect to time *t*, we have

$$\begin{aligned} \dot{s}_i &= c_i \dot{x}_{2i-1} + \dot{x}_{2i} \\ &= c_i x_{2i} + f_i(\mathbf{x}) + b_i(\mathbf{x})u + d_i(\mathbf{x},t) \end{aligned} \tag{10}$$

According to the equivalent control method, the equivalent control law of the subsystems can be obtained as

$$u_{eqi} = -\left(c_i x_{2i} + f_i(\mathbf{x})\right) / b_i(\mathbf{x}) \quad \text{for} \quad i = 1,2,\cdots,n. \tag{11}$$

Without loss of generality, the subsystem sliding surface s_1 is selected as S_1.

First, let the unknown nonlinear functions $f_i(\mathbf{x})$, $b_i(\mathbf{x})$, and $\rho_i(\mathbf{x})$, for $i=1,2,\ldots,n$, can be approximated, over a compact set $\Omega_{\mathbf{x}}$, by the fuzzy systems as follows:

$$\hat{f}_i(\mathbf{x}|\boldsymbol{\theta}_{f_i}) = \boldsymbol{\theta}_{f_i}^T \boldsymbol{\xi}(\mathbf{x}), \quad (12)$$

$$\hat{b}_i(\mathbf{x}|\boldsymbol{\theta}_{b_i}) = \boldsymbol{\theta}_{b_i}^T \boldsymbol{\xi}(\mathbf{x}), \quad (13)$$

$$\hat{\rho}_i(\mathbf{x}|\boldsymbol{\theta}_{\rho_i}) = \boldsymbol{\theta}_{\rho_i}^T \boldsymbol{\xi}(\mathbf{x}), \quad (14)$$

where $\boldsymbol{\xi}(\mathbf{x})$ is the fuzzy basis vector, $\boldsymbol{\theta}_{f_i}$, $\boldsymbol{\theta}_{b_i}$, and $\boldsymbol{\theta}_{\rho_i}$, for $i=1,2,\ldots,n$, are the corresponding adjustable parameter vectors of each fuzzy systems. It is assumed that $\boldsymbol{\theta}_{f_i}$, $\boldsymbol{\theta}_{b_i}$, and $\boldsymbol{\theta}_{\rho_i}$ belong to compact sets $\Omega_{\boldsymbol{\theta}_{f_i}}$, $\Omega_{\boldsymbol{\theta}_{b_i}}$, and $\Omega_{\boldsymbol{\theta}_{\rho_i}}$, respectively, which are defined as

$$\Omega_{\boldsymbol{\theta}_{f_i}} = \left\{\boldsymbol{\theta}_{f_i} \in R^M : \left\|\boldsymbol{\theta}_{f_i}\right\| \le N_{f_i} < \infty\right\}, \quad (15)$$

$$\Omega_{\boldsymbol{\theta}_{b_i}} = \left\{\boldsymbol{\theta}_{b_i} \in R^M : 0 < \delta \le \left\|\boldsymbol{\theta}_{b_i}\right\| \le N_{b_i} < \infty\right\}, \quad (16)$$

$$\Omega_{\boldsymbol{\theta}_{\rho_i}} = \left\{\boldsymbol{\theta}_{\rho_i} \in R^M : \left\|\boldsymbol{\theta}_{\rho_i}\right\| \le N_{\rho_i} < \infty\right\}, \quad (17)$$

where N_{f_i}, N_{b_i}, and N_{ρ_i} for $i=1,2,\ldots,n$, are the designed parameters, and M is the number of fuzzy inference rules. We require $\left\|\boldsymbol{\theta}_{b_i}\right\|$ to be bounded from below by $\delta > 0$ because from (28) we see that $\hat{b}_i(\mathbf{x}|\boldsymbol{\theta}_{b_i})$ must be nonzero. Let us define the optimal parameter vectors $\boldsymbol{\theta}_{f_i}^*$, $\boldsymbol{\theta}_{b_i}^*$, and $\boldsymbol{\theta}_{\rho_i}^*$, for $i=1,2,\ldots,n$, as follows:

$$\boldsymbol{\theta}_{f_i}^* = \arg\min_{\boldsymbol{\theta}_{f_i}\in\Omega_{\boldsymbol{\theta}_{f_i}}} \left\{\sup_{\mathbf{x}\in\Omega_{\mathbf{x}}} \left|\hat{f}_i(\mathbf{x}|\boldsymbol{\theta}_{f_i}) - f_i(\mathbf{x})\right|\right\}, \quad (18)$$

$$\boldsymbol{\theta}_{b_i}^* = \arg\min_{\boldsymbol{\theta}_{b_i}\in\Omega_{\boldsymbol{\theta}_{b_i}}} \left\{\sup_{\mathbf{x}\in\Omega_{\mathbf{x}}} \left|\hat{b}_i(\mathbf{x}|\boldsymbol{\theta}_{b_i}) - b_i(\mathbf{x})\right|\right\}, \quad (19)$$

$$\boldsymbol{\theta}_{\rho_i}^* = \arg\min_{\boldsymbol{\theta}_{\rho_i}\in\Omega_{\boldsymbol{\theta}_{\rho_i}}} \left\{\sup_{\mathbf{x}\in\Omega_{\mathbf{x}}} \left|\hat{\rho}_i(\mathbf{x}|\boldsymbol{\theta}_{\rho_i}) - \rho_i(\mathbf{x})\right|\right\}, \quad (20)$$

where $\boldsymbol{\theta}_{f_i}^*$, $\boldsymbol{\theta}_{b_i}^*$, and $\boldsymbol{\theta}_{\rho_i}^*$ for $i=1,2,\ldots,n$, are bounded in the suitable closed sets $\Omega_{\boldsymbol{\theta}_{f_i}}$, $\Omega_{\boldsymbol{\theta}_{b_i}}$, and $\Omega_{\boldsymbol{\theta}_{\rho_i}}$, respectively. The parameter estimation errors can be defined as

$$\tilde{\boldsymbol{\theta}}_{f_i} = \boldsymbol{\theta}_{f_i} - \boldsymbol{\theta}_{f_i}^*, \quad (21)$$

$$\tilde{\boldsymbol{\theta}}_{b_i} = \boldsymbol{\theta}_{b_i} - \boldsymbol{\theta}_{b_i}^*, \quad (22)$$

$$\tilde{\boldsymbol{\theta}}_{\rho_i} = \boldsymbol{\theta}_{\rho_i} - \boldsymbol{\theta}_{\rho_i}^*, \quad (23)$$

and

$$|w_1| + |w_2| \le w \quad (24)$$

where

$$w_1 = \sum_{r=1}^{i}\left(\prod\nolimits_{j=r}^{i} a_j\right)\left\{\left[f_r(\mathbf{x}) - \hat{f}_r(\mathbf{x}|\boldsymbol{\theta}_{f_r}^*)\right] + \left[b_r(\mathbf{x}) - \hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r}^*)\right]\right\} \quad (25)$$

$$w_2 = \sum_{r=1}^{i}\left|\left(\prod\nolimits_{j=r}^{i} a_j\right)\right|\left[\rho_r(\mathbf{x}) - \hat{\rho}_r(\mathbf{x}|\boldsymbol{\theta}_{\rho_r}^*)\right] \quad (26)$$

are the minimum approximation errors, which correspond to approximation errors obtained when optimal parameters are used.
Then, we define

$$\tilde{w} = \hat{w} - w \quad (27)$$

where $\hat{w}$ be as the estimate of w.

Based on the fuzzy systems, the equation (11) can be replaced as the following controller:

$$\hat{u}_{eqi} = -\left(c_i x_{2i} + \hat{f}_i(\mathbf{x}|\boldsymbol{\theta}_{f_i})\right)\Big/\hat{b}_i(\mathbf{x}|\boldsymbol{\theta}_{b_i}), \text{for } i=1,2,\cdots,n. \quad (28)$$

The ith-layer sliding surface S_i and its control law u_i can be defined as follows.

$$S_i = \lambda_{i-1}S_{i-1} + s_i \quad (29)$$

$$u_i = u_{i-1} + \hat{u}_{eqi} + \hat{u}_{swi} \quad (30)$$

Here λ_{i-1} $i=(1,2,\ldots,n)$ is a constant; $\lambda_0 = S_0 = 0$ $u_0 = 0$ u_{swi}, for $i=(1,2,\ldots,n)$ is the switching control of the ith-layer sliding surface. From the recursive formulas (29) and (30), we have

$$S_i = \sum_{r=1}^{i}\left(\prod\nolimits_{j=r}^{i} a_j\right)s_r \quad (31)$$

$$u_i = \sum_{r=1}^{i}(\hat{u}_{swr} + \hat{u}_{eqr}) \quad (32)$$

Here for a given i, $a_j = \lambda_j$ $(j \ne i)$ is a constant, $a_j = 1$ $(j = i)$, and $u = u_i$ for $i=(1,2,\cdots,n)$.
where

$$\hat{u}_{swi} = -\sum_{l=1}^{i-1}\hat{u}_{swl} - \frac{\sum_{l=1}^{i}\left[\sum_{r=1, r\ne l}^{i}\left(\prod\nolimits_{j=r}^{i} a_j\right)\hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})\right]\hat{u}_{eql}}{\sum_{r=1}^{i}\left(\prod\nolimits_{j=r}^{i} a_j\right)\hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})} - \frac{(k_i S_i) + \mathrm{sgn}(S_i)\hat{w}}{\sum_{r=1}^{i}\left(\prod\nolimits_{j=r}^{i} a_j\right)\hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})} - \frac{\mathrm{sgn}(S_i)\sum_{r=1}^{i}\left|\prod\nolimits_{j=r}^{i} a_j\right|\hat{\rho}_r(\mathbf{x}|\boldsymbol{\theta}_{\rho_r})}{\sum_{r=1}^{i}\left(\prod\nolimits_{j=r}^{i} a_j\right)\hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})} \quad (33)$$

where k_i is a positive constant, and the parameter update laws as follows:

$$\dot{\boldsymbol{\theta}}_{f_r} = \gamma_{f_r} S_i \left(\prod\nolimits_{j=r}^{i} a_j\right)\boldsymbol{\xi}(\mathbf{x}), \quad (34)$$

$$\dot{\boldsymbol{\theta}}_{b_r} = \gamma_{b_r} S_i \left(\prod\nolimits_{j=r}^{i} a_j\right) \boldsymbol{\xi}(\mathbf{x}) u, \tag{35}$$

$$\dot{\boldsymbol{\theta}}_{\rho_r} = \gamma_{\rho_r} \left|S_i\right| \left|\prod\nolimits_{j=r}^{i} a_j\right| \boldsymbol{\xi}(\mathbf{x}), \tag{36}$$

$$\dot{\hat{w}} = \gamma_w \left|S_i\right|, \tag{37}$$

where γ_{f_r}, γ_{b_r}, γ_{ρ_r}, for $r = 1, 2, \ldots, i$, and γ_w are positive constants.

Remark 1: Without loss of generality, the adaptive laws used in this paper are assumed that the parameter vectors are within the constraint sets or on the boundaries of the constraint sets but moving toward the inside of the constraint sets. If the parameter vectors are on the boundaries of the constraint sets but moving toward the outside of the constraint sets, we have to use the projection algorithm to modify the adaptive laws such that the parameter vectors will remain inside of the constraint sets. Readers can refer to reference [17]. The proposed adaptive laws (34)-(37) can be modified as the following form:

$$\dot{\boldsymbol{\theta}}_{f_r} = \begin{cases} \gamma_{f_r}\left(\prod_{j=r}^{i} a_j\right) S_i \boldsymbol{\xi}(\mathbf{x}), & \text{if } \left(\left\|\boldsymbol{\theta}_{f_r}\right\| < N_{f_r}\right) \text{ or } \\ & \left(\left\|\boldsymbol{\theta}_{f_r}\right\| = N_{f_r} \text{ and } \left(\prod_{j=r}^{i} a_j\right) S_i \boldsymbol{\theta}_{f_r}^T \boldsymbol{\xi}(\mathbf{x}) \le 0\right), \\ P\left\{\gamma_{f_r}\left(\prod_{j=r}^{i} a_j\right) S_i \boldsymbol{\xi}(\mathbf{x})\right\}, & \text{if } \begin{pmatrix} \left\|\boldsymbol{\theta}_{f_r}\right\| = N_{f_r} \text{ and} \\ \left(\prod_{j=r}^{i} a_j\right) S_i \boldsymbol{\theta}_{f_r}^T \boldsymbol{\xi}(\mathbf{x}) > 0 \end{pmatrix}, \end{cases} \tag{38}$$

where $P\left\{\gamma_{f_r}\left(\prod_{j=r}^{i} a_j\right) S_i \boldsymbol{\xi}(\mathbf{x})\right\}$ is defined as

$$P\left\{\gamma_{f_r}\left(\prod\nolimits_{j=r}^{i} a_j\right) S_i \boldsymbol{\xi}(\mathbf{x})\right\} = \gamma_{f_r}\left(\prod\nolimits_{j=r}^{i} a_j\right) S_i \boldsymbol{\xi}(\mathbf{x}) - \gamma_{f_r}\left(\prod\nolimits_{j=r}^{i} a_j\right) S_i \frac{\boldsymbol{\theta}_{f_r} \boldsymbol{\theta}_{f_r}^T}{\left\|\boldsymbol{\theta}_{f_r}\right\|^2} \boldsymbol{\xi}(\mathbf{x}). \tag{39}$$

$$\dot{\boldsymbol{\theta}}_{b_r} = \begin{cases} \gamma_{b_r}\left(\prod_{j=r}^{i} a_j\right) S_i \boldsymbol{\xi}(\mathbf{x}) u, & \text{if } \left(\left\|\boldsymbol{\theta}_{b_r}\right\| < N_{b_r}\right) \text{ or } \\ & \left(\left\|\boldsymbol{\theta}_{b_r}\right\| = N_{b_r} \text{ and } \left(\prod_{j=r}^{i} a_j\right) S_i \boldsymbol{\theta}_{b_r}^T \boldsymbol{\xi}(\mathbf{x}) u \le 0\right), \\ P\left\{\gamma_{b_r}\left(\prod_{j=r}^{i} a_j\right) S_i \boldsymbol{\xi}(\mathbf{x}) u\right\}, & \text{if } \begin{pmatrix} \left\|\boldsymbol{\theta}_{b_r}\right\| = N_{b_r} \text{ and} \\ \left(\prod_{j=r}^{i} a_j\right) S_i \boldsymbol{\theta}_{b_r}^T \boldsymbol{\xi}(\mathbf{x}) u > 0 \end{pmatrix}, \end{cases} \tag{40}$$

where $P\left\{\gamma_{b_r}\left(\prod_{j=r}^{i} a_j\right) S_i \boldsymbol{\xi}(\mathbf{x}) u\right\}$ is defined as

$$P\left\{\gamma_{b_r}\left(\prod\nolimits_{j=r}^{i} a_j\right) S_i \boldsymbol{\xi}(\mathbf{x}) u\right\} = \gamma_{b_r}\left(\prod\nolimits_{j=r}^{i} a_j\right) S_i \boldsymbol{\xi}(\mathbf{x}) u - \gamma_{b_r}\left(\prod\nolimits_{j=r}^{i} a_j\right) S_i \frac{\boldsymbol{\theta}_{b_r} \boldsymbol{\theta}_{b_r}^T}{\left\|\boldsymbol{\theta}_{b_r}\right\|^2} \boldsymbol{\xi}(\mathbf{x}) u. \tag{41}$$

$$\dot{\boldsymbol{\theta}}_{\rho_r} = \begin{cases} \gamma_{\rho_r}\left|\left(\prod_{j=r}^{i} a_j\right) S_i\right| \boldsymbol{\xi}(\mathbf{x}), & \text{if } \left(\left\|\boldsymbol{\theta}_{\rho_r}\right\| < N_{\rho_r}\right) \text{ or } \\ & \left(\left\|\boldsymbol{\theta}_{\rho_r}\right\| = N_{\rho_r} \text{ and } \left|\left(\prod_{j=r}^{i} a_j\right) S_i\right| \boldsymbol{\theta}_{\rho_r}^T \boldsymbol{\xi}(\mathbf{x}) \le 0\right), \\ P\left\{\gamma_{\rho_r}\left|\left(\prod_{j=r}^{i} a_j\right) S_i\right| \boldsymbol{\xi}(\mathbf{x})\right\}, & \text{if } \begin{pmatrix} \left\|\boldsymbol{\theta}_{\rho_r}\right\| = N_{\rho_r} \text{ and} \\ \left|\left(\prod_{j=r}^{i} a_j\right) S_i\right| \boldsymbol{\theta}_{\rho_r}^T \boldsymbol{\xi}(\mathbf{x}) > 0 \end{pmatrix}, \end{cases} \tag{42}$$

where $P\left\{\gamma_{\rho_r}\left|\left(\prod_{j=r}^{i} a_j\right) S_i\right| \boldsymbol{\xi}(\mathbf{x})\right\}$ is defined as

$$P\left\{\gamma_{\rho_r}\left|\left(\prod\nolimits_{j=r}^{i} a_j\right) S_i\right| \boldsymbol{\xi}(\mathbf{x})\right\} = \gamma_{\rho_r}\left|\left(\prod\nolimits_{j=r}^{i} a_j\right) S_i\right| \boldsymbol{\xi}(\mathbf{x}) - \gamma_{\rho_r}\left|\left(\prod\nolimits_{j=r}^{i} a_j\right) S_i\right| \frac{\boldsymbol{\theta}_{\rho_r} \boldsymbol{\theta}_{\rho_r}^T}{\left\|\boldsymbol{\theta}_{\rho_r}\right\|^2} \boldsymbol{\xi}(\mathbf{x}). \tag{43}$$

The main result of the robust adaptive tracking control scheme is summarized in the following theorem.

Theorem 1: Consider the single-input-multi-output uncertain under-actuated system (1). If ***Assumptions 1-2*** are satisfied, then the proposed adaptive fuzzy sliding mode controller defined by (32) with adaptive laws (34)-(37) guarantees that all signals of the closed-loop system are bounded, and the system states $\mathbf{x}(t)$ will converge to zero as $t \to \infty$.

Proof: Consider the Lyapunov function candidate

$$V_i = \frac{1}{2}\left(S_i^2 + \sum_{r=1}^{i} \frac{1}{\gamma_{f_r}} \tilde{\boldsymbol{\theta}}_{f_r}^T \tilde{\boldsymbol{\theta}}_{f_r} + \sum_{r=1}^{i} \frac{1}{\gamma_{b_r}} \tilde{\boldsymbol{\theta}}_{b_r}^T \tilde{\boldsymbol{\theta}}_{b_r} + \sum_{r=1}^{i} \frac{1}{\gamma_{\rho_r}} \tilde{\boldsymbol{\theta}}_{\rho_r}^T \tilde{\boldsymbol{\theta}}_{\rho_r} + \frac{1}{\gamma_w} \tilde{w}^2\right) \tag{44}$$

Differentiating the Lyapunov function V with respect to time, we can obtain

$$\dot{V}_i = S_i \dot{S}_i + \sum_{r=1}^{i} \frac{1}{\gamma_{f_r}} \tilde{\boldsymbol{\theta}}_{f_r}^T \dot{\tilde{\boldsymbol{\theta}}}_{f_r} + \sum_{r=1}^{i} \frac{1}{\gamma_{b_r}} \tilde{\boldsymbol{\theta}}_{b_r}^T \dot{\tilde{\boldsymbol{\theta}}}_{b_r} + \sum_{r=1}^{i} \frac{1}{\gamma_{\rho_r}} \tilde{\boldsymbol{\theta}}_{\rho_r}^T \dot{\tilde{\boldsymbol{\theta}}}_{\rho_r} + \frac{1}{\gamma_w} \tilde{w} \dot{\tilde{w}} \tag{45}$$

From (31) and by the fact $\dot{\tilde{\boldsymbol{\theta}}}_{f_r} = \dot{\boldsymbol{\theta}}_{f_r}$, $\dot{\tilde{\boldsymbol{\theta}}}_{b_r} = \dot{\boldsymbol{\theta}}_{b_r}$, $\dot{\tilde{\boldsymbol{\theta}}}_{\rho_r} = \dot{\boldsymbol{\theta}}_{\rho_r}$, and $\dot{\tilde{w}} = \dot{\hat{w}}$, the above equation becomes

$$\begin{aligned} \dot{V}_i &= S_i\left[\sum_{r=1}^{i}\left(\prod\nolimits_{j=r}^{i} a_j\right) \dot{s}_r\right] + \sum_{r=1}^{i} \frac{1}{\gamma_{f_r}} \tilde{\boldsymbol{\theta}}_{f_r}^T \dot{\boldsymbol{\theta}}_{f_r} + \sum_{r=1}^{i} \frac{1}{\gamma_{b_r}} \tilde{\boldsymbol{\theta}}_{b_r}^T \dot{\boldsymbol{\theta}}_{b_r} + \sum_{r=1}^{i} \frac{1}{\gamma_{\rho_r}} \tilde{\boldsymbol{\theta}}_{\rho_r}^T \dot{\boldsymbol{\theta}}_{\rho_r} + \frac{1}{\gamma_w} \tilde{w} \dot{\hat{w}} \\ &\le S_i \sum_{r=1}^{i}\left(\prod\nolimits_{j=r}^{i} a_j\right)\Big\{c_r x_{2r} + \left[f_r(\mathbf{x}) - \hat{f}_r(\mathbf{x}|\boldsymbol{\theta}_{f_r}^*)\right] + \left[\hat{f}_r(\mathbf{x}|\boldsymbol{\theta}_{f_r}^*) - \hat{f}_r(\mathbf{x}|\boldsymbol{\theta}_{f_r})\right] + \hat{f}_r(\mathbf{x}|\boldsymbol{\theta}_{f_r}) \\ &\qquad + \left[b_r(\mathbf{x})u - \hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r}^*)u\right] + \left[\hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r}^*)u - \hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})u\right] + \hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})u\Big\} \\ &\quad + \left|S_i\right| \sum_{r=1}^{i}\left|\prod\nolimits_{j=r}^{i} a_j\right|\Big\{\left[\rho_r(\mathbf{x}) - \hat{\rho}_r(\mathbf{x}|\boldsymbol{\theta}_{\rho_r}^*)\right] + \left[\hat{\rho}_r(\mathbf{x}|\boldsymbol{\theta}_{\rho_r}^*) - \hat{\rho}_r(\mathbf{x}|\boldsymbol{\theta}_{\rho_r})\right] + \hat{\rho}_r(\mathbf{x}|\boldsymbol{\theta}_{\rho_r})\Big\} \\ &\quad + \sum_{r=1}^{i} \frac{1}{\gamma_{f_r}} \tilde{\boldsymbol{\theta}}_{f_r}^T \dot{\boldsymbol{\theta}}_{f_r} + \sum_{r=1}^{i} \frac{1}{\gamma_{b_r}} \tilde{\boldsymbol{\theta}}_{b_r}^T \dot{\boldsymbol{\theta}}_{b_r} + \sum_{r=1}^{i} \frac{1}{\gamma_{\rho_r}} \tilde{\boldsymbol{\theta}}_{\rho_r}^T \dot{\boldsymbol{\theta}}_{\rho_r} + \frac{1}{\gamma_w} \tilde{w} \dot{\hat{w}} \end{aligned} \tag{46}$$

According to (12)-(14), (46) becomes

$$\dot{V}_i \le S_i\sum_{r=1}^{i}\left(\prod_{j=r}^{i} a_j\right)\left\{c_r x_{2r} + \hat{f}_r(\mathbf{x}|\boldsymbol{\theta}_{f_r}) + \hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})u\right\} + |S_i|\sum_{r=1}^{i}\left|\prod_{j=r}^{i} a_j\right|\hat{\rho}_r(\mathbf{x}|\boldsymbol{\theta}_{\rho_r})$$
$$+S_i\sum_{r=1}^{i}\left(\prod_{j=r}^{i} a_j\right)\left[\boldsymbol{\theta}_{f_r}^{*T}\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\theta}_{f_r}^{T}\boldsymbol{\xi}(\mathbf{x})\right] + S_i\sum_{r=1}^{i}\left(\prod_{j=r}^{i} a_j\right)\left[\boldsymbol{\theta}_{b_r}^{*T}\boldsymbol{\xi}(\mathbf{x})u - \boldsymbol{\theta}_{b_r}^{T}\boldsymbol{\xi}(\mathbf{x})u\right]$$
$$+|S_i|\sum_{r=1}^{i}\left|\prod_{j=r}^{i} a_j\right|\left[\boldsymbol{\theta}_{\rho_r}^{*T}\boldsymbol{\xi}(\mathbf{x}) - \boldsymbol{\theta}_{\rho_r}^{T}\boldsymbol{\xi}(\mathbf{x})\right]$$
$$+S_i\sum_{r=1}^{i}\left(\prod_{j=r}^{i} a_j\right)\left\{\left[f_r(\mathbf{x}) - \hat{f}_r(\mathbf{x}|\boldsymbol{\theta}_{f_r}^{*})\right] + \left[b_r(\mathbf{x})u - \hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r}^{*})u\right]\right\}$$
$$+|S_i|\sum_{r=1}^{i}\left|\prod_{j=r}^{i} a_j\right|\left[\rho_r(\mathbf{x}) - \hat{\rho}_r(\mathbf{x}|\boldsymbol{\theta}_{\rho_r}^{*})\right]$$
$$+\sum_{r=1}^{i}\frac{1}{\gamma_{f_r}}\tilde{\boldsymbol{\theta}}_{f_r}^{T}\dot{\boldsymbol{\theta}}_{f_r} + \sum_{r=1}^{i}\frac{1}{\gamma_{b_r}}\tilde{\boldsymbol{\theta}}_{b_r}^{T}\dot{\boldsymbol{\theta}}_{b_r} + \sum_{r=1}^{i}\frac{1}{\gamma_{\rho_r}}\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\dot{\boldsymbol{\theta}}_{\rho_r} + \frac{1}{\gamma_w}\tilde{w}\dot{\hat{w}} \quad (47)$$

According to (21)-(23) and (25)-(26), we have

$$\dot{V}_i \le S_i\sum_{r=1}^{i}\left(\prod_{j=r}^{i} a_j\right)\left\{c_r x_{2r} + \hat{f}_r(\mathbf{x}|\boldsymbol{\theta}_{f_r}) + \hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})u\right\} + |S_i|\sum_{r=1}^{i}\left|\prod_{j=r}^{i} a_j\right|\hat{\rho}_r(\mathbf{x}|\boldsymbol{\theta}_{\rho_r})$$
$$-S_i\sum_{r=1}^{i}\left(\prod_{j=r}^{i} a_j\right)\tilde{\boldsymbol{\theta}}_{f_r}^{T}\boldsymbol{\xi}(\mathbf{x}) - S_i\sum_{r=1}^{i}\left(\prod_{j=r}^{i} a_j\right)\tilde{\boldsymbol{\theta}}_{b_r}^{T}\boldsymbol{\xi}(\mathbf{x})u - |S_i|\sum_{r=1}^{i}\left|\prod_{j=r}^{i} a_j\right|\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\boldsymbol{\xi}(\mathbf{x})$$
$$+|S_i|w + \sum_{r=1}^{i}\frac{1}{\gamma_{f_r}}\tilde{\boldsymbol{\theta}}_{f_r}^{T}\dot{\boldsymbol{\theta}}_{f_r} + \sum_{r=1}^{i}\frac{1}{\gamma_{b_r}}\tilde{\boldsymbol{\theta}}_{b_r}^{T}\dot{\boldsymbol{\theta}}_{b_r} + \sum_{r=1}^{i}\frac{1}{\gamma_{\rho_r}}\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\dot{\boldsymbol{\theta}}_{\rho_r} + \frac{1}{\gamma_w}\tilde{w}\dot{\hat{w}} \quad (48)$$

Let us define v_{f_r}, v_{b_r}, and v_{ρ_r} as follows:

$$v_{f_r} = -S_i\sum_{r=1}^{i}\left(\prod_{j=r}^{i} a_j\right)\tilde{\boldsymbol{\theta}}_{f_r}^{T}\boldsymbol{\xi}(\mathbf{x}) + \sum_{r=1}^{i}\frac{1}{\gamma_{f_r}}\tilde{\boldsymbol{\theta}}_{f_r}^{T}\dot{\boldsymbol{\theta}}_{f_r} \quad (49)$$

$$v_{b_r} = -S_i\sum_{r=1}^{i}\left(\prod_{j=r}^{i} a_j\right)\tilde{\boldsymbol{\theta}}_{b_r}^{T}\boldsymbol{\xi}(\mathbf{x})u + \sum_{r=1}^{i}\frac{1}{\gamma_{b_r}}\tilde{\boldsymbol{\theta}}_{b_r}^{T}\dot{\boldsymbol{\theta}}_{b_r} \quad (50)$$

$$v_{\rho_r} = -|S_i|\sum_{r=1}^{i}\left|\prod_{j=r}^{i} a_j\right|\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\boldsymbol{\xi}(\mathbf{x}) + \sum_{r=1}^{i}\frac{1}{\gamma_{\rho_r}}\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\dot{\boldsymbol{\theta}}_{\rho_r} \quad (51)$$

If the condition in the first line of (38) is true, substituting (34) into (49), we have

$$v_{f_r} = -S_i\sum_{r=1}^{i}\left(\prod_{j=r}^{i} a_j\right)\tilde{\boldsymbol{\theta}}_{f_r}^{T}\boldsymbol{\xi}(\mathbf{x}) + S_i\sum_{r=1}^{i}\left(\prod_{j=r}^{i} a_j\right)\tilde{\boldsymbol{\theta}}_{f_r}^{T}\boldsymbol{\xi}(\mathbf{x}) = 0 \quad (52)$$

If the condition in the second line of (38) is true, we have

$$\left\|\boldsymbol{\theta}_{f_r}\right\| = N_{f_r} \text{ and } \left(\prod_{j=r}^{i} a_j\right)S_i\boldsymbol{\theta}_{f_r}^{T}\boldsymbol{\xi}(\mathbf{x}) > 0.$$

Then, substituting (39) into (49), we have

$$v_{fr} = -\sum_{r=1}^{i}\left[\frac{1}{2}\left\|\boldsymbol{\theta}_{f_r}\right\|^2 - \frac{1}{2}\left\|\boldsymbol{\theta}_{f_r}^{*}\right\|^2 + \frac{1}{2}\left\|\boldsymbol{\theta}_{f_r} - \boldsymbol{\theta}_{f_r}^{*}\right\|^2\right]\frac{\left(\prod_{j=r}^{i} a_j\right)S_i\boldsymbol{\theta}_{f_r}^{T}}{\left\|\boldsymbol{\theta}_{f_r}\right\|^2}\boldsymbol{\xi}(\mathbf{x}) \quad (53)$$

By the fact that $\left\|\boldsymbol{\theta}_{f_r}\right\| = N_{f_r}$, $\left\|\boldsymbol{\theta}_{f_r}^{*}\right\| \le N_{f_r}$, and $\left(\prod_{j=r}^{i} a_j\right)S_i\boldsymbol{\theta}_{f_r}^{T}\boldsymbol{\xi}(\mathbf{x}) > 0$, the above equation becomes

$$v_{f_r} \le 0. \quad (54)$$

Using the same method, we can prove that $v_{b_r} \le 0$ and $v_{\rho_r} \le 0$ for all $t \ge 0$. To show $\left\|\boldsymbol{\theta}_{b_r}\right\| \ge \delta$, we see from (40) that if $\left\|\boldsymbol{\theta}_{b_r}\right\| = \delta$, then $\dot{\boldsymbol{\theta}}_{b_r} \ge 0$; hence, we can guarantees $\left\|\boldsymbol{\theta}_{b_r}\right\| \ge \delta$. By applying (54), (37), and $v_{b_r} \le 0$ and $v_{\rho_r} \le 0$ into (48), it yields

$$\dot{V}_i \le S_i\sum_{r=1}^{i}\left(\prod_{j=r}^{i} a_j\right)\left\{c_r x_{2r} + \hat{f}_r(\mathbf{x}|\boldsymbol{\theta}_{f_r}) + \hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})u\right\}$$
$$+|S_i|\sum_{r=1}^{i}\left|\prod_{j=r}^{i} a_j\right|\hat{\rho}_r(\mathbf{x}|\boldsymbol{\theta}_{\rho_r}) + |S_i|\hat{w} \quad (55)$$

Using the control law (28) and (32), the above equation can be rewritten as

$$\dot{V}_i \le S_i\left\{\sum_{r=1}^{i}\left(\prod_{j=r}^{i} a_j\right)\left[\hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})\left(\sum_{\substack{l=1\\ l\ne r}}^{i}\hat{u}_{eql} + \sum_{l=1}^{i}\hat{u}_{swl}\right)\right]\right\}$$
$$+|S_i|\sum_{r=1}^{i}\left|\prod_{j=r}^{i} a_j\right|\hat{\rho}_r(\mathbf{x}|\boldsymbol{\theta}_{\rho_r}) + |S_i|\hat{w} \quad (56)$$

By applying (33), $\dot{V}_i$ can be obtained as follows:

$$\dot{V}_i \le -k_i S_i^2$$
$$= -k_i S_i^2 - \sum_{r=1}^{i}\frac{1}{2\gamma_{f_r}}\tilde{\boldsymbol{\theta}}_{f_r}^{T}\tilde{\boldsymbol{\theta}}_{f_r} - \sum_{r=1}^{i}\frac{1}{2\gamma_{b_r}}\tilde{\boldsymbol{\theta}}_{b_r}^{T}\tilde{\boldsymbol{\theta}}_{b_r} - \sum_{r=1}^{i}\frac{1}{2\gamma_{\rho_r}}\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\tilde{\boldsymbol{\theta}}_{\rho_r} - \frac{1}{2\gamma_w}\tilde{w}^2$$
$$+\sum_{r=1}^{i}\frac{1}{2\gamma_{f_r}}\tilde{\boldsymbol{\theta}}_{f_r}^{T}\tilde{\boldsymbol{\theta}}_{f_r} + \sum_{r=1}^{i}\frac{1}{2\gamma_{b_r}}\tilde{\boldsymbol{\theta}}_{b_r}^{T}\tilde{\boldsymbol{\theta}}_{b_r} + \sum_{r=1}^{i}\frac{1}{2\gamma_{\rho_r}}\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\tilde{\boldsymbol{\theta}}_{\rho_r} + \frac{1}{2\gamma_w}\tilde{w}^2 \quad (57)$$

Let $\mu = \sum_{r=1}^{i}\frac{1}{2\gamma_{f_r}}\tilde{\boldsymbol{\theta}}_{f_r}^{T}\tilde{\boldsymbol{\theta}}_{f_r} + \sum_{r=1}^{i}\frac{1}{2\gamma_{b_r}}\tilde{\boldsymbol{\theta}}_{b_r}^{T}\tilde{\boldsymbol{\theta}}_{b_r} + \sum_{r=1}^{i}\frac{1}{2\gamma_{\rho_r}}\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\tilde{\boldsymbol{\theta}}_{\rho_r} + \frac{1}{2\gamma_w}\tilde{w}^2$

$$\dot{V}_i \le -k_i S_i^2 - \sum_{r=1}^{i}\frac{1}{2\gamma_{f_r}}\tilde{\boldsymbol{\theta}}_{f_r}^{T}\tilde{\boldsymbol{\theta}}_{f_r} - \sum_{r=1}^{i}\frac{1}{2\gamma_{b_r}}\tilde{\boldsymbol{\theta}}_{b_r}^{T}\tilde{\boldsymbol{\theta}}_{b_r} - \sum_{r=1}^{i}\frac{1}{2\gamma_{\rho_r}}\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\tilde{\boldsymbol{\theta}}_{\rho_r} - \frac{1}{2\gamma_w}\tilde{w}^2 + \mu$$
$$= -k_i S_i^2 - \sum_{r=1}^{i}\frac{1}{2\gamma_{f_r}^2}\tilde{\boldsymbol{\theta}}_{f_r}^{T}\tilde{\boldsymbol{\theta}}_{f_r} - \sum_{r=1}^{i}\frac{1}{2\gamma_{b_r}^2}\tilde{\boldsymbol{\theta}}_{b_r}^{T}\tilde{\boldsymbol{\theta}}_{b_r} - \sum_{r=1}^{i}\frac{1}{2\gamma_{\rho_r}^2}\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\tilde{\boldsymbol{\theta}}_{\rho_r} - \frac{1}{2\gamma_w^2}\tilde{w}^2 + \mu$$
$$+\sum_{r=1}^{i}\frac{1-\gamma_{f_r}}{2\gamma_{f_r}^2}\tilde{\boldsymbol{\theta}}_{f_r}^{T}\tilde{\boldsymbol{\theta}}_{f_r} + \sum_{r=1}^{i}\frac{1-\gamma_{b_r}}{2\gamma_{b_r}^2}\tilde{\boldsymbol{\theta}}_{b_r}^{T}\tilde{\boldsymbol{\theta}}_{b_r} + \sum_{r=1}^{i}\frac{1-\gamma_{\rho_r}}{2\gamma_{\rho_r}^2}\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\tilde{\boldsymbol{\theta}}_{\rho_r} + \frac{1-\gamma_w}{2\gamma_w^2}\tilde{w}^2$$

Let $L = \sum_{r=1}^{i}\frac{1-\gamma_{f_r}}{2\gamma_{f_r}^2}\tilde{\boldsymbol{\theta}}_{f_r}^{T}\tilde{\boldsymbol{\theta}}_{f_r} + \sum_{r=1}^{i}\frac{1-\gamma_{b_r}}{2\gamma_{b_r}^2}\tilde{\boldsymbol{\theta}}_{b_r}^{T}\tilde{\boldsymbol{\theta}}_{b_r} + \sum_{r=1}^{i}\frac{1-\gamma_{\rho_r}}{2\gamma_{\rho_r}^2}\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\tilde{\boldsymbol{\theta}}_{\rho_r} + \frac{1-\gamma_w}{2\gamma_w^2}\tilde{w}^2 + \mu$

$$\dot{V}_i \le -k_i S_i^2 - \sum_{r=1}^{i}\frac{1}{2\gamma_{f_r}^2}\tilde{\boldsymbol{\theta}}_{f_r}^{T}\tilde{\boldsymbol{\theta}}_{f_r} - \sum_{r=1}^{i}\frac{1}{2\gamma_{b_r}^2}\tilde{\boldsymbol{\theta}}_{b_r}^{T}\tilde{\boldsymbol{\theta}}_{b_r} - \sum_{r=1}^{i}\frac{1}{2\gamma_{\rho_r}^2}\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\tilde{\boldsymbol{\theta}}_{\rho_r} - \frac{1}{2\gamma_w^2}\tilde{w}^2 + L$$
$$\le -c\left[\frac{1}{2}S_i^2 + \sum_{r=1}^{i}\frac{1}{2\gamma_{f_r}}\tilde{\boldsymbol{\theta}}_{f_r}^{T}\tilde{\boldsymbol{\theta}}_{f_r} + \sum_{r=1}^{i}\frac{1}{2\gamma_{b_r}}\tilde{\boldsymbol{\theta}}_{b_r}^{T}\tilde{\boldsymbol{\theta}}_{b_r} + \sum_{r=1}^{i}\frac{1}{2\gamma_{\rho_r}}\tilde{\boldsymbol{\theta}}_{\rho_r}^{T}\tilde{\boldsymbol{\theta}}_{\rho_r} + \frac{1}{2\gamma_w}\tilde{w}^2\right] + L$$

where $c = \min\left\{2k_i, \frac{1}{\gamma_{f_r}}, \frac{1}{\gamma_{b_r}}, \frac{1}{\gamma_{\rho_r}}, \frac{1}{\gamma_{w_r}}\right\}$. Then,

$$\dot{V}_i \le -cV_i + L. \quad (58)$$

Based on (58), we obtain

$$V_i(t) \le -e^{-ct}V_i(0) + \frac{L}{c}. \quad (59)$$

Then the tracking error converges to a region exponentially.

Substituting u_{swi} into $u_i = \sum_{r=1}^{i}(u_{swr} + u_{eqr})$ and letting i=n, we can obtain the hierarchical sliding mode control law

$$u_n = \sum_{l=1}^{n-1} \hat{u}_{swl} + \hat{u}_{swn} + \sum_{l=1}^{n} \hat{u}_{eql}$$

$$= \frac{\sum_{r=1}^{n}(\prod_{j=r}^{n} a_j)\hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})\hat{u}_{eqr}}{\sum_{r=1}^{n}(\prod_{j=r}^{n} a_j)\hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})} - \frac{(k_n S_n) + \mathrm{sgn}(S_n)\hat{w}}{\sum_{r=1}^{n}(\prod_{j=r}^{n} a_j)\hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})}$$

$$- \frac{\mathrm{sgn}(S_n)\sum_{r=1}^{i}\left|\prod_{j=r}^{i} a_j\right|\hat{\rho}_r(\mathbf{x}|\boldsymbol{\theta}_{\rho_r})}{\sum_{r=1}^{i}(\prod_{j=r}^{i} a_j)\hat{b}_r(\mathbf{x}|\boldsymbol{\theta}_{b_r})} \quad (60)$$

where $u = u_n$.

Remark 2: Note that the control law is discontinuous when the states across the sliding surface. Since the discontinuities in the control (60) give rise to chatter in the system, it has been proposed that the switching functions $\mathrm{sgn}(S_i)$ will be replaced by a continuous approximation in an ε-width region of S_i. Thus, replacing $\mathrm{sgn}(S_i)$ with $sat(S_i/\varepsilon)$, the $sat(S_i/\varepsilon)$ is described by

$$sat(S_i/\varepsilon) = \begin{cases} 1 & \text{if } S_i > \varepsilon \\ S_i/\varepsilon & \text{if } |S_i| \le \varepsilon, \quad \forall \varepsilon > 0. \\ -1 & \text{if } S_i < -\varepsilon \end{cases} \quad (61)$$

IV. AN EXAMPLE AND SIMULATION RESULTS

In this section, the inverted pendulum system is used to verify the performance of the proposed controller. Consider the inverted pendulum system described by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f_1(\mathbf{x}) + b_1(\mathbf{x})u + d_1(\mathbf{x},t) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = f_2(\mathbf{x}) + b_2(\mathbf{x})u + d_2(\mathbf{x},t) \end{cases}$$

$$\mathbf{y}(t) = [x_1, x_3]^T$$

(62)

where

$$f_1(\mathbf{x}) = \frac{m_t g \sin x_1 - m_p L \sin x_1 \cos x_1 x_2^2}{L \cdot \left(\frac{4}{3} m_t - m_p \cos^2 x_1\right)}$$

$$b_1(\mathbf{x}) = \frac{\cos x_1}{L \cdot \left(\frac{4}{3} m_t - m_p \cos^2 x_1\right)}$$

$$f_2(\mathbf{x}) = \frac{-\frac{4}{3} m_p L x_2^2 \sin x_1 + m_p g \sin x_1 \cos x_1}{\frac{4}{3} m_t - m_p \cos^2 x_1}$$

$$b_2(\mathbf{x}) = \frac{4}{3 \cdot \left(\frac{4}{3} m_t - m_p \cos^2 x_1\right)}$$

$$m_t = m_c + m_p$$

and $x_1 = \theta$, $x_2 = \dot{\theta}$, $x_3 = x$ and $x_4 = \dot{x}$ represent the angle of the pendulum with respect to the vertical axis, the angular velocity of the pendulum with respect to the vertical axis, the position of the cart and the velocity of the cart, respectively. $m_c = 1$ kg is the mass of cart, $m_p = 0.05$ kg is the mass of pole, $L = 0.5$ m is the half-length of pole, and $g = 9.8$ m/s^2 is the acceleration due to the gravity. u is the force input to move the cart. $d_1(\mathbf{x},t)$, and $d_2(\mathbf{x},t)$, are the external uncertainties. The control objective is to maintain the system states $\mathbf{x}(t)$ converge to zero.

In the implementation, six fuzzy sets are defined over interval [-3, 3] for x_1, x_2, x_3 and x_4, with labels *NB*, *NM*, *NS*, *PS*, *PM*, and *PB*, and their membership functions are

$$\mu_{NB}(x_i) = \frac{1}{1 + \exp(5(x_i + 2))},$$

$$\mu_{NM}(x_i) = \frac{1}{1 + \exp\left(-(x_i + 1.5)^2\right)},$$

$$\mu_{NS}(x_i) = \frac{1}{1 + \exp\left(-(x_i + 0.5)^2\right)},$$

$$\mu_{PS}(x_i) = \frac{1}{1 + \exp\left(-(x_i - 0.5)^2\right)},$$

$$\mu_{PM}(x_i) = \frac{1}{1 + \exp\left(-(x_i - 1.5)^2\right)},$$

$$\mu_{PB}(x_i) = \frac{1}{1 + \exp(-5(x_i - 2))}, i = 1,2,3,4.$$

A case is simulated in this inverted pendulum system, and we apply the hierarchical fuzzy sliding mode controller in Section 3 to deal with the control problem. In this case, the sliding surfaces are selected as $s_1 = c_1 x_1 + x_2$ and $s_2 = c_2 x_3 + x_4$, where $c_1 = 4$ and $c_2 = 1$, the hierarchical sliding surfaces are constructed as $S_1 = s_1$, $S_2 = a_1 s_1 + s_2$, where $a_1 = 1.1$. The initial values are chosen as $\mathbf{x}(0) = [-\frac{\pi}{6}, 0, 0, 0]^T$, $\boldsymbol{\theta}_{f_1}(0) = 0$, $\boldsymbol{\theta}_{f_2}(0) = 0$, $\boldsymbol{\theta}_{b_1}(0) = 5$, $\boldsymbol{\theta}_{b_2}(0) = 5$, $\boldsymbol{\theta}_{\rho_1}(0) = 0$, $\boldsymbol{\theta}_{\rho_2}(0) = 0$, and $\hat{w}(0) = 0$. The other parameters are selected as $k = 10$, $\gamma_{f_1} = 0.5$, $\gamma_{b_1} = 0.5$, $\gamma_{f_2} = 0.5$, $\gamma_{b_2} = 0.5$, $\gamma_{\rho_1} = 0.5$, $\gamma_{\rho_2} = 0.5$, and $\gamma_w = 0.2$, and the boundary layer $\varepsilon = 0.03$. $d_1(\mathbf{x},t) = \frac{3}{2}x_2^2$, and $d_2(\mathbf{x},t) = |x_1 x_2^2|$, are external uncertainties. The simulation results are shown in Figs. 1-5.

Figs. 1-2 reveal that the state trajectories of the states x_1 and x_3, respectively. From these simulation results, it is easily shown that the proposed controller ensures that the state trajectories converge asymptotically to zero. The performance of sliding dynamics and the control signal are shown in Figs. 3-4 and Fig. 5, respectively. The simulation results verify the effectiveness of the proposed robust hierarchical fuzzy sliding mode controller.

V. CONCLUSION

In this paper, a hierarchical fuzzy sliding mode controller is constructed to deal with the problems for a class of SIMO under-actuated systems with uncertainties. Within the scheme, the fuzzy logic systems and some adaptive laws are used to approximate the unknown nonlinear functions and the unknown upper bounds of uncertainties. Based on Lyapunov stability theorem and the theory of sliding mode control, the presented controller can not only guarantee the convergence to zero of each sliding surface, but also ensure the robust stability of the uncertain nonlinear under-actuated system. Finally, some simulation results are illustrated to confirm the effectiveness of the proposed control method.

REFERENCES

[1] D. Qian, J. Yi, and D. Zhao, "Control of a class of under-actuated systems with saturation using hierarchical sliding mode," *IEEE International Conference on Robotics and Automation*, pp. 2429 - 2434, May, 2008.

[2] F. Nafa, S. Labiod, and H. Chekireb, "A structured sliding mode controller for a class of under-actuated mechanical systems," *International Workshop on Systems, Signal Processing and their Applications (WOSSPA)*, pp. 243 - 246, May, 2011.

[3] W. Wang, J. Yi, D. Zhao, and D. Liu, "Design of a stable sliding-mode controller for a class of second-order underactuated systems," *IEE Proc.-Control Theory Appl.,* Vol. 151, No. 6, pp. 683 - 690, Nov., 2004.

[4] J. X. Xu, Z. Q. Guo, and T. H. Lee, "A synthesized integral sliding mode controller for an underactuated unicycle," *International Workshop on Variable Structure Systems*, pp. 352 - 357, June., 2010.

[5] M. Zhang and T. J. Tarn, "A hybrid switching control strategy for nonlinear and underactuated mechanical Systems," *IEEE Trans. Automatic Control*, vol. 48, no. 10,pp. 1777 - 1782, Oct., 2003.

[6] I. Hussein and M. Bloch, "Optimal control of underactuated nonholonomic mechanical systems," *IEEE Trans. Automatic Control*, vol. 53, no. 3, pp. 668 - 682, Apr., 2008.

[7] Y. Fang, B. Ma, P. Wang, and X. Zhang, "A motion planning-based adaptive control method for an underactuated crane system," *IEEE Trans. Control Systems Technology,* vol. 20, no. 1, pp. 241 - 248, Jan., 2012.

[8] T. Li, B. Yu, and B. Hong,"A novel adaptive fuzzy design for path following for underactuated ships with actuator dynamics," *IEEE Conference on Industrial Electronics and Applications*, pp.2796 - 2800, May, 2009.

[9] V. Sankaranarayanan and A. D. Mahindrakar, "Control of a class of underactuated mechanical systems using sliding modes," *IEEE Trans. Robotics*, vol. 25, no. 2, pp. 459-467, Apr., 2009.

[10] J. M. Yang and J. H. Kim, "Sliding mode control for trajectory tracking of nonholonomic wheeled mobile robots," *IEEE Trans. Robotic and Automation*, vol. 15, no. 3, pp. 578-587, June., 1999.

[11] W. Wang, X. D. Liu, and J. Q. Yi, "Structure design of two types of sliding-mode controllers for a class of underactuated mechanical systems," *IET Control Theory Appl.*, vol. 1, no. 1, pp. 163-172, Jan., 2007.

[12] R. Martinez, J. Alvarez, and Y. Orlov, "Hybrid sliding-mode-based Control of underactuated systems with dry friction," *IEEE Trans. Industrial Electronics*, vol. 55, no. 11, pp. 3998 - 4003, Nov., 2008.

[13] C. L. Hwang, H. M. Wu, and C. L. Shih, " Fuzzy sliding-Mode underactuated control for autonomous dynamic balance of an electrical bicycle," *IEEE Trans. control systems technology*, vol. 17, no. 3, pp. 783-795, May., 2009.

[14] C. C. Kung, T. H. Chen, and L. C. Huang, "Adaptive fuzzy sliding mode control for a class of underactuated Systems," *IEEE International Conference on Fuzzy Systems*, pp. 1791 - 1796, Aug., 2009.

[15] S. Y. Shin, J. Y. Lee, M. Sugisaka, and J. J. Lee, "Decoupled fuzzy adaptive sliding mode control for under-actuated systems with mismatched Uncertainties," *IEEE International Conference on Information and Automation*, vol.3, pp. 599 - 604, June., 2010.

[16] Y. Yang and J. Ren, "Adaptive fuzzy robust tracking controller design via small gain approach and its application," *IEEE Trans. Fuzzy Systems*, vol. 11, no. 6, pp. 783-795, Dec., 2003.

[17] L. X. Wang, " *A course in fuzzy systems and control*, " Englewood Cliffs, NJ: Prentice-Hall, 1997.

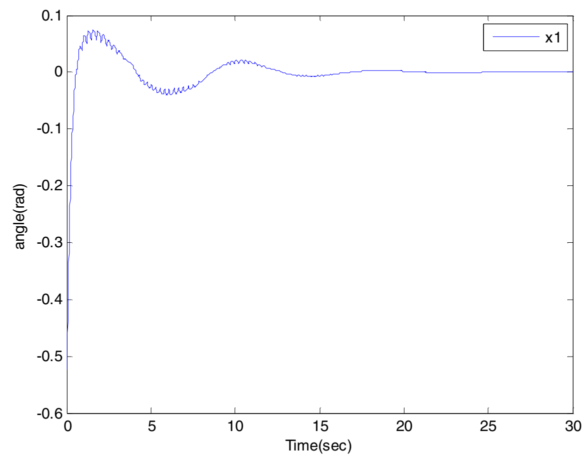


Fig. 1. The state trajectory of the state x_1 .

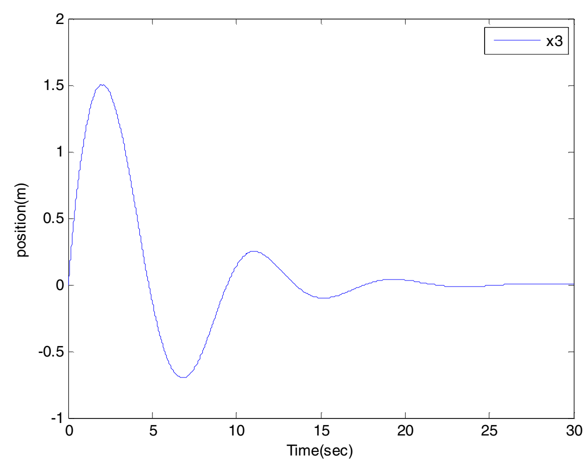
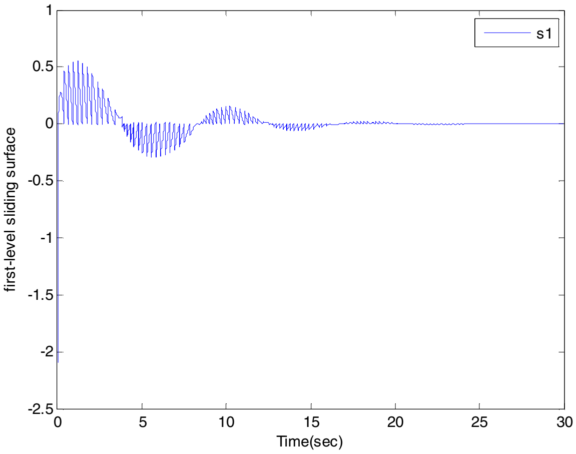


Fig. 2. The state trajectory of the state x_3 .



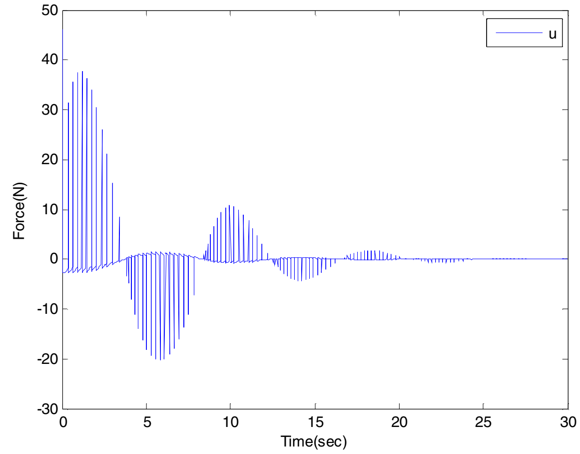


Fig. 3. The sliding surface $s_1 = c_1 x_1 + x_2$.
Fig. 5. The control input u.

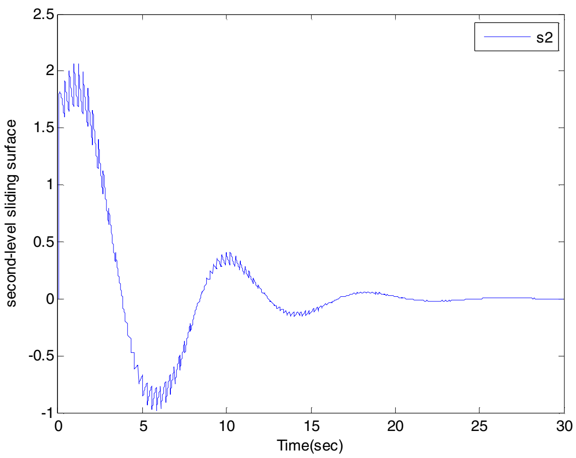


Fig. 4. The sliding surface $s_2 = c_2 x_3 + x_4$.