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Distributed Fuzzy Proportional-spatial Integral Control Design for a Class of Nonlinear Distributed Parameter Systems

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Abstract—**The fuzzy feedback control design problem is addressed in this paper by using the distributed proportional-spatial integral (P-sI) control approach for a class of nonlinear distributed parameter systems represented by semi-linear parabolic partial differential-integral equations (PDIEs). The objective of this paper is to develop a fuzzy distributed P-sI controller for the semi-linear parabolic PDIE system such that the resulting closed-loop system is exponentially stable. To do this, the semi-linear parabolic PDIE system is first assumed to be exactly represented by a Takagi-Sugeno (T-S) fuzzy parabolic PDIE model. A new vector-valued integral inequality is established via the vector-valued Wirtinger's inequality. Then, based on the T-S fuzzy PDIE model and this new integral inequality, a distributed fuzzy P-sI state feedback controller is proposed such that the closed-loop PDIE system is exponentially stable. The sufficient condition on the existence of this fuzzy controller is given in terms of a set of standard linear matrix inequalities (LMIs), which can be effectively solved by using the existing convex optimization techniques. Finally, the developed design methodology is successfully applied to solve the feedback control design of a semi-linear reaction-diffusion system with a spatial integral term.**

I. INTRODUCTION

FUZZY control approach offers a systematic way to deal with the control synthesis of nonlinear systems. Over the past few decades, various fuzzy control approaches have been proposed and fruitful results have been achieved [1]. In particular, the so-called Takagi-Sugeno (T-S) fuzzy model [2] has been widely employed for the control design of nonlinear systems represented by ordinary differential equations (ODE) or delay differential equations (DDEs) (see e.g., [3], [4] and the references therein for a survey of recent development), since it can combine the merits of both fuzzy logic theory and linear system theory. This T-S fuzzy-model-based control technique is conceptually simple and effective for controlling complex nonlinear systems modeled by ODEs or DDEs. Based on the T-S fuzzy model, the fruitful linear system theory can be applied to the analysis and controller synthesis of nonlinear ODE systems or nonlinear DDE systems [1]-[8]. However, these results only focus on nonlinear ODE systems or nonlinear DDE systems.

It is well-known that most industrial processes are spatiotemporal ones, i.e., their behavior not only depends on time but also is concerned with spatial position, for example, thermal diffusion processes, fluid heat exchangers, and chemical engineering processes [9]-[12]. The mathematical models describing these processes are typically obtained from the dynamical conservation laws and take the form of parabolic partial differential equations (PDEs) [9] or partial differential-integral equations (PDIEs) [13]-[15]. The key characteristic of PDE systems and PDIE systems is that their outputs, inputs, and process states and relevant parameters may vary temporally as well as spatially. Due to the spatial distribution feature, the existing fuzzy control design for nonlinear ODE system or nonlinear DDE systems cannot be directly applied to address the control synthesis of the nonlinear parabolic PDE systems or nonlinear PDIE systems.

Over the past decade, the existing fuzzy-model-based control techniques have been extended to address the control design of nonlinear PDE systems. Based on the approximated ODE models derived by using the model reduction techniques and the existing T-S fuzzy ODE model-based control techniques, some important results have been reported [16]-[19]. A potential drawback of these results is that the inherent loss of physical features of the problem due to the truncation before the control design, and the ultimate controller design may be negative and fail to take advantage of natural property of the system. More recently, Wang *et al.* have developed a distributed fuzzy control design method for a class of semi-linear parabolic PDE systems through a fuzzy PDE modeling approach [20] and [21]. However, to the best of authors' knowledge, the distributed fuzzy proportional-spatial integral (P-sI) control design has not been reported for semi-linear parabolic PDIE systems based on directly the T-S fuzzy PDIE model, which motivates this study.

In this paper, we will study the distributed fuzzy P-sI control design for a class of semi-linear parabolic PDIE systems via the fuzzy PDIE modeling approach, where the control actuators are continuously distributed in space. A T-S fuzzy parabolic PDIE model is assumed to be used to accurately represent the semi-linear parabolic PDIE system. Then, based on the T-S fuzzy PDIE model and the parallel distributed compensation (PDC) scheme [3], a distributed fuzzy P-sI controller is developed such that the closed-loop PDIE system is exponentially stable. A new integral inequality is established via the vector-valued Wirtinger's inequality. A sufficient condition of the closed-loop exponential stability is derived with the help of this new integral inequality, and presented in terms of a set of standard

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linear matrix inequalities (LMIs), which can be solved via the existing convex optimization techniques [22] and [23]. Finally, the effectiveness of the proposed design method in this paper is illustrated by the simulation results on the feedback control of a semi-linear parabolic PDIE system.

Notations: The following notations will be used throughout this paper. $\Re$, $\Re^n$ and $\Re^{m\times n}$ denote the set of all real numbers, n-dimensional Euclidean space and the set of all real $m\times n$ matrices, respectively. $\|\cdot\|$ and $\langle\cdot,\cdot\rangle_{\Re^n}$ denote the Euclidean norm and inner product for vectors, respectively. Identity matrix, of approximate dimension, will be denoted by $\boldsymbol{I}$. For a symmetric matrix $\boldsymbol{M}$, $\boldsymbol{M}>0$ ($\boldsymbol{M}<0$, respectively) means that $\boldsymbol{M}$ is positive definite (negative definite, respectively). $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ stand for the minimum and maximum eigenvalues of a square matrix, respectively. $\mathcal{H}^n \triangleq \mathcal{L}_2([l_1,l_2];\Re^n)$ is a Hilbert space of n-dimensional square integrable vector functions $\boldsymbol{\omega}(z)\in\Re^n$, $z\in[l_1,l_2]\subset\Re$ with the inner product and norm:

$$\langle\boldsymbol{\omega}_1,\boldsymbol{\omega}_2\rangle=\int_{l_1}^{l_2}\langle\boldsymbol{\omega}_1(z),\boldsymbol{\omega}_2(z)\rangle_{\Re^n}\,dz \text{ and } \|\boldsymbol{\omega}_1\|_2=\langle\boldsymbol{\omega}_1,\boldsymbol{\omega}_1\rangle^{1/2},$$

where $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2\in\mathcal{H}^n$. $\mathcal{W}^{l,2}([l_1,l_2];\Re^n)$ is a Sobolev space of absolutely continuous n-dimensional vector functions $\boldsymbol{\omega}(x)$: $[l_1,l_2]\to\Re^n$ with square integrable derivatives $\frac{d^l\boldsymbol{\omega}(x)}{dx^l}$ of the order $l\ge 1$ and with the norm $\|\boldsymbol{\omega}(\cdot)\|^2_{\mathcal{W}^{l,2}}=\int_{l_1}^{l_2}\sum_{i=0}^{l}\left(\frac{d^i\boldsymbol{\omega}(x)}{dx^i}\right)^T\cdot\left(\frac{d^i\boldsymbol{\omega}(x)}{dx^i}\right)dx$. The superscript '$T$' is used for the transpose of a vector or a matrix. The symbol $*$ is used as an ellipsis in matrix expressions that are induced by symmetry, e.g.,

$$\begin{bmatrix}\boldsymbol{S}+[\boldsymbol{M}+\boldsymbol{N}+*] & \boldsymbol{X}\\ * & \boldsymbol{Y}\end{bmatrix}\triangleq\begin{bmatrix}\boldsymbol{S}+[\boldsymbol{M}+\boldsymbol{N}+\boldsymbol{M}^T+\boldsymbol{N}^T] & \boldsymbol{X}\\ \boldsymbol{X}^T & \boldsymbol{Y}\end{bmatrix}.$$

II. Preliminary and Problem Formulation

Consider the following semi-linear PDIE systems in one spatial dimension with a state-space description:

$$\begin{cases}\boldsymbol{y}_t(z,t)=\Theta\boldsymbol{y}_{zz}(z,t)+\boldsymbol{f}(\boldsymbol{y}(z,t))\\ \qquad +\boldsymbol{G}_u\boldsymbol{u}(z,t)+\boldsymbol{G}_I\int_{l_1}^{z}\boldsymbol{y}(s,t)ds\\ \boldsymbol{y}(l_1,t)=\boldsymbol{y}(l_2,t)=0\\ \boldsymbol{y}(z,0)=\boldsymbol{y}_0(z)\end{cases}\tag{1}$$

where $\boldsymbol{y}(z,t)\in\Re^n$ is the state, the subscripts z and t stand for the partial derivatives with respect to z, t, respectively, $z\in[l_1,l_2]\subset\Re$ and $t\in[0,\infty)$ is the spatial position and time, respectively, $\boldsymbol{u}(z,t)\in\Re^m$ is the control input. $\boldsymbol{f}(\boldsymbol{y}(z,t))$ is a locally Lipschitz continuous function in $\boldsymbol{y}(z,t)$ and satisfies $\boldsymbol{f}(0)=0$. Θ, $\boldsymbol{G}_u$, and $\boldsymbol{G}_I$ are real known matrices with approximate dimensions.

For brevity, set

$$\mathcal{A}\boldsymbol{y}(z,t)\triangleq\Theta\boldsymbol{y}_{zz}(z,t) \text{ and } \boldsymbol{\upsilon}(z,t)\triangleq\int_{l_1}^{z}\boldsymbol{y}(s,t)ds,\tag{2}$$

then the PDIE system (1) can be rewritten as

$$\begin{cases}\boldsymbol{y}_t(z,t)=\mathcal{A}\boldsymbol{y}(z,t)+\boldsymbol{f}(\boldsymbol{y}(z,t))+\boldsymbol{G}_u\boldsymbol{u}(z,t)+\boldsymbol{G}_I\boldsymbol{\upsilon}(z,t)\\ \boldsymbol{y}(l_1,t)=\boldsymbol{y}(l_2,t)=0\\ \boldsymbol{y}(z,0)=\boldsymbol{y}_0(z).\end{cases}\tag{3}$$

When $\boldsymbol{u}(z,t)\equiv 0$, the system (3) is referred to as an *unforced* system. We introduce the following definition of exponential stability on the Hilbert space $\mathcal{H}^n$:

Definition 1. The *unforced* system of (3) (i.e., $\boldsymbol{u}(z,t)\equiv 0$) is said to be *exponentially stable*, if there exist constants $\rho>0$ and $\sigma\ge 1$ such that the following expression holds:

$$\|\boldsymbol{y}(\cdot,t)\|_2^2\le\sigma\|\boldsymbol{y}_0(\cdot)\|_2^2\exp(-\rho t),\ \forall\, t\ge 0.\tag{4}$$

It has been pointed out that an exact T-S fuzzy ODE model construction from a given nonlinear dynamical ODE model can be obtained by applying the sector nonlinearity approach [3]. More recently, a T-S fuzzy PDE model has been proposed in [20] and [21] by extending this approach to a class of semi-linear parabolic PDE systems. Similar to [20] and [21], by extending this sector nonlinearity approach to the semi-linear PDIE system (3), we can derive an exact T-S fuzzy PDIE model construction. In this study, we also assume that the semi-linear PDIE system (3) can be exactly represented by the following T-S fuzzy PDIE model:

Plant Rule i:

IF $\xi_1(z,t)$ is F_{i1} and $\cdots$ and $\xi_l(z,t)$ is F_{il}

THEN

$$\begin{cases}\boldsymbol{y}_t(z,t)=\mathcal{A}\boldsymbol{y}(z,t)+\boldsymbol{A}_i\boldsymbol{y}(z,t)+\boldsymbol{G}_u\boldsymbol{u}(z,t)\\ \qquad +\boldsymbol{G}_I\boldsymbol{\upsilon}(z,t)\\ \boldsymbol{y}(l_1,t)=\boldsymbol{y}(l_2,t)=0\\ \boldsymbol{y}(z,0)=\boldsymbol{y}_0(z), i\in\mathcal{S}\triangleq\{1,2,\cdots,r\}\end{cases}\tag{5}$$

where F_{ij}, $i\in\mathcal{S}$, $j=1,2,\cdots,l$ are fuzzy sets, $\boldsymbol{A}_i\in\Re^{n\times n}$, $i\in\mathcal{S}$ are real known matrices, r is the number of IF-THEN rules, and $\xi_j(z,t)$, $j=1,2,\cdots,l$ are the known premise variables. In order to avoid a complicated defuzzification process of fuzzy controller, in this study, these premise variables are assumed to be functions of only the state $\boldsymbol{y}(z,t)$.

By applying the center-average defuzzifier, product interference and singleton fuzzifier, the overall dynamics of T-S fuzzy PDIE (5) can be expressed as:

$$\begin{cases} \boldsymbol{y}_t(z,t) = \mathcal{A}\boldsymbol{y}(z,t) + \sum_{i=1}^{r} h_i(\boldsymbol{\xi}(z,t))\boldsymbol{A}_i\boldsymbol{y}(z,t) + \boldsymbol{G}_u\boldsymbol{u}(z,t) \\ \qquad + \boldsymbol{G}_I\boldsymbol{\upsilon}(z,t) \\ \boldsymbol{y}(l_1,t) = \boldsymbol{y}(l_2,t) = 0 \\ \boldsymbol{y}(z,0) = \boldsymbol{y}_0(z). \end{cases} \tag{6}$$

where $\boldsymbol{\xi}(z,t) = [\xi_1(z,t) \quad \cdots \quad \xi_l(z,t)]^T$ and

$$w_i(\boldsymbol{\xi}(z,t)) = \prod_{j=1}^{l} F_{ij}(\xi_j(z,t)),$$
$$h_i(\boldsymbol{\xi}(z,t)) = w_i(\boldsymbol{\xi}(z,t)) \Big/ \sum_{i=1}^{r} w_i(\boldsymbol{\xi}(z,t)),\ i \in \mathcal{S}.$$

The term $F_{ij}(\xi_j(z,t))$ is the grade of the membership of $\xi_j(z,t)$ in F_{ij}, $i \in \mathcal{S}$. In this paper, it is assumed that $w_i(\boldsymbol{\xi}(z,t)) \ge 0$, $i \in \mathcal{S}$ and $\sum_{i=1}^{r} w_i(\boldsymbol{\xi}(z,t)) > 0$, for all $z \in [l_1, l_2]$ and $t \ge 0$. Then we can obtain the following conditions:

$$h_i(\boldsymbol{\xi}(z,t)) \ge 0,\ i \in \mathcal{S} \text{ and } \sum_{i=1}^{r} h_i(\boldsymbol{\xi}(z,t)) = 1 \tag{7}$$

for all $z \in [l_1, l_2]$ and $t \ge 0$.

Based on the T-S fuzzy PDIE model (6), we consider the following fuzzy P-sI controller via the PDC scheme [3] for the semi-linear PDIE system (1):

Control Rule j:
IF $\xi_1(z,t)$ is F_{j1} and $\cdots$ and $\xi_l(z,t)$ is F_{jl}
THEN $\boldsymbol{u}(z,t) = \boldsymbol{K}_{P,j}\boldsymbol{y}(z,t) + \boldsymbol{K}_{I,j}\boldsymbol{\upsilon}(z,t)$, $j \in \mathcal{S}$

where $\boldsymbol{K}_{P,j}$ and $\boldsymbol{K}_{I,j}$, $j \in \mathcal{S}$ are $m \times n$ real matrices to be determined. Obviously, from (2), we have

$$\boldsymbol{\upsilon}_z(z,t) = \boldsymbol{y}(z,t) \text{ and } \boldsymbol{\upsilon}(l_1,t) = 0. \tag{8}$$

The overall fuzzy P-sI controller can be represented by

$$\boldsymbol{u}(z,t) = \sum_{j=1}^{r} h_j(\boldsymbol{\xi}(z,t))[\boldsymbol{K}_{P,j}\boldsymbol{y}(z,t) + \boldsymbol{K}_{I,j}\boldsymbol{\upsilon}(z,t)]. \tag{9}$$

From (6) and (9), we have the following closed-loop fuzzy PDIE system:

$$\begin{cases} \boldsymbol{y}_t(z,t) = \mathcal{A}\boldsymbol{y}(z,t) + \sum_{i=1}^{r} [\boldsymbol{A}_i + \boldsymbol{G}_u\boldsymbol{K}_{P,i}]\boldsymbol{y}(z,t) \\ \qquad + \boldsymbol{G}_u \sum_{i=1}^{r} h_i(\boldsymbol{\xi}(z,t))\boldsymbol{K}_{I,i}\boldsymbol{\upsilon}(z,t) + \boldsymbol{G}_I\boldsymbol{\upsilon}(z,t) \\ \boldsymbol{y}(l_1,t) = \boldsymbol{y}(l_2,t) = 0 \\ \boldsymbol{y}(z,0) = \boldsymbol{y}_0(z). \end{cases} \tag{10}$$

Based on the T-S fuzzy PDIE model (6), the design purpose of this study is to seek a distributed P-sI controller of the form (9) for the semi-linear PDIE system (1) such that the resulting closed-loop system (10) is exponentially stable. To this end, the following lemmas are useful for the developments of control design in this study:

Lemma 1. (Vector-valued Wirtinger's inequalities [24]). Let $\bar{\boldsymbol{y}} \in \mathcal{W}^{1,2}([l_1, l_2]; \Re^n)$ be a vector function with $\bar{\boldsymbol{y}}(l_1) = \bar{\boldsymbol{y}}(l_2) = 0$. Then, for a matrix $\boldsymbol{S} > 0$, we have

$$\int_{l_1}^{l_2} \bar{\boldsymbol{y}}^T(s)\boldsymbol{S}\bar{\boldsymbol{y}}(s)ds \le (l_2 - l_1)^2 \pi^{-2} \int_{l_1}^{l_2} \left(d\bar{\boldsymbol{y}}(s)/ds\right)^T \boldsymbol{S} \left(d\bar{\boldsymbol{y}}(s)/ds\right) ds. \tag{11}$$

Moreover, if $\bar{\boldsymbol{y}}(l_1) = 0$ or $\bar{\boldsymbol{y}}(l_2) = 0$, we have

$$\int_{l_1}^{l_2} \bar{\boldsymbol{y}}^T(s)\boldsymbol{S}\bar{\boldsymbol{y}}(s)ds \le \frac{4(l_2 - l_1)^2}{\pi^2} \int_{l_1}^{l_2} \left(d\bar{\boldsymbol{y}}(s)/ds\right)^T \boldsymbol{S} \left(d\bar{\boldsymbol{y}}(s)/ds\right) ds. \tag{12}$$

Based on the Lemma 1, we have the following lemma:

Lemma 2. Let $\bar{\boldsymbol{y}} : [l_1, l_2] \to \Re^n$ be a square integrable vector function and $\bar{\boldsymbol{\upsilon}}(z) = \int_{l_1}^{z} \bar{\boldsymbol{y}}(s)ds$ be square integrable. Then for a given matrix $\boldsymbol{S} > 0$, we have

$$\int_{l_1}^{l_2} \boldsymbol{\upsilon}^T(s)\boldsymbol{S}\boldsymbol{\upsilon}(s)ds \le 4(l_2 - l_1)^2 \pi^{-2} \int_{l_1}^{l_2} \bar{\boldsymbol{y}}^T(s)\boldsymbol{S}\bar{\boldsymbol{y}}(s)ds. \tag{13}$$

Proof. Since (8), using the inequality (12) in Lemma 1, we get

$$\int_{l_1}^{l_2} \boldsymbol{\upsilon}^T(s)\boldsymbol{S}\boldsymbol{\upsilon}(s)ds \le \frac{4(l_2 - l_1)^2}{\pi^2} \int_{l_1}^{l_2} (d\boldsymbol{\upsilon}(s)/ds)^T \boldsymbol{S}\, d\boldsymbol{\upsilon}(s)/ds\, ds.$$

The inequality (13) can be derived from the property $d\boldsymbol{\upsilon}(s)/ds = \bar{\boldsymbol{y}}(s)$ and the above inequality. The proof is complete. □

Remark 1. If we set $\boldsymbol{\upsilon}(z) = -\int_{l_1}^{z} \bar{\boldsymbol{y}}(s)ds$ ($\boldsymbol{\upsilon}(z) = -\int_{z}^{l_2} \bar{\boldsymbol{y}}(s)ds$ or $\boldsymbol{\upsilon}(z) = \int_{z}^{l_2} \bar{\boldsymbol{y}}(s)ds$), similar to the proof of Lemma 2, the same result (i.e., the inequality (13)) can be established using the fact $\boldsymbol{\upsilon}(l_1) = 0$ (or $\boldsymbol{\upsilon}(l_2) = 0$).

III. Main Result

Based on Lemmas 1 and 2, this section will present a simple LMI-based design method of a distributed fuzzy P-sI controller of the form (9) exponentially stabilizing the semi-linear parabolic PDIE system (1).

Consider the following Lyapunov functional for the fuzzy parabolic PDIE system (10):

$$V(t) = \int_{l_1}^{l_2} \boldsymbol{y}^T(z,t)\boldsymbol{P}\boldsymbol{y}(z,t)dz \tag{14}$$

where $\boldsymbol{P} > 0$ is a real $n \times n$ matrix to be determined. The time derivative of $V(t)$ along the solution of the system given by (10) is represented as

$$\dot{V}(t) = 2\int_{l_1}^{l_2} \boldsymbol{y}^T(z,t)\boldsymbol{P}\boldsymbol{y}_t(z,t)dz$$
$$= 2\int_{l_1}^{l_2} \boldsymbol{y}^T(z,t)\boldsymbol{P}\mathcal{A}\boldsymbol{y}(z,t)dz + 2\int_{l_1}^{l_2} \boldsymbol{y}^T(z,t)\boldsymbol{P}\boldsymbol{G}_I\boldsymbol{\upsilon}(z,t)dz$$
$$+\int_{l_1}^{l_2} \boldsymbol{y}^T(z,t)\sum_{i=1}^{r} h_i(\boldsymbol{\xi}(z,t))[\boldsymbol{P}(\boldsymbol{A}_i+\boldsymbol{G}_u\boldsymbol{K}_{P,i})+*]\boldsymbol{y}(z,t)dz$$
$$+2\int_{l_1}^{l_2} \boldsymbol{y}^T(z,t)\boldsymbol{P}\boldsymbol{G}_u\sum_{i=1}^{r} h_i(\boldsymbol{\xi}(z,t))\boldsymbol{K}_{I,i}\boldsymbol{\upsilon}(z,t)dz\,. \qquad (15)$$

Utilizing (2), integrating by parts and taking into account of boundary conditions of (10), we can find that

$$2\int_{l_1}^{l_2} \boldsymbol{y}^T(z,t)\boldsymbol{P}\mathcal{A}\boldsymbol{y}(z,t)dz = 2\int_{l_1}^{l_2} \boldsymbol{y}^T(z,t)\boldsymbol{P}\Theta\boldsymbol{y}_{zz}(z,t)dz$$
$$= -\int_{l_1}^{l_2} \boldsymbol{y}_z^T(z,t)[\boldsymbol{P}\Theta+*]\boldsymbol{y}_z(z,t)dz\,. \quad (16)$$

Therefore, we have the following theorem based on Lemmas 1 and 2:

Theorem 1. Consider the fuzzy PDIE system (6) and the fuzzy P-sI controller (9). If there exist a real $n\times n$ matrix $\boldsymbol{Q}>0$ and real $m\times n$ matrices $\boldsymbol{Z}_{P,i}$, $\boldsymbol{Z}_{I,i}$, $i\in\mathcal{S}$ such that the following LMIs are satisfied:

$$\boldsymbol{\Psi}_{11,i} \triangleq \begin{bmatrix} [\boldsymbol{A}_i\boldsymbol{Q}+\boldsymbol{G}_u\boldsymbol{Z}_{P,i}+*] & \boldsymbol{G}_u\boldsymbol{Z}_{I,i}+\boldsymbol{G}_I\boldsymbol{Q} \\ * & -0.25\pi^4(l_2-l_1)^{-4}[\Theta\boldsymbol{Q}+*] \end{bmatrix} < 0\,,$$
$$i\in\mathcal{S}\,, \qquad (17)$$

then the closed-loop fuzzy PDIE system (10) is exponentially stable, i.e., the fuzzy P-sI controller (9) can exponentially stabilize the semi-linear PDIE system (1). In this case, the control gain matrices $\boldsymbol{K}_{P,i}$ and $\boldsymbol{K}_{I,i}$, $i\in\mathcal{S}$ are given as

$$\boldsymbol{K}_{P,i} = \boldsymbol{Z}_{P,i}\boldsymbol{Q}^{-1}\,,\ \boldsymbol{K}_{I,i} = \boldsymbol{Z}_{I,i}\boldsymbol{Q}^{-1}\,,\ i\in\mathcal{S}\,. \qquad (18)$$

Proof. Assume that LMIs (17) are fulfilled. Let

$$\boldsymbol{Q} = \boldsymbol{P}^{-1} \text{ and } \boldsymbol{Z}_{P,i} = \boldsymbol{K}_{P,i}\boldsymbol{Q}\,,\ \boldsymbol{Z}_{I,i} = \boldsymbol{K}_{I,i}\boldsymbol{Q}\,,\ i\in\mathcal{S}\,. \qquad (19)$$

The LMIs (17) imply

$$[\Theta\boldsymbol{Q}+*] > 0\,. \qquad (20)$$

From (19), we have $[\boldsymbol{P}\Theta+*] = \boldsymbol{P}[\Theta\boldsymbol{Q}+*]\boldsymbol{P}$, we can further get from (20)

$$[\boldsymbol{P}\Theta+*] > 0\,. \qquad (21)$$

Since the boundary condition of (1), (8), and (21), using Lemmas 1 and 2, the equality (16) can be written as

$$2\int_{l_1}^{l_2} \boldsymbol{y}^T(z,t)\boldsymbol{P}\mathcal{A}\boldsymbol{y}(z,t)dz = -\int_{l_1}^{l_2} \boldsymbol{y}_z^T(z,t)[\boldsymbol{P}\Theta+*]\boldsymbol{y}_z(z,t)dz$$
$$\le -\pi^2(l_2-l_1)^{-2}\int_{l_1}^{l_2} \boldsymbol{y}^T(z,t)[\boldsymbol{P}\Theta+*]\boldsymbol{y}(z,t)dz$$
$$\le -0.25\pi^4(l_2-l_1)^{-4}\int_{l_1}^{l_2} \boldsymbol{\upsilon}^T(z,t)[\boldsymbol{P}\Theta+*]\boldsymbol{\upsilon}(z,t)dz\,. \qquad (22)$$

From (15) and (22), we have

$$\dot{V}(t) \le \int_{l_1}^{l_2} \boldsymbol{y}^T(z,t)\sum_{i=1}^{r} h_i(\boldsymbol{\xi}(z,t))[\boldsymbol{P}(\boldsymbol{A}_i+\boldsymbol{G}_u\boldsymbol{K}_{P,i})+*]\boldsymbol{y}(z,t)dz$$
$$+2\int_{l_1}^{l_2} \boldsymbol{y}^T(z,t)\boldsymbol{P}\boldsymbol{G}_I\boldsymbol{\upsilon}(z,t)dz$$
$$+2\int_{l_1}^{l_2} \boldsymbol{y}^T(z,t)\boldsymbol{P}\boldsymbol{G}_u\sum_{i=1}^{r} h_i(\boldsymbol{\xi}(z,t))\boldsymbol{K}_{I,i}\boldsymbol{\upsilon}(z,t)dz$$
$$-\frac{\pi^4}{4(l_2-l_1)^4}\int_{l_1}^{l_2} \boldsymbol{\upsilon}^T(z,t)[\boldsymbol{P}\Theta+*]\boldsymbol{\upsilon}(z,t)dz$$
$$= \int_{l_1}^{l_2}\sum_{i=1}^{r} h_i(\boldsymbol{\xi}(z,t))\tilde{\boldsymbol{y}}^T(z,t)\boldsymbol{\Phi}_{11,i}\tilde{\boldsymbol{y}}(z,t)dz \qquad (23)$$

where $\tilde{\boldsymbol{y}}(z,t) \triangleq [\boldsymbol{y}^T(z,t) \quad \boldsymbol{\upsilon}^T(z,t)]^T$ and

$$\boldsymbol{\Phi}_{11,i} \triangleq \begin{bmatrix} [\boldsymbol{P}(\boldsymbol{A}_i+\boldsymbol{G}_u\boldsymbol{K}_{P,i})+*] & \boldsymbol{P}\boldsymbol{G}_u\boldsymbol{K}_{I,i}+\boldsymbol{P}\boldsymbol{G}_I \\ * & -0.25\pi^4(l_2-l_1)^{-4}[\boldsymbol{P}\Theta+*] \end{bmatrix}. \qquad (24)$$

Using (19), and pre- and post-multiplying the matrix $\boldsymbol{\Phi}_{11,i}$ given in (24) with the block diagonal matrix $\text{diag}\{\boldsymbol{Q} \quad \boldsymbol{Q}\}$, respectively, we have

$$\text{diag}\{\boldsymbol{Q} \quad \boldsymbol{Q}\}\boldsymbol{\Phi}_{11,i}\text{diag}\{\boldsymbol{Q} \quad \boldsymbol{Q}\} = \boldsymbol{\Psi}_{11,i}\,,\ i\in\mathcal{S}\,. \qquad (25)$$

Hence, we can conclude from (25) that if the LMIs (17) hold, then

$$\boldsymbol{\Phi}_{11,i} < 0\,,\ i\in\mathcal{S}\,. \qquad (26)$$

For the inequalities (26), we can find an appropriate scalar $\mu>0$ such that the following inequalities are fulfilled:

$$\boldsymbol{\Phi}_{11,i} + \mu\boldsymbol{I} < 0\,,\ i\in\mathcal{S}\,. \qquad (27)$$

Substituting (27) into (23) and using (7), we get for non-zero $\boldsymbol{y}(\cdot,t)$,

$$\dot{V}(t) \le -\mu\left\|\tilde{\boldsymbol{y}}(\cdot,t)\right\|_2^2 \le -\mu\left\|\boldsymbol{y}(\cdot,t)\right\|_2^2\,. \qquad (28)$$

It is clear for the Lyapunov function $V(t)$ given by (14) that there exist two positive scalars $p_1 \triangleq \lambda_{\min}(\boldsymbol{P})$ and $p_2 \triangleq \lambda_{\max}(\boldsymbol{P})$ such that

$$p_1\left\|\boldsymbol{y}(\cdot,t)\right\|_2^2 \le V(t) \le p_2\left\|\boldsymbol{y}(\cdot,t)\right\|_2^2\,. \qquad (29)$$

Using (28) and (29), we can get the following relation:

$$p_1\|\boldsymbol{y}(\cdot,t)\|_2^2 \le V(t) \le V(0)\exp(-\mu p_2^{-1}t)$$
$$\le p_2\|\boldsymbol{y}_0(\cdot)\|_2^2 \exp(-\mu p_2^{-1}t) \ . \quad (30)$$

Therefore, we have from (30) that

$$\|\boldsymbol{y}(\cdot,t)\|_2^2 \le p_2 p_1^{-1}\|\tilde{\boldsymbol{y}}_0(\cdot)\|_2^2 \exp(-\mu p_2^{-1}t)\ , \ t\ge 0$$

which implies that the fuzzy closed-loop PDIE system (10) is exponentially stable. From (19), we have (18). The proof is complete. □

Based on Lemmas 1 and 2, Theorem 1 provides an LMI-based sufficient condition on the existence of a distributed fuzzy P-sI controller (9) guaranteeing the exponential stability of the fuzzy PDIE system (10). The desired control gain matrices $\boldsymbol{K}_{P,i}$ and $\boldsymbol{K}_{I,i}$, $i\in\mathcal{S}$ can be constructed as (18) via the feasible solutions to LMIs (17). These feasible solutions can be directly solved using the *feasp* solver in the convex optimization techniques [22], [23].

Remark 2. Notice that if the simple fuzzy proportional controller is used, i.e.,

$$\boldsymbol{u}(z,t)=\sum\nolimits_{j=1}^{r} h_j(\boldsymbol{\xi}(z,t))\boldsymbol{K}_{P,j}\boldsymbol{y}(z,t)\ , \quad (31)$$

then by letting $\boldsymbol{K}_{I,j}\equiv 0$, $i\in\mathcal{S}$, it is immediate from Theorem 1 to obtain an LMI-based exponential stabilization condition via the controller (31) for the semi-linear PDIE system (1).

IV. NUMERICAL SIMULATION

In this section, in order to illustrate the effectiveness of the proposed result, we consider the control problem of a class of semi-linear reaction-diffusion system with distributed control inputs and a spatially integral term:

$$\begin{cases} y_{1,t}(z,t)=y_{1,zz}(z,t)+\tilde{a}(y_1(z,t)-y_2(z,t)) \\ \qquad +\tilde{c}\int_0^z y_1(s,t)ds+u_1(z,t) \\ y_{2,t}(z,t)=y_{2,zz}(z,t)+\tilde{b}y_1(z,t)-y_2(z,t)-y_1(z,t)y_3(z,t) \\ y_{3,t}(z,t)=y_{3,zz}(z,t)+\tilde{d}y_3(z,t)+y_1(z,t)y_2(z,t)+u_2(z,t) \\ y_i(l_1,t)=y_i(l_2,t)=0, \\ y_i(z,0)=y_{i,0}(z), z\in[l_1,l_2], i\in\{1,2,3\} \end{cases} \quad (32)$$

where $y_i(z,t)\in\Re$, $i\in\{1,2,3\}$ are the state variables, $u_1(z,t)$, $u_2(z,t)\in\Re$ are distributed control inputs, t, z and l_2-l_1 denote the independent time, spatial position and the length of spatial domain, respectively. $\tilde{a}\ge 0$, $\tilde{b}$, $\tilde{c}$, and $\tilde{d}$ are process parameters. $y_{i,0}(z)$, $i\in\{1,2,3\}$ are initial variables.

The values of process parameters are given as: $l_1=0$, $l_2=0.5\pi$, $\tilde{a}=2$, $\tilde{b}=-5$, $\tilde{c}=0.5$, and $\tilde{d}=2$.

For the above process parameter values, it will be verified via the numerical simulation that the equilibrium points $y_i(z,t)=0$, $i\in\{1,2,3\}$ of the semi-linear PDIE system (32) are unstable ones. Set $y_{1,0}(z)=0.5\sin(2z)$, $y_{2,0}(z)=0.3\times\sin(2z)$ and $y_{3,0}(z)=0.1\sin(2z)$. Fig. 1 shows open-loop profiles of evolution $y_i(z,t)$, $i\in\{1,2,3\}$. It is clear from Fig. 1 that the equilibrium points $y_i(z,t)=0$, $i\in\{1,2,3\}$ of the system (32) are unstable ones and $y_1(z,t)\in[0,1.5]$, $z\in[0,0.5\pi]$.

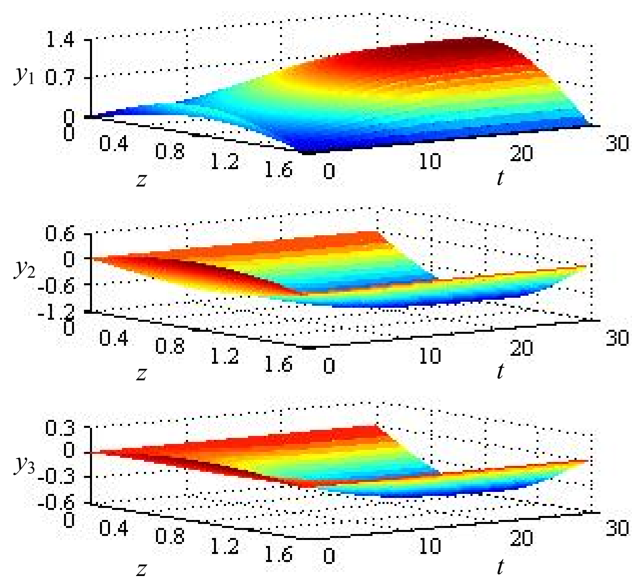

Fig. 1 Open-loop profiles of evolution of $y_i(z,t)$, $i\in\{1,2,3\}$

Set $\boldsymbol{y}(z,t)\triangleq[y_1(z,t)\quad y_2(z,t)\quad y_3(z,t)]^T\in\Re^3$ and $\boldsymbol{u}(z,t)\triangleq[u_1(z,t)\quad u_2(z,t)]^T\in\Re^2$, the system (32) can be written as (1), where

$$\boldsymbol{\Theta}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \boldsymbol{G}_u=\begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}, \boldsymbol{G}_I=\begin{bmatrix}\tilde{c}&0&0\\0&0&0\\0&0&0\end{bmatrix},$$
$$\boldsymbol{f}(\boldsymbol{y}(z,t))=\begin{bmatrix}\tilde{a}&-\tilde{a}&0\\\tilde{b}&-1&y_1(z,t)\\0&y_1(z,t)&\tilde{d}\end{bmatrix}\boldsymbol{y}(z,t) \quad (33)$$

where $y_1(z,t)$ is the nonlinear term.

Assume $y_1(z,t)\in[\bar{\alpha}_1,\bar{\alpha}_2]$, where $\bar{\alpha}_1=-0.5$ and $\bar{\alpha}_2=1.5$. Under this assumption, let $\xi(y_1(z,t))=y_1(z,t)$, calculating the maximum and minimum values of $\xi(y_1(z,t))$ gives

$\min_{y_1(z,t)} \xi(y_1(z,t)) = \bar{\alpha}_1$ and $\max_{y_1(z,t)} \xi(y_1(z,t)) = \bar{\alpha}_2$.

Using the above values, $\xi(y_1(z,t))$ can be written as

$$\begin{aligned}\xi(y_1(z,t)) &= y_1(z,t) \\ &= h_1(\xi(y_1(z,t)))\cdot\bar{\alpha}_1 + h_2(\xi(y_1(z,t)))\cdot\bar{\alpha}_2 \end{aligned} \tag{34}$$

where $h_1(\xi(y_1(z,t)))$, $h_2(\xi(y_1(z,t))) \in [0,1]$, and

$$h_1(\xi(y_1(z,t))) + h_2(\xi(y_1(z,t))) = 1 . \tag{35}$$

Solving equations (34) and (35) derives the following membership functions:

$h_1(\xi(y_1(z,t))) = (\bar{\alpha}_2 - \xi(y_1(z,t)))/(\bar{\alpha}_2 - \bar{\alpha}_1)$ and
$h_2(\xi(y_1(z,t))) = (\xi(y_1(z,t)) - \bar{\alpha}_1)/(\bar{\alpha}_2 - \bar{\alpha}_1)$.

Define the fuzzy sets "Big" and "Small". Then, the system (32) can be represented by the following fuzzy PDIE of two rules:

Plant Rule 1:
IF $\xi(y_1(z,t))$ is "Small", **THEN**

$$\begin{cases} \boldsymbol{y}_t(z,t) = \boldsymbol{\mathcal{A}}\boldsymbol{y}(z,t) + \boldsymbol{A}_1\boldsymbol{y}(z,t) + \boldsymbol{G}_u\boldsymbol{u}(z,t) \\ \qquad + \boldsymbol{G}_I\int_{l_1}^{z}\boldsymbol{y}(s,t)ds \\ \boldsymbol{y}(l_1,t) = \boldsymbol{y}(l_2,t) = 0 \\ \boldsymbol{y}(z,0) = \boldsymbol{y}_0(z), \end{cases}$$

Plant Rule 2:
IF $\xi(y_1(z,t))$ is "Big", **THEN**

$$\begin{cases} \boldsymbol{y}_t(z,t) = \boldsymbol{\mathcal{A}}\boldsymbol{y}(z,t) + \boldsymbol{A}_2\boldsymbol{y}(z,t) + \boldsymbol{G}_u\boldsymbol{u}(z,t) \\ \qquad + \boldsymbol{G}_I\int_{l_1}^{z}\boldsymbol{y}(s,t)ds \\ \boldsymbol{y}(l_1,t) = \boldsymbol{y}(l_2,t) = 0 \\ \boldsymbol{y}(z,0) = \boldsymbol{y}_0(z), \end{cases}$$

where

$$\boldsymbol{A}_1 = \begin{bmatrix} \tilde{a} & -\tilde{a} & 0 \\ \tilde{b} & -1 & \bar{\alpha}_1 \\ 0 & \bar{\alpha}_1 & \tilde{d} \end{bmatrix} \text{ and } \boldsymbol{A}_2 = \begin{bmatrix} \tilde{a} & -\tilde{a} & 0 \\ \tilde{b} & -1 & \bar{\alpha}_2 \\ 0 & \bar{\alpha}_2 & \tilde{d} \end{bmatrix}.$$

Then, the overall fuzzy PDIE model is given as

$$\begin{cases} \boldsymbol{y}_t(z,t) = \boldsymbol{\mathcal{A}}\boldsymbol{y}(z,t) + \sum_{i=1}^{2} h_i(\xi(y_1(z,t)))\boldsymbol{A}_i\boldsymbol{y}(z,t) \\ \qquad + \boldsymbol{G}_u\boldsymbol{u}(z,t) + \boldsymbol{G}_I\int_{l_1}^{z}\boldsymbol{y}(s,t)ds \\ \boldsymbol{y}(l_1,t) = \boldsymbol{y}(l_2,t) = 0 \\ \boldsymbol{y}(z,0) = \boldsymbol{y}_0(z). \end{cases} \tag{36}$$

Based on T-S fuzzy PDIE model (36), we consider the following P-sI controller:

$$\boldsymbol{u}(z,t) = \sum_{j=1}^{2} h_j(\xi(y_1(z,t))[\boldsymbol{K}_{P,j}\boldsymbol{y}(z,t) + \boldsymbol{K}_{I,j}\int_{l_1}^{z}\boldsymbol{y}(s,t)ds] . \tag{37}$$

Solving LMIs (17) and using (18), the control gain matrices $\boldsymbol{K}_{P,1}$, $\boldsymbol{K}_{P,2}$, $\boldsymbol{K}_{I,1}$, and $\boldsymbol{K}_{I,2}$ can be derived as follows:

$$\boldsymbol{K}_{P,1} = \begin{bmatrix} -7.1742 & 7.8768 & -0.9235 \\ 1.0251 & 0.6373 & -5.7910 \end{bmatrix},$$
$$\boldsymbol{K}_{P,2} = \begin{bmatrix} -7.1768 & 7.8878 & -0.4213 \\ 0.9489 & -3.1755 & -5.8323 \end{bmatrix},$$
$$\boldsymbol{K}_{I,1} = \boldsymbol{K}_{I,2} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

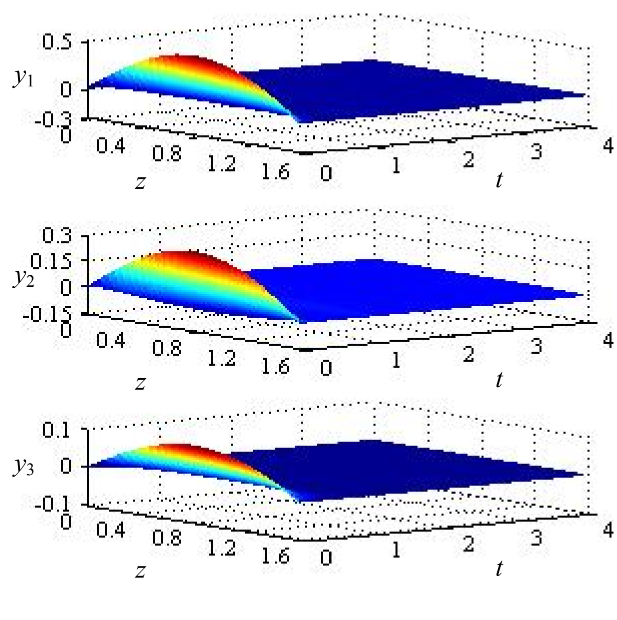

Fig. 2 Closed-loop profiles of evolution of $y_i(z,t)$, $i \in \{1,2,3\}$

Now, the fuzzy P-sI controller (37) with the above gain matrices is applied to the semi-linear PDIE system (32). Under the same initial condition (i.e., $y_{1,0}(z) = 0.5\sin(2z)$, $y_{2,0}(z) = 0.3\sin(2z)$, and $y_{3,0}(z) = 0.1\sin(2z)$), Fig. 2 indicates the closed-loop profiles of evolution of $y_i(z,t)$, $i \in \{1,2,3\}$. Clearly, the proposed fuzzy P-sI controller (37) with above gain matrices can stabilize the system (32). The profile of evolution of the fuzzy P-sI controller $\boldsymbol{u}(z,t)$ is given in Fig. 3.

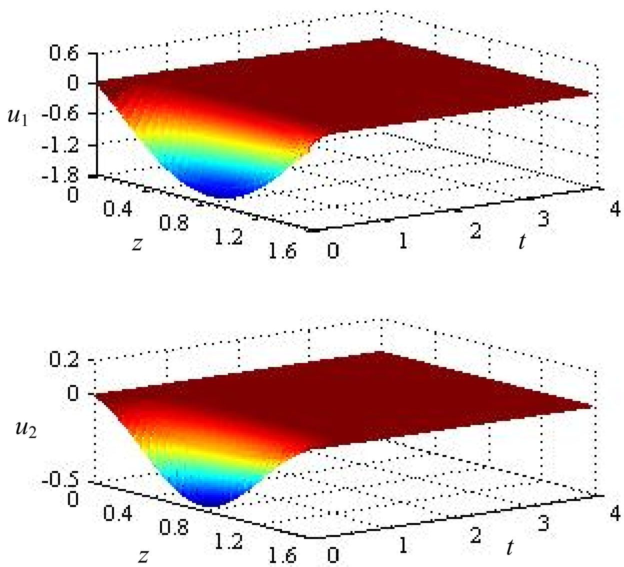

Fig. 3 Profiles of evolution of fuzzy P-sI controller $u_i(z,t)$, $i \in \{1,2\}$

V. Conclusion

In this paper, we have considered the problem of distributed fuzzy P-sI control design for a class of semi-linear PDIE systems via the fuzzy PDIE modeling approach. A T-S fuzzy parabolic PDIE model is utilized to accurately describe the semi-linear parabolic PDIE system. A new vector-valued integral inequality is established based on the vector-valued Wirtinger's inequality. Based on the T-S fuzzy PDIE model and this new integral inequality, an LMI-based distributed fuzzy P-sI state feedback control design has been developed. Numerical simulation results on feedback control of a semi-linear parabolic PDIE system illustrate the effectiveness of the proposed design method.

References

[1] L.-X. Wang, *A Course in Fuzzy Systems and Control*, New Jersey: Prentice-Hall, 1996.

[2] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Trans. Syst. Man, Cybern.*, vol. 15, no. 1, pp. 116-132, 1985.

[3] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*. New York: Wiley, 2001.

[4] G. Feng, *Analysis and Synthesis of Fuzzy Control Systems: A Model Based Approach*, New York: Taylor & Francis, 2010.

[5] H. Gao and T. Chen, "Stabilization of nonlinear systems under variable sampling: a fuzzy control approach," *IEEE Trans. Fuzzy Syst.*, vol. 15, no. 5, pp. 972-983, 2007.

[6] V. F. Montagner, R. C. L. F. Oliveira, and P. L. D. Peres, "Convergent LMI relaxations for quadratic stabilizability and H_∞ control of Takagi-Sugeno fuzzy systems," *IEEE Trans. Fuzzy Syst.,* vol. 17, no. 4, pp. 863-873, 2009.

[7] J.-W. Wang, H.-N. Wu, L. Guo and Y.-S. Luo, "Robust H_∞ fuzzy control for uncertain nonlinear Markovian jump systems with time-varying delay," *Fuzzy Set. Syst.*, vol. 212, pp. 41-61, 2013.

[8] Q. Gao, G. Feng, Y. Wang, and J. Qiu, "Universal fuzzy models and universal fuzzy controllers for stochastic nonaffine nonlinear systems," *IEEE Trans. Fuzzy Syst.*, vol. 21, no. 2, pp. 328-341, 2013.

[9] W. H. Ray, *Advanced Process Control*, New York: McGraw-Hill, 1981.

[10] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-dimensional Linear Systems Theory*, New York: Springer-Verlag, 1995.

[11] P. D. Christofides, *Nonlinear and Robust Control of PDE Systems: Methods and Applications to Transport-Reaction Processes*, Boston, MA: Birkhäuser, 2001.

[12] R. Padhi and Sk. F. Ali, "An account of chronological developments in control of distributed parameter systems," *Annu. Rev. Control*, vol. 33, no. 1, pp. 59-68, 2009.

[13] J. Baillieul, "Linearized models for the control of rotating beams," in *Proc. 27th Conf. Decis. Control*, Austin, Texas, December, 1988, pp. 1726-1731.

[14] F. Najar, S. Choura, E. Abdel-Rahman *et al*., "Dynamic analysis of variable-geometry electrostatic microactuators," *J. Micromech. Microeng.*, vol. 16, no. 11, pp. 2449-2457, 2006.

[15] C. Ly and B. Doiron, "Divisive gain modulation with dynamic stimuli in integrate-and- fire neurons," *PLoS Comput. Biol.*, vol. 5, no. 4, pp. 1-12, 2009.

[16] H.-N. Wu and H.-X. Li, "Finite-dimensional constrained fuzzy control of a class of nonlinear distributed process systems," *IEEE Trans. Syst., Man, Cybern.: Part B*, vol. 37, no. 5, pp. 1422-1430, 2007.

[17] H.-N. Wu and H.-X. Li, "H_∞ fuzzy observer-based control for a class of nonlinear distributed parameter systems with control constraints," *IEEE Trans. Fuzzy Syst.,* vol. 16, no. 2, pp. 502-516, 2008.

[18] B.-S. Chen and Y.-T. Chang, "Fuzzy state-space modeling and robust observer-based control design for nonlinear partial differential systems," *IEEE Trans. Fuzzy Syst.,* vol. 17, no. 5, pp. 1025-1043, 2009.

[19] Y.-T. Chang and B.-S. Chen, "A fuzzy approach for robust reference-tracking-control design of nonlinear distributed parameter time-delayed systems and its application," *IEEE Trans. Fuzzy Syst.*, vol. 18, no. 6, pp. 1041-1057, 2010.

[20] J.-W. Wang, H.-N. Wu and H.-X. Li, "Distributed proportional-spatial derivative control of nonlinear parabolic systems via fuzzy PDE modeling approach," *IEEE Trans. Syst., Man, Cybern.: Part B*., vol. 42, no. 3, pp. 927-938, 2012.

[21] H.-N. Wu, J.-W. Wang and H.-X. Li, "Exponential stabilization for a class of nonlinear parabolic PDE systems via fuzzy control approach," *IEEE Trans. Fuzzy Syst.*, vol. 20, no. 2, pp. 318-329, 2012.

[22] S. Boyd, L. EI Ghaoui, E. Feron., and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, Philadelphia, PA: SIAM, 1994.

[23] P. Gahinet, A. Nemirovskii, A. J. Laub, and M. Chilali, *LMI Control Toolbox for Use with Matlab*, Natick, MA : The Math.Works Inc., 1995.

[24] A. Seuret and F. Gouaisbaut, "Jensen's and Wirtinger's inequalities for time-delay systems," in *11th IFAC Workshop on Time-Delay Systems*, Grenoble, France, 2013.