

# Bayesian Games with Ambiguous Type Players

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**Abstract**—Bayesian games can handle the incomplete information about players' types. However, in real life, the information could be not only incomplete but also ambiguous for lack of sufficient evidence, i.e., a player cannot have a probability precisely about each type of the other players. To address this issue, we extend the Bayesian games to ambiguous Bayesian games. We also illustrate and analyse our game model.

## I. INTRODUCTION

Bayesian games [1]–[3] can handle incomplete information about players' types. That is, in a Bayesian game, players know the probability precisely about each type of other players although they do not know the exact types of each other. However, in real-life games, sometimes a player even cannot know the precise probability of each type of other players, but can only know the overall probability for several types of others for lack of sufficient evidence. For example, in a security game [4], the available intelligence shows that airport attackers could be the curbside bomb attackers, the large truck bomb attackers, or the sniper attackers. However, the available evidence only shows the probability that this attacker takes the curbside bomb attack or the large truck bomb attack is 0.4, and the probability that he takes the curbside bomb attack or the sniper attack is 0.6. So, the police could not have the accurate belief about the probability of each single type.

On the other hand, Dempster-Shafer (D-S) theory [5], [6] is an effective tool to handle the ambiguous information, i.e., the imprecise probabilities, by mass function. Based on D-S theory, Strat [7] figures out how to calculate interval-valued expected utilities for the ambiguous decision problems. He further derives a point-valued expected utility by using the Hurwicz criterion [8]. Recently, Ma *et al.* [9] extend the work of Strat [7] to form an ambiguous decision framework.

Now based on the ambiguous decision framework [9], in this paper, we propose an ambiguous Bayesian games model to extend Bayesian games to handle ambiguous information about player types. That is, in our ambiguous Bayesian games, the belief about the players' types is represented by mass functions in D-S theory [5], [6]. Thus, the probability could be not only for one type but also for several types together. If the expected utility of a strategy chosen by a player is determined by different types of the other players but no precise probability of each single type is available, then the expected utility of a strategy becomes an interval [7]. Fortunately, from the

interval, the player can get a point-valued expected utility by using the methods of [7], [9]. Further, some players prefer the choice with the less ambiguous belief (i.e., they are ambiguity-averse [9]–[11]), but others prefer the choice with the more ambiguous belief (i.e., they are ambiguity-seeking [10], [11]). So, in this paper we propose different methods to model these different situations by extending the model of [9] that can only reflect ambiguity-averse [12] decision makers. In addition, similar to the Harsanyi transformation [13], we propose a method to find the ambiguous Nash equilibrium in our model.

The main contributions of this paper are: (i) we identify the ambiguous information about players' types; (ii) we introduce the concept of the ambiguous Bayesian Nash equilibrium; (iii) we provide methods to model ambiguity-averse and ambiguity-seeking players; (iv) we reveal some insights into our model; and (v) we show that our ambiguous Bayesian games are the Bayesian games when the probability distribution among the players' types becomes precise.

The rest of this paper is organised as follows. Section II recaps the ambiguous decision framework and Bayesian games. Section III defines our game model. Section IV illustrates our model. Section V analyses our model. Section VI discusses the related work. Finally, Section VII concludes this paper.

## II. PRELIMINARIES

This section will recap an ambiguous decision framework and Bayesian games.

### A. A Decision Model Based on Dempster-Shafer Theory

D-S theory [5] can model the ambiguous belief when the evidence is insufficient. Formally, we have:

*Definition 1:* Let  $\Theta$  be a finite set of mutually disjoint atomic elements, called a **frame of discernment**, and  $2^\Theta$  be the set of all the subsets of  $\Theta$ . Then a **basic probability assignment**, or called a **mass function**, is a mapping of  $m : 2^\Theta \rightarrow [0, 1]$ , which satisfies  $m(\emptyset) = 0$  and  $\sum_{A \subseteq \Theta} m(A) = 1$ . Subset  $A (\subset \Theta)$  satisfying  $m(A) > 0$  is called a **focal element** of mass function  $m$ .

The more elements in focal elements of a mass function and the bigger the mass function values of focal elements, the more ambiguous the belief is. This can be captured by the following definition [9], [14]–[16]:

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**Definition 2:** The **ambiguity degree** of a mass function  $m$  over discernment frame  $\Theta$ , denoted as  $\delta$ , is given by

$$\delta(m) = \frac{\sum_{A \subseteq \Theta} m(A) \log_2 |A|}{\log_2 |\Theta|}, \quad (1)$$

where  $|A|$  and  $|\Theta|$  are the cardinality of sets  $A$  and  $\Theta$ , respectively. Especially, if  $|\Theta| = 1$ ,  $\delta(m) = 0$ .

Formula (1) reflects well that the more ambiguous the belief, the higher the ambiguity degree. In particular,  $\forall A \subseteq \Theta$ , if  $|A| = 1$  when  $m(A) > 0$ , then  $\delta(m) = 0$ , which represents the precise belief; and if  $m(\Theta) = 1$ , then  $\delta(m) = 1$ , which represents the most ambiguous belief.

Strat [7] defines the expected utility interval as follows:

**Definition 3:** Given a choice of  $c$  corresponding to mass function  $m$  over the possible consequence set, denoted as  $\Theta$ , of all the choices, let  $u(a_i, c)$  be the utility of choice  $c$  corresponding to element  $a_i$  in focal element  $A$ . Then the **expected utility interval** of choice  $c$  is  $EUI(c) = [\underline{E}(c), \overline{E}(c)]$ , where

$$\underline{E}(c) = \sum_{A \subseteq \Theta} \min\{u(a_i, c) \mid a_i \in A\} m(A), \quad (2)$$

$$\overline{E}(c) = \sum_{A \subseteq \Theta} \max\{u(a_i, c) \mid a_i \in A\} m(A). \quad (3)$$

Here  $\underline{E}(c)$  and  $\overline{E}(c)$  are called the lower and upper boundaries of the expected utility of choice  $c$ , respectively.

If two choices' expected utility intervals do not overlap, it is easy to make a choice (i.e., just choose the one which lower boundary is higher than the other's upper boundary); otherwise, the choice is unclear [7], [9]. In this case, more evidence is required in order to make the expected utility intervals no longer overlap. However, what should we do if we cannot have more evidence? In this case, the Hurwicz criterion [8] can help. In fact, Strat [7] extends the Hurwicz criterion to handle the expected utility interval:

**Definition 4:** Given expected utility interval  $EUI(c) = [\underline{E}(c), \overline{E}(c)]$  of choice  $c$ , the **point-valued expected utility** of choice  $c$  is defined as:

$$E(c) = (1 - \alpha)\underline{E}(c) + \alpha\overline{E}(c), \quad (4)$$

where  $\alpha \in [0, 1]$  represents the decision maker's ambiguity attitude.

In the above definition, the *ambiguity attitude* refers to the decision maker's attitude towards the ambiguous belief. Dimmock *et al.* [10] argue that the ambiguity attitude could be *ambiguity averse*, *neutral*, or *seeking*. If a decision maker takes the ambiguity averse attitude, he prefers the choice with the more precise belief to the choice with the more ambiguous belief; conversely, if a decision maker takes the ambiguity seeking attitude, he prefers the choice with the more ambiguous belief to the choice with the more precise belief. Dimmock *et al.* [10] also show that  $\alpha < 0.5$  reflects ambiguity aversion,  $\alpha > 0.5$  reflects ambiguity seeking, and  $\alpha = 0.5$  reflects ambiguity neutral.

In the ambiguous environment, after getting the expected utility interval by formulas (2) and (3), we can get a point-valued preference degree about each choice by formula (4) if the ambiguity attitude of a decision maker is known.

## B. Bayesian Games

In a game, when every player has several types and thus several possible utilities functions, a player's utility of taking a strategy is uncertain if he does not know others' types, but every player can have a belief about other players' types. This kind of games are Bayesian games. Formally, we have:

**Definition 5:** A **Bayesian game** is a tuple of  $(N, \{S_i, T_i, p_i, u_i\}_{i \in N})$ , where

- (i)  $N = \{1, \dots, n\}$  is the set of players;
- (ii)  $S_i$  is a set of strategies of player  $i \in N$ ;
- (iii)  $T_i$  is a set of types of player  $i \in N$ ;
- (iv)  $p_i(t_{-i} \mid t_i)$  is player  $i$ 's probability distribution over other players' possible types  $t_{-i}$ , given player  $i$ 's own type  $t_i$ ; and
- (v)  $u_i(s_1, \dots, s_n, t_i)$  is player  $i$ 's utility function with his own type  $t_i$ .

With the Harsanyi transformation [13], the Bayesian Nash equilibrium can be defined as follows:

**Definition 6:** Given a Bayesian game of  $(N, \{S_i, T_i, p_i, u_i\}_{i \in N})$ , the strategy profile  $s^* = (s_1^*, \dots, s_n^*)$  is a **Bayesian Nash equilibrium** if  $\forall i \in N, t_i \in T_i$ ,

$$s_i^*(t_i) = \arg \max_{s_i \in S_i} \sum_{t_{-i} \in T_{-i}} u_i(s_1^*(t_1), \dots, s_{i-1}^*(t_{i-1}), s_i, s_{i+1}^*(t_{i+1}), \dots, s_n^*(t_n), t_i) p_i(t_{-i} \mid t_i). \quad (5)$$

## III. GAMES WITH AMBIGUOUS INFORMATION

This section proposes our ambiguous Bayesian games.

**Definition 7:** An **ambiguous Bayesian game** is defined as a 5-tuple of  $(N, \{T_i, S_i, M_i, u_i\}_{i \in N})$ , where:

- (i)  $N = \{1, \dots, n\}$  is a finite set of players;
- (ii)  $T_i = \{t_{i,1}, \dots, t_{i,m_i}\}$  is player  $i$ 's disjoint type set, and player  $i$  knows his type  $t_i$  and is uncertain about the types of the other players;
- (iii)  $S_i = \{s_{i,1}, \dots, s_{i,n_i}\}$  is the strategy set for player  $i \in N$ , and  $s_i(t_{i,k}) \in S_i$  is the strategy chosen by player  $i$  with type  $t_{i,k}$  and  $S_i(T_i) = \prod_{t_{i,k} \in T_i} s_i(t_{i,k}) = (s_i(t_{i,1}), \dots, s_i(t_{i,m_i}))$  is the strategy profile chosen by every type of player  $i$ ;
- (iv)  $M_i = \{m_{i,j} \mid j \in N - \{i\}\}$ , where  $m_{i,j} : 2^{T_j} \rightarrow [0, 1]$  is player  $i$ 's mass function over the frame of discernment  $T_j$  (i.e., the player  $j$ 's type set); and
- (v)  $u_i(s_1(t_{1,h}), \dots, s_{i-1}(t_{i-1,k}), s_{i,j}, s_{i+1}(t_{i+1,l}), \dots, s_n(t_{n,r}), t_{i,k})$  with  $s_i(t_{i,k}) = s_{i,j}$  is the utility function of player  $i$  with type  $t_{i,k}$  over strategy profile  $(s_1(t_{1,h}), \dots, s_{i-1}(t_{i-1,k}), s_{i,j}, s_{i+1}(t_{i+1,l}), \dots, s_n(t_{n,r}))$  to  $\mathbb{R}$ , given the other players' types  $t_{1,h}, \dots, t_{i-1,k}, t_{i+1,l}, \dots, t_{n,r}$ , respectively.

In the above ambiguous Bayesian games, every player has the ambiguous belief about the other players' types. That is, in an ambiguous Bayesian game, each player  $i$  knows his type  $t_{i,k}$  and the type set,  $T_j$ , that player  $j$  could be in, but he does

not know the precise probability distribution over these types. What player  $i$  knows may be only the probability distribution over  $T_j$ 's power set  $2^{T_j}$ , (i.e., he may not know the probability of each type of player  $j$ ). Thus, the information for player  $i$  about player  $j$ 's types is ambiguous. Thus, player  $i$  have mass function  $m_{i,j}$  to measure this ambiguous information. Each player  $i$  has strategy set  $S_i$ , but a player of each type  $t_{i,k}$  may have a different utility function  $u_i(s_1(t_{1,h}), \dots, s_{i-1}(t_{i-1,k}), s_{i,j}, s_{i+1}(t_{i+1,l}), \dots, s_n(t_{n,r}), t_{i,k})$ . To find the best response strategy, player  $i$  needs to know the expected utility. However, the utility obtained from a chosen strategy is determined by all the types of other players. Thus, player  $i$  has to make a decision in the light of the ambiguous information about the other players' types. Moreover, in ambiguous decision environments, the player could be ambiguity averse, seeking, or neutral [10]. So, formally we have:

**Definition 8:** In ambiguous Bayesian game  $(N, \{T_i, S_i, M_i, u_i\}_{i \in N})$ ,

(i) for the ambiguity-averse players, the ambiguity attitude, called his ambiguity-averse degree, is given by:

$$\beta(m) = \frac{1 - \delta(m)}{2}; \quad (6)$$

(ii) for the ambiguity-seeking player, the ambiguity attitude, called his ambiguity-seeking degree, is given by:

$$\beta(m) = \frac{1 + \delta(m)}{2}; \quad (7)$$

(iii) for the ambiguity-neutral player, the ambiguity attitude is given by:

$$\beta(m) = 0.5, \quad (8)$$

where  $\delta(m) \in [0, 1]$  is mass function  $m$ 's ambiguity degree.

By the above definition,  $\beta < 0.5$  is for ambiguity averse,  $\beta > 0.5$  is for ambiguity seeking, and  $\beta = 0.5$  is for ambiguity neutral, which is equal to the range of  $\alpha$  in the Hurwicz criterion. And we calculate the ambiguity attitude degree from the ambiguity degree because it can capture the following intuitions. First, for the same ambiguity degree, the ambiguity-seeking degree is higher than the ambiguity-averse degree, which reflects that an ambiguity-seeking player prefers the strategy with the ambiguous belief more than an ambiguity-averse player. Second, the higher the ambiguity degree, the lower the ambiguity-averse degree and the higher the ambiguity-seeking degree. That is, an ambiguity-averse player will avoid the strategy with the more ambiguous belief, but an ambiguity-seeking player will prefer it more.

Furthermore, in the ambiguous environment, every player  $i$  with the certain ambiguity attitude can make a decision to find his best response strategy. To find the best response strategy, the best way is to find the preference ordering over all strategies. Then player  $i$  needs to have the point-valued expected utility to represent his utility of the chosen strategy in the ambiguous environment. Formally, we have:

**Definition 9:** In ambiguous Bayesian game  $(N, \{T_i, S_i, M_i, u_i\}_{i \in N})$ , let player  $i$ 's type be  $t_{i,k}$ ,  $s_{i,j} = s_i(t_{i,k})$ , and strategy profile  $\vec{S}(s_{i,j}) = (S_1(T_1), \dots, S_{i-1}(T_{i-1}), s_{i,j}, S_{i+1}(T_{i+1}), \dots, S_n(T_n))$ , which is chosen by player  $i$  of

$t_{i,k}$  and other players with every type. Then the point-valued expected utility of player  $i$  with type  $t_{i,k}$  is

$$u_i(\vec{S}(s_{i,j}), t_{i,k}) = (1 - \beta(m_{i,n})) \underline{u}_i(\vec{S}(s_{i,j}), t_{i,k}, T_n) + \beta(m_{i,n}) \bar{u}_i(\vec{S}(s_{i,j}), t_{i,k}, T_n), \quad (9)$$

where:

$$\underline{u}_i(\vec{S}(s_{i,j}), t_{i,k}, T_n) = \sum_{\tau \subseteq T_n} \min\{u_i(\vec{S}(s_{i,j}), t_{i,k}, t_{n,h}) | t_{n,h} \in \tau\} m_{i,n}(\tau), \quad (10)$$

$$\bar{u}_i(\vec{S}(s_{i,j}), t_{i,k}, T_n) = \sum_{\tau \subseteq T_n} \max\{u_i(\vec{S}(s_{i,j}), t_{i,k}, t_{n,h}) | t_{n,h} \in \tau\} m_{i,n}(\tau), \quad (11)$$

where:

$$u_i(\vec{S}(s_{i,j}), t_{i,k}, t_{n,h}) = (1 - \beta(m_{i,n-1})) \underline{u}_i(\vec{S}(s_{i,j}), t_{i,k}, T_{n-1}) + \beta(m_{i,n-1}) \bar{u}_i(\vec{S}(s_{i,j}), t_{i,k}, T_{n-1}), \quad (12)$$

where  $\underline{u}_i(\vec{S}(s_{i,j}), t_{i,k}, T_{n-1})$  and  $\bar{u}_i(\vec{S}(s_{i,j}), t_{i,k}, T_{n-1})$  are calculated by the formulas that have the same forms as formulas (10) and (11) but replace  $T_n$ ,  $u_i(\vec{S}(s_{i,j}), t_{i,k}, t_{n,h})$ , and  $m_{i,n}$ , respectively, with  $T_{n-1}$ ,  $u_i(\vec{S}(s_{i,j}), t_{i,k}, t_{n-1,h})$ , and  $m_{i,n-1}$ ; and  $\beta$  is player  $i$ 's ambiguity attitude (ambiguity averse, seeking, or neutral) to the mass function. With the similar method and the decreasing of  $n$ , player  $i$  considers all others players' types and gets his expected utility.

Actually, Definition 9 gives a point-valued expected utility by setting ambiguity attitude in the Hurwicz criterion. And formulas (10) and (11) are the variations of formulas (2) and (3). In our model, according to the belief about every other player's types, corresponding to each type of each player there is a subgame with each player. In each subgame, by Definition 9, every player  $i$  could have a lower expected utility and an upper expected utility based on the different utilities for each type of every other player. That is, every player can obtain an expected utility interval from their beliefs about the other players' types on every strategy profile by formulas (10) and (11). Then, by formula (9), they can have point-valued utility over every strategy profile to represent their preference. Specifically, formula (9) uses the ambiguity attitude, calculated by formulas (6), (7) or (8), to get the value between the lower utility and the upper utility, obtained by formulas (10) and (11).

Now based on the point-valued expected utilities, players can get linear preferences over their strategies to find their best response strategies. Thus, they can find the Nash equilibrium in the ambiguous Bayesian game. Formally, we have:

**Definition 10:** In ambiguous Bayesian game  $(N, \{T_i, S_i, M_i, u_i\}_{i \in N})$ , strategy profile  $\vec{S}^* = (S_1^*(T_1), \dots, S_n^*(T_n))$  is an **ambiguous Bayesian Nash equilibrium** if for each player  $i$  and for each of his types  $t_{i,k} \in T_i$ ,  $\vec{S}^*$  satisfies:

$$\forall s_{i,-j} \in S_i, u_i(\vec{S}^*(s_{i,-j}), t_{i,k}) \geq u_i(\vec{S}^*(s_{i,-j}), t_{i,k}) \quad (13)$$

where  $\vec{S}^*(s_{i,-j}) = (S_1^*(T_1), \dots, S_{i-1}^*(T_{i-1}), s_{i,-j}, S_{i+1}^*(T_{i+1}), \dots, S_n^*(T_n))$  and  $S_i^*(T_i) = (s_i^*(t_{i,1}), \dots, s_i^*(t_{i,m_i}))$ .

That is, if every strategy, for every player and every type, in a strategy profile is a best response strategy, then the strategy

profile is an ambiguous Bayesian Nash equilibrium. When every type of players is regarded as a player, the ambiguous Bayesian Nash equilibrium is a Nash equilibrium.

#### IV. ILLUSTRATION

In this section, we illustrate how to play an ambiguous Bayesian game for a famous security problem [4].

Now we solve the security game as shown in Table I. There are two players: (i) the defender who has only one type  $T_d = \{t_{d,1}\}$ , and should protect three locations  $s_1, s_2$ , and  $s_3$  (for example, three terminals at the Los Angeles International Airport [4]); and (ii) the attacker who has three types  $T_a = \{t_{a,1}, t_{a,2}, t_{a,3}\}$  (for example, the curbside bomb attacker, the large truck bomb attacker, and the sniper attacker [4]). The attacker can launch an attack to any location of  $s_1, s_2$ , and  $s_3$ . Here the defender knows that the attacker has three types, but he is unsure which type the attack is exactly as shown in Table I. Let  $A_1 = \{t_{a,1}, t_{a,2}\}$  and  $A_2 = \{t_{a,1}, t_{a,3}\}$ . Then  $m_{d,a}(A_1) = 0.4$  and  $m_{d,a}(A_2) = 0.6$  are the defender's mass function about the attacker's types. And the attacker knows that the defender has only one type. That is,  $m_{a,d}(T_d) = 1$  is the attacker's mass function about the defender's type. We suppose all the players are ambiguity-averse and the defender and the attacker take the strategy simultaneously. Then, for the belief, by formula (1), the ambiguity degree of mass functions  $m_{d,a}$  is:

$$\begin{aligned}\delta(m_{d,a}) &= \frac{m_{d,a}(A_1) \log_2 |A_1| + m_{d,a}(A_2) \log_2 |A_2|}{\log_2 |T_a|} \\ &= \frac{0.4 \log_2 2 + 0.6 \log_2 2}{\log_2 3} \\ &= 0.6309.\end{aligned}$$

That is, the defender's belief about the attacker's types is ambiguous. For  $|T_d| = 1$ ,  $\delta(m_{a,d}) = 0$ . That is, the attacker's belief about the defender's type is precise. Then, by formula (6) the corresponding ambiguity-averse degrees are:

$$\begin{aligned}\beta(m_{d,a}) &= \frac{1 - \delta(m_{d,a})}{2} = \frac{1 - 0.6309}{2} = 0.1845, \\ \beta(m_{a,d}) &= \frac{1 - \delta(m_{a,d})}{2} = \frac{1 - 0}{2} = 0.5.\end{aligned}$$

By formulas (9)–(11), the defender can get the utility interval first, and then gets the corresponding point-valued utility. When  $s_a(t_{a,1}) = s_a(t_{a,2}) = s_a(t_{a,3}) = s_1$  (i.e.,  $\vec{S}(s_1) = (s_1, s_1, s_1, s_1)$  (the first is the strategy of the defender, the second is the strategy of the attacker with type  $t_{a,1}$ , the third is the strategy of the attacker with type  $t_{a,2}$ , and the fourth is the strategy of the attacker with type  $t_{a,3}$ ), to obtain the point-valued utility  $u_d(\vec{S}(s_1), t_{d,1}) = u_d(s_1, s_1, s_1, s_1, t_{d,1})$ , by formulas (10) and (11), the defender first gets the following

expected utility interval:

$$\begin{aligned}&\underline{u}_d(s_1, s_1, s_1, s_1, t_{d,1}, T_a) \\ &= \sum_{\tau \subseteq T_a} \min\{u_d((s_1, s_a(t_{a,j}), t_{d,1}) \mid t_{a,j} \in \tau) m_{d,a}(\tau) \\ &= \min\{1, 1\} \times 0.4 + \min\{1, 2\} \times 0.6 \\ &= 1; \\ &\bar{u}_d(s_1, s_1, s_1, s_1, t_{d,1}, T_a) \\ &= \sum_{\tau \subseteq T_a} \max\{u_d((s_1, s_a(t_{a,j}), t_{d,1}) \mid t_{a,j} \in \tau) m_{d,a}(\tau) \\ &= \max\{1, 1\} \times 0.4 + \max\{1, 2\} \times 0.6 \\ &= 1.6.\end{aligned}$$

Then, by formula (9), we get:

$$\begin{aligned}u_d(s_1, s_1, s_1, s_1, t_{d,1}) &= (1 - \beta(m_{d,a})) \underline{u}_d(s_1, s_1, s_1, s_1, t_{d,1}, T_a) \\ &\quad + \beta(m_{d,a}) \bar{u}_d(s_1, s_1, s_1, s_1, t_{d,1}, T_a) \\ &= (1 - 0.1845) \times 1 + 0.1845 \times 1.6 \\ &= 1.1107.\end{aligned}$$

Similarly, we can get  $u_d(s_2, s_1, s_1, s_1, t_{d,1}) = 0$  and  $u_d(s_3, s_1, s_1, s_1, t_{d,1}) = 0$ . So,  $u_d(s_1, s_1, s_1, s_1, t_{d,1}) > u_d(s_2, s_1, s_1, s_1, t_{d,1}) = u_d(s_3, s_1, s_1, s_1, t_{d,1})$ . That is,  $s_1$  is the defender's best response strategy given strategy  $s_1$  of each type of the attacker. Similarly, we can also find that  $u_d(s_1, s_1, s_1, s_2, t_{d,1}) = 0.5107 < 0.6643 = u_d(s_2, s_1, s_1, s_2, t_{d,1}) > u_d(s_3, s_1, s_1, s_2, t_{d,1}) = 0$ . That is,  $s_2$  is the defender's best response strategy given the attacker's strategy  $s_1$  for type  $t_{a,1}$  and type  $t_{a,2}$ , and  $s_2$  for type  $t_{a,3}$ . We also find that given the strategy  $s_2$  of the defender,  $s_1$  is the best response strategy of type  $t_{a,1}$  and type  $t_{a,2}$  of the attacker, and  $s_2$  is the best response of type  $t_{a,3}$  of the attacker. By formula (13), strategy profile  $(s_2, s_1, s_1, s_2)$  is an ambiguous Bayesian Nash equilibrium in the ambiguous Bayesian game of Table I.

#### V. PROPERTIES

This section will reveal some insights into our model.

##### A. The Equilibrium Existence

The existence of the Nash Equilibrium has been well studied [17], i.e., the normal Nash game with finite strategies always has an equilibrium point. An ambiguous Bayesian game with finite types can be reduced to a normal form Nash game, which has finite dimensional strategy space. Thus, we can give the following theorem for the existence of ambiguous Bayesian Nash equilibrium.

*Theorem 1: Every finite ambiguous Bayesian game  $G = (N, \{T_i, S_i, M_i, u_i\}_{i \in N})$  has an ambiguous Bayesian Nash equilibrium.*

*Proof:* Given the other players' information for each type  $t_{i,k}$ , player  $i$  can get the ambiguous belief  $M_i$  about other players' types  $T_{-i}$ . By one of formulas (6)–(8), the player can obtain his degree of ambiguity attitude. Thus, by formula (9), he can get every strategy's point-valued expected utility from the expected utility interval, which is obtained, by formulas (10) and (11), from the utility  $u_i$  and his ambiguous belief  $M_i$ . So, given other players' chosen strategy profile

Defender			
↓			
Attacker: $t_{a,1}$	$s_1$	$s_2$	$s_3$
$s_1$	1,0	0,2	0,3
$s_2$	0,5	2,0	0,3
$s_3$	0,5	0,2	3,0

Attacker: $t_{a,2}$	$s_1$	$s_2$	$s_3$
$s_1$	1,0	0,6	0,1
$s_2$	0,4	5,0	0,2
$s_3$	0,4	0,6	3,0

Attacker: $t_{a,3}$	$s_1$	$s_2$	$s_3$
$s_1$	2,0	0,5	0,4
$s_2$	0,0	6,0	0,0
$s_3$	0,2	0,5	2,0

TABLE I. A SECURITY GAME WITH THE PROBABILITY FOR TYPES  $t_1$  AND  $t_2$  OF THE ATTACKER IS 0.4, AND THE PROBABILITY FOR TYPES  $t_1$  AND  $t_3$  IS 0.6: THE PAYOFFS FOR BOTH PLAYERS OVER PURE STRATEGY PROFILES.

$(S_1(T_1), \dots, S_{i-1}(T_{i-1}), S_{i+1}(T_{i+1}), \dots, S_n(T_n))$ , each type  $t_{i,k}$  has a normal form Nash game. Then, each type  $t_{i,k}$  can choose the best response strategy by formula (13). All normal form Nash games of all types of all players form a normal form Nash game with  $|T_1| \times \dots \times |T_n|$  dimensions. The number of each player's types is finite, so this normal form Nash game is finite. Thus, this normal form Nash game has an equilibrium point [17]. That is, every strategy for every player and every type in an equilibrium point is a best response strategy. By Definition 10, this strategy profile is an ambiguous Bayesian Nash equilibrium. Then, every finite ambiguous Bayesian game has an ambiguous Bayesian Nash equilibrium. ■

### B. The Ambiguity Effect on Equilibrium

Now, we show that how the ambiguous information influences the equilibrium.

In the decision making under ambiguity [12], different people with different ambiguity attitudes have different behaviors [10]. The following theorem reveals how the ambiguity degree and ambiguity attitude influence the outcomes of our games.

**Theorem 2:** For two-person ambiguous Bayesian games  $G_1$  and  $G_2$ , let  $s_{1,1}$  and  $s_{1,2}$  be the two strategies of player 1 with unique type  $t_1$ ,  $S_2^*(T_2)$  be the best response strategy profile of player 2's all types in both games, and the strategy profile  $s_{1,j} = (s_{1,j}, S_2(T_2))$ ,  $m_1$  and  $m_2$  be mass functions over player 2's type set  $T_2$  with respect to  $G_1$  and  $G_2$ ,  $[\underline{u}(s_{1,1}), \bar{u}(s_{1,1})]$  and  $[\underline{u}(s_{1,2}), \bar{u}(s_{1,2})]$  be player 1's expected utility intervals in both games,  $\underline{u}(s_{1,1}) < \underline{u}(s_{1,2}) < \bar{u}(s_{1,1}) < \bar{u}(s_{1,2})$ , and  $u_1(s_{1,j})$  and  $u'_1(s_{1,j})$  be the point-valued utilities from these intervals with respect to  $G_1$  and  $G_2$ . Then:

- (i) When player 1 is ambiguity-seeking and  $\delta_0 \in (0, 1)$  is the ambiguity degree of the mass function such that  $u_1(s_{1,1}) = u_1(s_{1,2}) = u'_1(s_{1,1}) = u'_1(s_{1,2})$ ,
  - a) if  $\delta(m_1) < \delta_0 < \delta(m_2)$  then  $(s_{1,1}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_1$  and  $(s_{1,2}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_2$ ; and
  - b) if  $\delta(m_1) > \delta_0 > \delta(m_2)$  then  $(s_{1,2}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_1$  and  $(s_{1,1}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_2$ .
- (ii) When player 1 is ambiguity-averse and  $\delta_0 \in (0, 1)$  is the ambiguity degree of the mass function such that  $u_1(s_{1,1}) = u_1(s_{1,2}) = u'_1(s_{1,1}) = u'_1(s_{1,2})$ ,
  - a) if  $\delta(m_1) < \delta_0 < \delta(m_2)$  then  $(s_{1,2}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_1$  and

$(s_{1,1}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_2$ ; and

- b) if  $\delta(m_1) > \delta_0 > \delta(m_2)$  then  $(s_{1,1}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_1$  and  $(s_{1,2}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_2$ .

*Proof:* For the ambiguity-seeking player, by formulas (7) and (9), we have

$$u_1(s_{1,1}) = u'_1(s_{1,1}) = (1 - \frac{1 + \delta_0}{2})\underline{u}(s_{1,1}) + \frac{1 + \delta_0}{2}\bar{u}(s_{1,1}),$$

$$u_1(s_{1,2}) = u'_1(s_{1,2}) = (1 - \frac{1 + \delta_0}{2})\underline{u}(s_{1,2}) + \frac{1 + \delta_0}{2}\bar{u}(s_{1,2}).$$

By the above formulas with  $u_1(s_{1,1}) = u_1(s_{1,2})$ , we can get

$$\delta_0 = \frac{(\underline{u}(s_{1,1}) - \underline{u}(s_{1,2})) - (\bar{u}(s_{1,2}) - \bar{u}(s_{1,1}))}{(\underline{u}(s_{1,1}) - \underline{u}(s_{1,2})) + (\bar{u}(s_{1,2}) - \bar{u}(s_{1,1}))}.$$

Since  $\underline{u}(s_{1,1}) < \underline{u}(s_{1,2}) < \bar{u}(s_{1,1}) < \bar{u}(s_{1,2})$  and  $\bar{u}(s_{1,2}) - \bar{u}(s_{1,1}) < \underline{u}(s_{1,1}) - \underline{u}(s_{1,2})$ , we can have  $0 < \delta_0 < 1$ , which satisfies the definition of ambiguity degree. By formula (7), we can have the ambiguity-seeking degree:

$$\beta(\delta_0) = \frac{\underline{u}(s_{1,1}) - \underline{u}(s_{1,2})}{(\underline{u}(s_{1,1}) - \underline{u}(s_{1,2})) + (\bar{u}(s_{1,2}) - \bar{u}(s_{1,1}))}.$$

So, we have  $0.5 < \beta(\delta_0) < 1$ , which satisfies the definition of ambiguity-seeking degree. Thus, if player 1 is ambiguity-seeking and  $\bar{u}(s_{1,2}) - \bar{u}(s_{1,1}) < \underline{u}(s_{1,1}) - \underline{u}(s_{1,2})$ , then there exists a mass function's ambiguity degree  $\delta_0$  such that  $u_1(s_{1,1}) = u_1(s_{1,2}) = u'_1(s_{1,1}) = u'_1(s_{1,2})$ .

If  $\delta(m_1) < \delta_0$ , by formula (7), we have  $\beta(\delta(m_1)) < \beta(\delta_0)$ . By formula (9), we have:

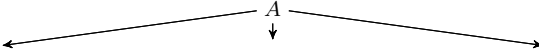
$$u_1(s_{1,1}) = (1 - \beta(\delta(m_1)))\underline{u}(s_{1,1}) + \beta(\delta(m_1))\bar{u}(s_{1,1}),$$

$$u_1(s_{1,2}) = (1 - \beta(\delta(m_1)))\underline{u}(s_{1,2}) + \beta(\delta(m_1))\bar{u}(s_{1,2}).$$

So, we have:

$$\begin{aligned} u_1(s_{1,1}) - u_1(s_{1,2}) &= \underline{u}(s_{1,1}) - \underline{u}(s_{1,2}) + \beta(\delta(m_1))((\bar{u}(s_{1,1}) \\ &\quad - \underline{u}(s_{1,1})) - (\bar{u}(s_{1,2}) - \underline{u}(s_{1,2}))) \\ &> \underline{u}(s_{1,1}) - \underline{u}(s_{1,2}) - \beta(\delta_0)((\underline{u}(s_{1,1}) \\ &\quad - \underline{u}(s_{1,2})) + (\bar{u}(s_{1,2}) - \bar{u}(s_{1,1}))) \\ &= (\underline{u}(s_{1,1}) - \underline{u}(s_{1,2})) - (\underline{u}(s_{1,1}) - \underline{u}(s_{1,2})) \\ &= 0. \end{aligned}$$

Thus,  $u_1(s_{1,1}) > u_1(s_{1,2})$ . Similarly, if  $\delta_0 < \delta(m_2)$  we have  $u'_1(s_{1,1}) < u'_1(s_{1,2})$ ; and if  $\delta(m_1) > \delta_0 > \delta(m_2)$ , then  $u_1(s_{1,1}) < u_1(s_{1,2})$  and  $u'_1(s_{1,1}) > u'_1(s_{1,2})$ . And  $S_2^*(T_2)$  is the best response strategy profile of player 2's all types in  $G_1$  and  $G_2$ . By Definition 10,  $(s_{1,1}, S_2^*(T_2))$  is the ambiguous

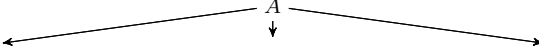


		B: $t_{2,1}$	
		L	R
A	U	101, 40	100, 10
	D	91, 50	120, 10

		B: $t_{2,2}$	
		L	R
A	U	110, 50	80, 40
	D	130, 70	105, 10

		B: $t_{2,3}$	
		L	R
A	U	105, 70	1000/9, 90
	D	100, 80	100, 150

TABLE II. AN AMBIGUOUS BAYESIAN GAME WITH THE PROBABILITY FOR TYPES  $t_{2,1}$  AND  $t_{2,2}$  OF PLAYER 2 IS 1, AND THE PROBABILITY FOR TYPE  $t_{2,3}$  IS 0.



		B: $t_{2,1}$	
		L	R
A	U	10, 40	100, 10
	D	10, 50	120, 10

		B: $t_{2,2}$	
		L	R
A	U	100, 50	80, 40
	D	400, 70	105, 10

		B: $t_{2,3}$	
		L	R
A	U	105, 70	1000/9, 90
	D	100, 80	100, 150

TABLE III. AN AMBIGUOUS BAYESIAN GAME WITH THE PROBABILITY FOR TYPE 2' TYPES  $t_{2,1}$  AND  $t_{2,2}$  IS 0.1, AND THE PROBABILITY FOR TYPE  $t_{2,3}$  IS 0.9.

Bayesian Nash equilibrium in  $G_1$  and  $(s_{1,2}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_2$ . Similarly, we know that if  $\delta(m_1) > \delta_0 > \delta(m_2)$ , then  $(s_{1,2}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_1$  and  $(s_{1,1}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_2$ .

For the ambiguity-averse player and  $\bar{u}(s_{1,2}) - \bar{u}(s_{1,1}) > \underline{u}(s_{1,1}) - \underline{u}(s_{1,2})$ , using the same method, we can show:

$$\delta_0 = \frac{(\bar{u}(s_{1,2}) - \bar{u}(s_{1,1})) - (\underline{u}(s_{1,1}) - \underline{u}(s_{1,2}))}{(\underline{u}(s_{1,1}) - \underline{u}(s_{1,2})) + (\bar{u}(s_{1,2}) - \bar{u}(s_{1,1}))}.$$

And if  $\delta(m_1) < \delta_0 < \delta(m_2)$  then  $u_1(s_{1,1}) < u_1(s_{1,2})$  and  $u'_1(s_{1,1}, t_1) > u'_1(s_{1,2}, t_1)$ ; if  $\delta(m_1) > \delta_0 > \delta(m_2)$  then  $u_1(s_{1,1}) > u_1(s_{1,2}, t_1)$  and  $u'_1(s_{1,1}) < u'_1(s_{1,2}, t_1)$ . And  $S_2^*(T_2)$  is the best response strategy profile of player 2's all types in  $G_1$  and  $G_2$ . So, if  $\delta(m_1) < \delta_0 < \delta(m_2)$ , then  $(s_{1,2}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_1$  and  $(s_{1,1}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_2$ ; and if  $\delta(m_1) > \delta_0 > \delta(m_2)$ , then  $(s_{1,1}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_1$  and  $(s_{1,2}, S_2^*(T_2))$  is the ambiguous Bayesian Nash equilibrium in  $G_2$ . ■

Now, we give an example to show how the ambiguous Bayesian Nash equilibria change with the players' ambiguity attitudes and ambiguity degrees as shown in the above theorem. So, we consider our model with ambiguity-averse players. To find the equilibria in ambiguous Bayesian games as shown in Tables II and III, we find the best response strategy of every type of each player first. Now we start to discuss that of player 2 with each type. Similarly to the ambiguous Bayesian game as shown in Table I, by formulas (9) and (12), since player 2 knows that player 1 has only one type  $t_1$ , player 2 can get the point-valued expected utility because of this precise belief. That is, the point-valued expected utility of player 2 with each type  $t_{2,k}$  over strategy profile  $\vec{S}(s_{2,j})$  ( $s_{2,j} \in S_2 = \{L, R\}$ ) is:  $u_2(\vec{S}(s_{2,j}), t_{2,k}) = u_2(s_1(t_1), s_{2,j}, t_{2,k})$ , where  $s_1(t_1) \in S_1 = \{U, D\}$ . So, the point-valued expected utility of player 2 with each type  $t_{2,k}$  is shown in Table II. For example,  $u_2(\vec{S}(s_{2,L}), t_{2,1}) = u_2(U, L, t_{2,1}) = 40$  when  $s_1(t_1) = U$ . So, in the ambiguous Bayesian games as shown in Tables II and III, whatever player 1 with a unique type chooses, player 2's best response strategy profiles are always  $(L, L, R)$  (i.e.,

$L, L$ , and  $R$  are the best response strategies of type  $t_{2,1}$ ,  $t_{2,2}$ , and  $t_{2,3}$ , respectively) in both ambiguous Bayesian games as shown in Tables II and III.

Now we calculate player 1's point-valued utilities on the strategy profile when player 2's best response strategy profile is  $(L, L, R)$  in both games of Tables II and III. In the game of Table II, by formula (1), the ambiguity degree of mass function  $m_{1,2}$  is  $\delta_1(m_{1,2}) \approx 0.6309$ . For player 1 with ambiguity aversion, by formula (6) the ambiguity-averse degree is  $\beta_1(m_{1,2}) = 0.1845$ . By formulas (10) and (11), the utility intervals of profiles  $(U, L, L, R)$  and  $(D, L, L, R)$  are:

$$[\underline{u}_1(U, L, L, R, t_1, T_2), \bar{u}_1(U, L, L, R, t_1, T_2)] = [101, 110];$$

$$[\underline{u}_1(D, L, L, R, t_1, T_2), \bar{u}_1(D, L, L, R, t_1, T_2)] = [91, 130].$$

Further, by formula (9), player 1's point-valued utilities are:

$$u_1(U, L, L, R, t_1) = (1 - 0.1845) \times 101 + 0.1845 \times 110 = 102.66,$$

$$u_1(D, L, L, R, t_1) = (1 - 0.1845) \times 91 + 0.1845 \times 130 = 98.20.$$

That is,  $u_1(U, L, L, R, t_1) > u_1(D, L, L, R, t_1)$ . So, strategy  $U$  is player 1's best response strategy. From the above discussion, we know that if player 1 chooses strategy  $U$ , then  $L, L$ , and  $R$  are the best response strategies of player 2's type  $t_{2,1}$ ,  $t_{2,2}$ , and  $t_{2,3}$ , respectively. So, by Definition 10,  $(U, L, L, R)$  is the ambiguous Nash equilibrium in the game of Table II.

In the game of Table III, by formulas (10) and (11), the utility intervals of  $(U, L, L, R)$  and  $(D, L, L, R)$  still are:

$$[\underline{u}_1(U, L, L, R, t_1, T_2), \bar{u}_1(U, L, L, R, t_1, T_2)] = [101, 110];$$

$$[\underline{u}_1(D, L, L, R, t_1, T_2), \bar{u}_1(D, L, L, R, t_1, T_2)] = [91, 130].$$

However, by formula (1), the ambiguity degree of mass function  $m_{1,2}$  is  $\delta_2(m_{1,2}) \approx 0.0631$ . For player 1 with ambiguity aversion, by formula (6) the ambiguity-averse degree is  $\beta_2(m_{1,2}) = 0.47$ . By formula (9), player 1 has point-valued expected utilities as follows:

$$u_1(U, L, L, R, t_1) = (1 - 0.47) \times 101 + 0.47 \times 110 = 105.21,$$

$$u_1(D, L, L, R, t_1) = (1 - 0.47) \times 91 + 0.47 \times 130 = 109.27.$$

That is,  $u_1(U, L, L, R, t_1) < u_1(D, L, L, R, t_1)$ . So, strategy  $D$  is player 1's best response strategy. From the above discussion, we know that if player 1 chooses strategy  $D$ , then  $L, L$ , and  $R$  are the best response strategy of player 2's type  $t_{2,1}$ ,

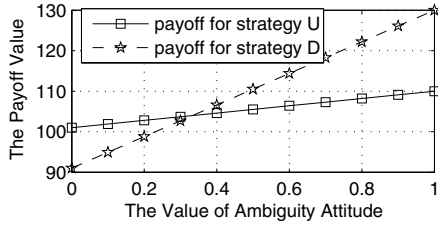


Fig. 1. Player 1's utility when player 2's response strategy profile is  $(L, L, R)$  in both games of Tables II and III.

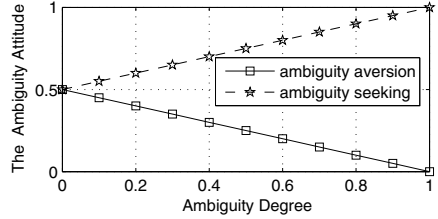


Fig. 2. The ambiguity attitude with ambiguity degree.

$t_{2,2}$ , and  $t_{2,3}$ , respectively. So, by Definition 10,  $(D, L, L, R)$  is the ambiguous Nash equilibrium in the game of Table III.

However, we have found that  $(U, (L, L, R))$  is the ambiguous Nash equilibrium in the game of Table II. So, given player 2's unique best response strategy profile  $(L, L, R)$ , even though the player 1's utility intervals (both are  $[110, 130]$ ) are the same on the same strategy profile, the ambiguous Nash equilibria are different in the games of Tables II and III. Actually, in both games, the ambiguity degrees of player 1's belief are different. Player 1's belief about player 2's types in the game of Table II is more ambiguous than the one in the game of Table III (i.e.,  $\delta_1(m_{1,2}) > \delta_2(m_{1,2})$ ). In our model, players have different values of ambiguity attitude for different ambiguity degrees. As shown in Figs. 1 and 2, the higher the ambiguity degree, the lower the ambiguity-averse degree; conversely, the higher the ambiguity degree, the higher the ambiguity-seeking degree. However, the higher the ambiguity-averse or ambiguity-seeking degree, the higher the utility for players as shown in formula (9).

Furthermore, the ambiguous Bayesian Nash equilibrium changes with the ambiguity degree of the belief. For example, in Fig. 1, when  $\beta = \frac{1}{3}$  (ambiguity degree  $\delta(m) = \frac{1}{3}$  in Fig. 2), player 1's point-valued utility of strategy  $U$  is equal to the point-valued utility of strategy  $D$  given player 2's response strategy profile  $(L, L, R)$ . If  $\beta < \frac{1}{3}$  (the corresponding ambiguity degree  $\delta(m) > \frac{1}{3}$  in Fig. 2), player 1's point-valued expected utility of strategy  $U$  is more than the point-valued expected utility of strategy  $D$ . That is, player 1 prefers strategy  $U$  given player 2's response strategy profile  $(L, L, R)$ . Actually, in the game of Table II, the value of player 1's ambiguity attitude is  $\beta_1(m_{1,2}) = 0.1845 < \frac{1}{3}$  (the corresponding ambiguity degree is  $\delta_1(m_{1,2}) = 0.6039 > \frac{1}{3}$ ) and we know that  $(L, L, R)$  is the best response strategy profile of player 2's all types given player 1's strategy  $U$ . So,  $(U, L, L, R)$  is the ambiguous Bayesian Nash equilibrium. If  $\beta > \frac{1}{3}$  (the corresponding ambiguity degree  $\delta(m) < \frac{1}{3}$  in Fig. 2), the point-valued utility of strategy  $D$  is more than the point-valued expected utility of strategy  $U$ . Thus, player 1 prefers strategy  $D$  given player 2's response strategy profile  $(L, L, R)$ . Actually, in the game of Table III, the value

of player 1's ambiguity attitude is  $\beta_2(m_{1,2}) = 0.47 > \frac{1}{3}$  (ambiguity degree  $\delta_2(m_{1,2}) = 0.0631 < \frac{1}{3}$ ) and we know that  $(L, L, R)$  is the best response strategy profile of player 2's all types given player 1's strategy  $D$ . So,  $(D, L, L, R)$  is the ambiguous Bayesian Nash equilibrium.

So, the above example confirms Theorem 2: different mass functions and ambiguity attitudes will cause different ambiguous Bayesian Nash equilibria. That is the reason why the ambiguous Bayesian Nash equilibria are different in games of Tables II and III.

## VI. RELATED WORK

To deal with incomplete information (i.e., the precise probability distribution over players' types), a standard model is Bayesian games [1]. The following theorem reveals the relation between our ambiguous Bayesian games and the Bayesian games.

**Theorem 3:** In ambiguous Bayesian game  $(N, \{T_i, S_i, M_i, u_i\}_{i \in N})$ , if for every player  $i$ , his belief about each type of other players is precise (i.e., given every focal elements  $\tau_j \subseteq T_j$  and player  $i$ 's mass function  $m_{i,j}$ ,  $|\tau_j| = 1$  when  $m(\tau_j) > 0$ ), the ambiguous Bayesian game is a Bayesian game and the ambiguous Bayesian Nash equilibrium is the Bayesian Nash equilibrium.

*Proof:* Because the belief about the types in Bayesian games are precise, i.e., if  $|\tau_j| = 1$  when  $m_{i,j}(\tau_j) > 0$ , the ambiguous Bayesian game is the Bayesian game. If  $|\tau_j| = 1$  when  $m_{i,j}(\tau_j) > 0$ , let  $\tau_j = t_{j,h}$  and  $m_{i,j} = p_{i,j}$ , then by formulas (10) and (11), we have

$$\begin{aligned} \underline{u}_n(\vec{S}(s_{i,j}), T_n) &= \bar{u}_n(\vec{S}(s_{i,j}), T_n) \\ &= \sum_{t_{n,h} \in T_n} u_n(\vec{S}(s_{i,j}), t_{n,h}) p_{i,n}(t_{n,h}). \end{aligned} \quad (14)$$

By formula (9), we have:

$$u_i(\vec{S}(s_{i,j}), t_{i,k}) = \sum_{t_{n,h} \in T_n} u_n(\vec{S}(s_{i,j}), t_{n,h}) p_{i,n}(t_{n,h}). \quad (15)$$

By formula (12), we have:

$$\begin{aligned} u_i(\vec{S}(s_{i,j}), t_{i,k}) &= \sum_{t_{n,h} \in T_n} \sum_{t_{n-1,h} \in T_{n-1}} u_{n-1}(\vec{S}(s_{i,j}), t_{n-1,h}) p_{i,n-1}(t_{n-1,h}) p_{i,n}(t_{n,h}). \end{aligned}$$

Recursively, we have:

$$\begin{aligned} u_i(\vec{S}(s_{i,j}), t_{i,k}) &= \sum_{t_{n,h} \in T_n} \dots \sum_{t_{i+1,h} \in T_{i+1}} \sum_{t_{i-1,h} \in T_{i-1}} \dots \sum_{t_{1,h} \in T_1} u_i(s_1(t_{1,h}), \dots, \\ &\quad s_{i-1}(t_{i-1,h}), s_{i,j}, s_{i+1}(t_{i+1,h}), \dots, s_n(t_{n,h}), t_{i,k}) p_{i,1}(t_{1,h}) \\ &\quad \dots p_{i,i-1}(t_{i-1,h}) p_{i,i+1}(t_{i+1,h}) \dots p_{i,n}(t_{n,h}). \end{aligned}$$

Let  $t_{-i} = (t_{1,h}, \dots, t_{i-1,h}, t_{i+1,h}, \dots, t_{n,h}) \in T_{-i}$  be the types of the other players,  $T_{-i}$  be the set of all possible values of  $t_{-i}$ , and the probability distribution  $p_i(t_i | t_{i,k}) = p_{i,1}(t_{1,h}) \dots p_{i,i-1}(t_{i-1,h}) p_{i,i+1}(t_{i+1,h}) \dots p_{i,n}(t_{n,h})$  be player  $i$ 's belief about  $t_{-i}$ . Then

$$\begin{aligned} u_i(\vec{S}(s_{i,j}), t_{i,k}) &= \sum_{t_{-i} \in T_{-i}} u_i(s_1(t_{1,h}), \dots, s_{i-1}(t_{i-1,h}), s_{i,j}, \\ &\quad s_{i+1}(t_{i+1,h}), \dots, s_n(t_{n,h}), t_{i,k}) p(t_{-i} | t_{i,k}), \end{aligned}$$

By formula (13), for each player  $i$  and for each of his types  $t_{i,k} \in T_i$ , the ambiguous Bayesian Nash equilibrium  $\bar{S}^* = (S_1^*(T_1), \dots, S_n^*(T_n))$  ( $S_i^*(T_i) = \prod_{t_{i,k} \in T_i} s_i^*(t_{i,k})$ ) is

$$s_i^*(t_{i,k}) = \arg \max_{s_{i,j} \in S_i} \sum_{t_{-i} \in T_{-i}} u_i(s_1^*(t_{1,h}), \dots, s_{i-1}^*(t_{i-1,h}), s_{i,j}, s_{i+1}^*(t_{i+1,h}), \dots, s_n^*(t_{n,h}), t_{i,k}) p(t_{-i} | t_{i,k}),$$

which is the same as the Bayesian Nash equilibrium obtained by using formula (5). So, the theorem holds. ■

So, ambiguous Bayesian games are a generalised model of Bayesian games. Wang *et al.* [18] handle players with fuzzy types and so can be regarded as a generalisation of the Bayesian game as well. However, they cannot deal with players' types with imprecise probability, but we can.

In the field of games under ambiguity, Eichberger *et al.* [19] define a notion of equilibrium under ambiguity to explain the hypothesis that the result from changing an apparently irrelevant parameter contradicts Nash equilibrium. In their two-person games, they view their opponents' behavior as ambiguous based on non-additive beliefs. And Marco and Romaniello [20] try to use the ambiguity model to remedy defects of Nash equilibrium. In their model, the belief depends on the strategy profile and then affects the equilibrium. Moreover, Xiong *et al.* also [21] consider the ambiguous utility in game theory. Furthermore, Ma *et al.* [22] extend the ambiguity decision framework [9], [21] to deal with ambiguous utilities in the static security games. However, all these studies do not concern the players' types, but we handle the ambiguous information about players' types.

Zhang *et al.* [16] propose a model of security games with ambiguous information about attacker types. However, that is the dynamic game with two players, but our model is the static game with  $n$  players by extending the Bayesian game and can discriminate different ambiguity attitudes.

## VII. CONCLUSION AND FUTURE WORK

This paper extends the Bayesian game model to handle the ambiguous information about the player's types. Moreover, our model can handle ambiguity-averse, ambiguity-seeking, and ambiguity-neutral players. We also define a new solution concept: the ambiguous Nash equilibrium, and prove the existence of the solution of the game. In addition, we reveal how the ambiguity degrees influence the outcomes of the games of this kind. In the future, it is interesting to use the idea behind our model to extend signaling games to ambiguous signaling games.

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