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Hierarchy of Lattice-valued Fuzzy Automata and Decidability of Their Languages

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***Abstract*—In this paper, the role of local finiteness of truth values domain of fuzzy automata is analyzed, in which the truth value domain of fuzzy automata is the (commutative) lattice-ordered monoid. We introduce a hierarchy of lattice-valued fuzzy finite automata and the languages which were recognized by these automata. Besides, the role of local finiteness of truth value domain of fuzzy languages to the hierarchy of fuzzy automata, the role of some special archimedean t-norms in the hierarchy of fuzzy automata and the decidability of lattice-valued languages are also discussed.**

I. Introduction

Classical computation theory aims at the deterministic computing environment, but the reality of computing environment or computing system, especially the environment that has relation to human activities, often contains uncertainty. Since 1960s, the research of computing theory under uncertainty environment has become an important research topic. The computing problem of fuzzy environment was first introduced by L.A.Zadeh who put forward the question of fuzzy language and fuzzy computing models. The notion of fuzzy automata was produced by Wee in 1967 (see[1] for the detail introduction of classical fuzzy automata). The notion of lattice-valued fuzzy automata put forward by Li and Pedrycz is one of the most well-known fuzzy automata currently (see [2],[3],[4],[5]). The lattice-valued fuzzy automata have many unique properties. For instance, it holds that nondeterministic fuzzy finite automata is equivalent to deterministic fuzzy automata, but it is not valid for lattice-valued fuzzy automata; The family of lattice-valued fuzzy languages is not closed under complement operation and so on. The languages accepted by lattice-valued automata are called lattice-valued fuzzy regular languages. Klimann and S.Lombardy have studied the hierarchy of weighted automata and their languages in tropical semiring (see[6]). As we all know, for two regular languages L_1 and L_2, the problems $L_1 = L_2$ and $L_1 \leq L_2$ are decidable, KROB has discussed these problems for weighted regular languages in the tropical semiring (see[7]). In this paper, wee will study these problems for the lattice-valued regular languages and the hierarchy of lattice-valued fuzzy automata.

II. Preliminaries

Definition 2.1: [2] Let L be a bounded lattice, the least and largest elements will be denoted by 0 and 1. There is a binary operation $\cdot$ on L such that $(L, \cdot, e)$ is a monoid with identity $e \in L$, if it satisfies the following three conditions for arbitrary $a, b, x \in L$:

(1) $a \cdot 0 = 0 \cdot a = 0$;

(2) $a \leq b \Rightarrow a \cdot x \leq b \cdot x$ and $x \cdot a \leq x \cdot b$;

(3) $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$ and $(b \vee c) \cdot a = (b \cdot a) \vee (c \cdot a)$.

then we call L a lattice-ordered monoid or l-monoid.

Definition 2.2: [2] Let $*$ is a binary operation on $[0, 1]$, if it satisfies the following conditions:

(1) $\forall a, b \in [0, 1], a * b = b * a$;

(2) $\forall a, b, c \in [0, 1], (a * b) * c = a * (b * c)$;

(3) $\forall a, b, c, d \in [0, 1]$ and $a \leq c, b \leq d \Rightarrow a * b \leq c * d$;

(4) $\forall a \in [0, 1], 1 * a = a$.

then we call $*$ a t-norm, if $*$ is continuous and $\forall a \in (0, 1), a * a < a$, then $*$ is called an Archimedean t-norm.

Here are a few examples of l-monoid:

(1) $([0, 1], \vee, \wedge, 0, 1)$;

(2) $([0, 1], \vee, *, 0, 1)$,in which$*$ is a t-norm.

Definition 2.3: [2] An $L-$fuzzy finite automata ($L-$FFA, for short) is a five tuple, $\mathscr{A} = (Q, \Sigma, \delta, I, T)$, where Q, Σ are finite nonempty sets, the elements of Q, Σ denote states and input symbols respectively, $\delta : Q \times \Sigma \times Q \to L$ is called a fuzzy transition function and $I, T : Q \to L$ are respectively called fuzzy initial and final mappings.

Definition 2.4: [2] Let $\mathscr{A}$ be an $L-$FFA, the language recognized by $\mathscr{A}$ is denoted by $|\mathscr{A}| : \Sigma^* \to L$, which is defined as, $\forall \theta = \sigma_1\sigma_2 \cdots \sigma_k$, $|\mathscr{A}|(\theta) = \vee\{I(q_0) \cdot \delta(q_0, \sigma_1, q_1) \cdot \cdots \cdot \delta(q_{k-1}, \sigma_k, q_k) \cdot T(q_k) | q_0, q_1, \ldots, q_k \in Q\}$.

For a fuzzy language $f:\Sigma^*\to L$, if there exists a finite automaton $\mathscr{A}$ such that $|\mathscr{A}|=f$, then f is called a lattice-valued fuzzy regular language, we use $L-Reg$ to represent the set of fuzzy regular languages.

III. Hierarchy of lattice-valued fuzzy automata

Suppose that $\mathscr{A}=(Q,\Sigma,\delta,I,T)$ is an $L-$FFA, we can classify it as follows:

(1) Deterministic fuzzy automaton $L-$DFA: $\delta: Q\times\Sigma\to Q$ is deterministic transition function, and $I=\{q_0\}, q_0\in Q$.

(2) Sequential fuzzy automaton $L-$Seq: $\mathscr{A}=(Q,\Sigma,\delta,I,T)$ such that, there exists a unique $q\in Q$ satisfies $I(q)>0$, and for any $q\in Q$, $\sigma\in\Sigma$, there exists a unique $p\in Q$ such that $\delta(q,\sigma,p)>0$.

(3) Unambiguous fuzzy automaton $L-$NAmb: There is a unique successful path ρ for any word $\theta\in supp(|\mathscr{A}|)$, and $\rho=q_0q_1\dots q_k$ such that $|\mathscr{A}|(\theta)=I(q_0)\cdot\delta(q_0,\sigma_1,q_1)\cdot\dots\cdot\delta(q_{k-1},\sigma_k,q_k)\cdot T(q_k)$, where $\theta=\sigma_1\sigma_2\dots\sigma_k$.

(4) Finitely ambiguous automaton $L-$FAmb: If there exists some $n\in N$ such that for any word $\theta\in supp(|\mathscr{A}|)$, there are at most n successful paths of label $\theta,\rho_1=q_{1_0}q_{1_1}\dots q_{1_k},\rho_l=q_{l_0}q_{l_1}\dots q_{l_k}, l\le n$, such that $r_i=I(q_{i_0})\cdot\delta(q_{i_0},\sigma_1,q_{i_1})\cdot\dots\cdot\delta(q_{i_{k-1}},\sigma_k,q_{i_k})\cdot T(q_{i_k})>0(1\le i\le n)$, where $\theta=\sigma_1\sigma_2\dots\sigma_k$, and $|\mathscr{A}|(\theta)=\vee_{i=1}^n r_i$. The minimal such n is called the ambiguity degree of the automaton.

Considering an $L-$Reg f. The language f is deterministic (resp. sequential, unambiguous, finitely ambiguous) if there exists a deterministic (resp. sequential, unambiguous, finitely ambiguous) fuzzy automaton recognizing it. The language f is infinitely ambiguous if there exists no finitely ambiguous fuzzy automaton recognizing it. The degree of ambiguity of a finitely ambiguous language is the minimal degree of ambiguity of an automaton recognizing it. The set of deterministic, sequential, unambiguous, finitely ambiguous language are denoted, respectively, by $\overline{L-DFA},\overline{L-Seq},\overline{L-NAmb},\overline{L-FAmb}$. In the following, we will discuss the relationship between these languages.

IV. Hierarchy of lattice-valued fuzzy regular language

Theorem 4.1: For any $L-NFA$ $\mathscr{A}$, there exists an $L-NFA$ $\mathscr{B}$ with crisp initial state such that $|\mathscr{A}|=|\mathscr{B}|$.

Proof: Assume that $\mathscr{A}=(Q,\Sigma,\delta,I,T)$ is an $L-NFA$, we construct a automaton $\mathscr{B}=(Y,\Sigma,\eta,y_0,T_Y)$, let $Y=I\cup Q, y_0=I,\eta: Y\times\Sigma\times Y\to Y$, such that:

$$\eta(y_1,\varepsilon,y_2)=\begin{cases} e, & y_1=y_2;\\ 0, & y_1\ne y_2.\end{cases}$$

$$\eta(y_1,\sigma,y_2)=\begin{cases} \vee_{q\in Q}I(q)\cdot\delta(q,\sigma,y_2), & y_1=I, y_2\in Q;\\ \delta(y_1,\sigma,y_2), & y_1,y_2\in Q;\\ 0, & y_1\in Q, y_2=I.\end{cases}$$

$$T_Y(y)=\begin{cases} \vee_{q\in Q}I(q)\cdot T(q), & y=I;\\ T(y), & y\in Q.\end{cases}$$

then $|\mathscr{B}|(\varepsilon)=\vee_{y\in Y}\eta^*(y_0,\varepsilon,y)\cdot T_Y(y)=\eta^*(y_0,\varepsilon,y_0)\cdot T_Y(y_0)=\vee_{q\in Q}I(q)\cdot T(q)=|\mathscr{A}|(\varepsilon)$.

When $\theta=\mu_1$, $|\mathscr{B}|(\theta)=\vee_{y\in Y}\eta(y_0,\mu_1,y)\cdot T_Y(y)=\vee_{y\in Q}\eta(y_0,\mu_1,y)\cdot T_Y(y)\bigvee\vee_{y=I}\eta(y_0,\ \mu_1,y)\cdot T_Y(y)=\vee I(q)\cdot\delta(q,\theta,y)\cdot T(y)=|\mathscr{A}|(\theta)$.

Assume that it holds for $|\theta|=n$, so when $|\theta|=n+1$,

$$\begin{aligned}|\mathscr{B}|(\theta)&=\vee_{y',y_1,\dots y_{n+1}\in Y}I(y')\cdot\eta^*(y'),\theta,y)\cdot T_Y(y)\\ &=\vee_{y_1,\dots,y_{n+1}\in Q}I(y_0)\cdot\eta^*(y_0),\theta,y_{n+1})\cdot T_Y(y_{n+1})\bigvee 0\\ &=\vee_{q\in Q}I(q)\cdot\delta^*(q,\theta,y_{n+1})\cdot T(y_{n+1})\\ &=|\mathscr{A}|(\theta)\end{aligned}$$

So we can also get that $|\mathscr{B}|(\theta)=|\mathscr{A}|(\theta)$.

Theorem 4.2: [2] The following conditions are equivalent for an $L-$Reg f.

(1) f can be recognized by an $L-DFA$;

(2) $Im(f)$ is finite and f_γ is regular for any $\gamma\in Im(f)$, where $f_\gamma=\{\theta\in\Sigma^*|f(\theta)\ge\gamma\}$;

(3) $Im(f)$ is finite and $f_{[\gamma]}$ is regular for any $\gamma\in Im(f)$, where $f_{[\gamma]}=\{\theta\in\Sigma^*|f(\theta)=\gamma\}$.

Theorem 4.3: [2] The following conditions are equivalent.

(1) For any $L-NFA$ $\mathscr{A}$, there exists an $L-DFA$ $\mathscr{B}$, such that $|\mathscr{A}|=|\mathscr{B}|$.

(2) L is locally finite.

Lemma 4.4: Assume L is a commutative l-monoid, then L is not locally finite iff there is $a\in L$ such that $a^i\ne a^j$ whenever $i\ne j$.

Proof: "If" part is obvious. Conversely, let $L''=\langle L'\rangle$, where L' is a finite subset of L. Then $L''=\{\vee_{i=1}^n m_i|m_i\in T, i=1,2,\cdots,n,n\ge 1\}, T=\{a_1^{l_1}\cdot a_2^{l_2}\cdot\dots\cdot a_k^{l_k}: l_1,l_2,\cdots,l_k\ge 0\}, L'=\{a_1,a_2,\cdots,a_k:0\le k\le n\}$. So L'' is infinite iff T is infinite iff there exists $a\in L'$ such that $\{a^l:l\ge 0\}$ is infinite. The necessary condition is equivalent to $a^i\ne a^j$ whenever $i\ne j$.

Clearly, "deterministic" implies "sequential", which implies "unambiguous", the last implies "finitely ambiguous".

In the sequel we assume that L is commutative, and Σ is an alphabet.

Theorem 4.5: If L is commutative, we can get $\overline{L-DFA}\subseteq\overline{L-Seq}\subseteq\overline{L-NAmb}\subseteq\overline{L-FAmb}\subseteq L-Reg$.

Theorem 4.6: If L is not locally finite, then the above inclusion is proper, i.e., $\overline{L-DFA}\subsetneq\overline{L-Seq}\subsetneq\overline{L-NAmb}\subsetneq\overline{L-FAmb}\subsetneq L-Reg$.

Proof: Since Theorem 4.5 is obviously holds, so we only need to give examples to explain the above inequalities hold. Since L is not locally finite, we can choose a in L such that $a^i\ne a^j$ whenever $i\ne j$ as declared by Lemma 4.4.

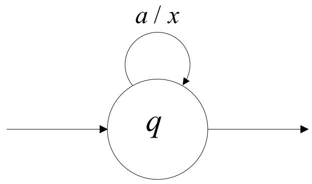

Fig. 1. An automaton that shows $\overline{L-DFA} \neq \overline{L-Seq}$

(i) The automata in Fig.1 is an $L-Seq$, and

$$|\mathscr{A}|(x^k) = \begin{cases} 1, & k = 0; \\ a^k, & k \neq 0. \end{cases}$$

Obviously, $Im(|\mathscr{A}|)$ is infinite, but the image of language which is accepted by $L-DFA$ is finite, so $|\mathscr{A}|$ can not be accepted by any $L-DFA$.

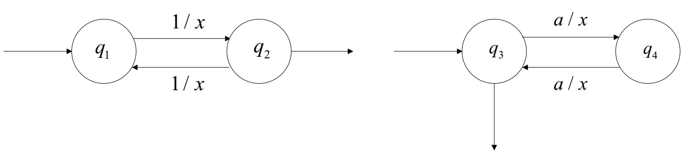

Fig. 2. An automaton that shows $\overline{L-Seq} \neq \overline{L-NAmb}$

(ii) The automata in Fig.2 is an $L-NAmb$, for any word ω, there is only one successful path can be accepted by the automaton, and

$$|\mathscr{A}|(x^k) = \begin{cases} a^k, & k = 2m; \\ 1, & k = 2m+1. \end{cases}$$

Assume there is an $L-Seq\mathscr{B}$ such that $|\mathscr{A}| = |\mathscr{B}|$, let $\mathscr{B} = (Q, \Sigma, \delta, I, T)$ and $I = b/q_0$, hence

$$b \cdot \delta(q_0, x, q_1) \cdot \ldots \cdot \delta(q_{n-1}, x, q_n) \cdot T(q_n) = \begin{cases} a^k, & k = 2m; \\ 1, & k = 2m+1. \end{cases}$$

so $|\mathscr{B}|(x^n) = b \cdot \delta(q_0, x, q_1) \cdot \ldots \cdot \delta(q_{n-1}, x, q_n) \cdot T(q_n) = 1$, whenever $n = 2m+1$, then $b = 1, \delta(q_0, x, q_1) = \ldots = \delta(q_{n-1}, x, q_n) = T(q_n) = 1$, thus $T(q_{n+1}) = a^{n+1}$. And, $T: Q \to L$, Q is finite, so $Im(f)$ is finite. However, $\{a^{n+1} | n = 2m+1\}$ is infinite set and a contradictory occurs. Thus, there is no $L-Seq\mathscr{B}$ such that $|\mathscr{A}| = |\mathscr{B}|$.

(iii) The automaton in Fig.3 is an $L-FAmb$, and $|\mathscr{A}|(\mu) = a^{min\{|\mu|_x, |\mu|_y\}}$. For any $\mu \in \Sigma^*$, there are at most two successful paths: $q_1 \overrightarrow{\mu} q_1, q_2 \overrightarrow{\mu} q_2$, i.e., the ambiguity degree of the automaton is 2. Now we assume there exists an $L-NAmb$ $\mathscr{B} = (Q, \Sigma, \delta, I, T)$ such that $|\mathscr{A}| = |\mathscr{B}|$, thus

$$|\mathscr{B}|(\mu) = \begin{cases} 1, & \mu = x^k or y^k; \\ \mu = x^k y^l, & l \leq k. \end{cases}$$

So there is only one path $q_0 q_1 \ldots q_k$ can be accepted by

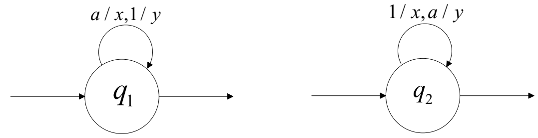

Fig. 3. An automaton that shows $\overline{L-NAmb} \neq \overline{L-FAmb}$

$\mathscr{B}$ whenever $\mu = x^k$. Thus, if $\mu = x^k, |\mathscr{B}|(\mu) = I(q_0) \cdot \delta(q_0, x, q_1) \cdot \ldots \cdot \delta(q_{k-1}, x, q_k) \cdot T(q_k) = 1$, then we can get $I(q_0) = \delta(q_0, x, q_1) = \ldots = \delta(q_{k-1}, x, q_k) = T(q_k) = 1$; If $\mu = x^k y, |\mathscr{B}|(\mu) = I(q_0) \cdot \delta(q_0, x, q_1) \cdot \ldots \cdot \delta(q_k, y, q_{k+1}) \cdot T(q_{k+1}) = a$, so $\delta(q_k, y, q_{k+1}) \cdot T(q_{k+1}) = a$; If $\mu = x^{k+1}, T(q_{k+1}) = 1$, then we can get $\delta(q_k, y, q_{k+1}) = a$. But, if $\mu = y^{k+1}, |\mathscr{B}|(\mu) = |\mathscr{B}|(y^k) \cdot \delta(q_k, y, q_{k+1}) \cdot T(q_{k+1}) = 1 \cdot a \cdot 1 = a$, however this is contradict with $|\mathscr{B}|(\mu) = 1$, thus $|\mathscr{A}|$ can not be accepted by an $L-Seq$, i.e., the number of initial state is larger than 2. And, $|\mathscr{B}|(\varepsilon) = 1$, thus there is only one state q_0 satisfies $I(q_0) = T(q_0) = 1$. Besides, $|\mathscr{B}|(x^k) = |\mathscr{B}|(y^k) = 1$, here we assume $k \geq |Q| = n$, so there must be $i \leq j$ satisfies $q_i = q_j$, and the membership of each transfer value is 1; $|\mathscr{B}|(x^k y^l) = a^l, |\mathscr{B}|(y^k x^l) = a^l, n \leq l \leq k$, then there exists cycles and the membership of each transfer value is a. So if there exists an $L-NAmb$ $\mathscr{B}$ such that $|\mathscr{A}| = |\mathscr{B}|$ iff for any $i, j, \eta(q_i, x, q_j) = \eta(q_i, y, q_j) = 1$, and $I(q_i), T(q_i) = \{1, a, a^2, \ldots\}$, but $|Q|$ is finite, so $Im(I), Im(T)$ is finite. Hence, there doesn't exist an $L-NAmb$ $\mathscr{B}$ such that $|\mathscr{A}| = |\mathscr{B}|$.

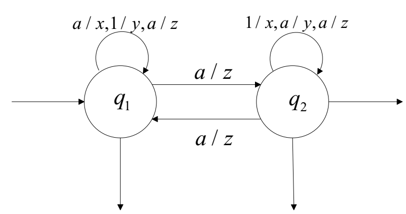

Fig. 4. An automaton that shows $\overline{L-FAmb} \neq L-Reg$

(iv) The automaton in Fig.4 is an $L-NFA$, so we only need to show $\mathscr{A}$ is infinite unambiguous. The successful paths of $\mathscr{A}$ is 2^{k+1} when $\mu = x^n y^m z^k$, but $\{2^{k+1}\}$ is an infinite set, thus there is no integer n satisfies n is larger than all the elements of $\{2^{k+1}\}$. Hence, we can get $\mathscr{A}$ is not an $L-FAmb$. So, $\overline{L-FAmb} \neq L-Reg$.

Theorem 4.7: If L is commutative and L is locally finite, then the above inclusion is equality, i.e., $\overline{L-DFA} = \overline{L-Seq} = \overline{L-NAmb} = \overline{L-FAmb} = L-Reg$.

Proof: According to Theorem 4.5, it is available that $\overline{L-DFA} \subseteq \overline{L-Seq} \subseteq \overline{L-NAmb} \subseteq \overline{L-FAmb} \subseteq L-Reg$, so we just need to prove $L-Reg \subseteq \overline{L-DFA}$. By Theorem 4.3, $\overline{L-NFA} \subseteq \overline{L-DFA}$, so the theorem holds.

Theorem 4.8: The following conditions are equivalent:

(1) For any $L-NFA$ $\mathscr{A}$, there exists an $L-Seq$ $\mathscr{B}$ such

that $|\mathscr{A}| = |\mathscr{B}|$;

(2) L is locally finite.

Proof: $(2) \Rightarrow (1)$: According to Theorem 4.3, for any $L-NFA\ \mathscr{A}$, there must be an $L-DFA\ \mathscr{B}$ such that $|\mathscr{A}| = |\mathscr{B}|$, and $L-DFA$ is a special $L-Seq$, so the condition (1) holds.

$(1) \Rightarrow (2)$: Assume that L is not locally finite, then there exists $a \in L$ such that $a^i \neq a^j$ whenever $i \neq j$. Define $f : \Sigma^* \to L$ as follows:

$$f(a^n) = \begin{cases} a^n, & n = 2m; \\ 1, & n = 2m+1. \end{cases}$$

and for any $\theta \neq \sigma^n, f(\theta) = 0$. Then f can be accepted by the automata in Fig.2, but it can not be accepted by any $L-Seq$.

Theorem 4.9: The following conditions are equivalent:

(1) For any $L-Seq\ \mathscr{A}$, there exists an $L-DFA\ \mathscr{B}$ such that $|\mathscr{A}| = |\mathscr{B}|$.

(2) L is locally finite.

Proof: $(2) \Rightarrow (1)$: It is obviously by [2], because $L-Seq$ is a special $L-NFA$.

$(1) \Rightarrow (2)$: Assume that L is not locally finite, then there exists $a \in L$ such that $a^i \neq a^j$ whenever $i \neq j$. Constructing an $L-Seq\mathscr{A} = (\{q\}, \Sigma, \delta, \{q\}, \{q\})$ such that for any $\sigma \in \Sigma, \delta(q, \sigma, q) = a$, then

$$|\mathscr{A}|(\sigma_1\sigma_2\cdots\sigma_k) = \begin{cases} 1, & k = 0; \\ a^k, & k > 0. \end{cases}$$

Clearly, $Im(|\mathscr{A}|)$ is an infinite set, but the image of $L-DFA$ is finite, so the language accepted by $\mathscr{A}$ can not be accepted by any $L-DFA$. This shows that L is locally finite.

In the following, we assume that L is the unit interval $[0,1]$ equipped with an Archimedean t-norm.

Theorem 4.10: Let $f \in L-Reg$, then for any $\alpha > 0, f_\alpha$ is regular and $Im(f \vee \alpha)$ is finite.

Proof: According to [9] we can know f_α is regular for any $\alpha > 0$.

$Im(f \vee \alpha)$ =Im$(f) \vee \alpha \subseteq \{f(\theta)|\theta \in \Sigma^*, f(\theta) \geq \alpha\} = Im_\alpha$. Assume D =Im$(\delta)\cup$Im(σ), then $Im(f) \subseteq S(D)$, thus $Im_\alpha(f) \subseteq S_\alpha(D)$, where $S(D)$ denotes the subset of [0,1] generated by D with the Archimedean t-norm and maximum operations, and $S_\alpha(D) = \{a \in [0,1]|a \geq \alpha\}$. For Archimedean t-norm, it is weakly finitely generated(for any finite subset D of $[0,1]$ and any $a \in [0,1], S_\alpha(D)$ is finite), so $S_\alpha(D)$ is finite, and $Im(f \vee \alpha) \subseteq Im_\alpha(f)$, hence $Im(f \vee \alpha)$ is finite.

Theorem 4.11: The following conditions are equivalent for an $L-$ fuzzy regular language f:

(1) f can be recognized by an $L-DFA\ \mathscr{A}$;

(2) $Im(f)$ is finite.

Proof: $(1) \Rightarrow (2)$: It is obvious.

$(2) \Rightarrow (1)$: It is reality to see that f_α is regular for any $\alpha > 0$ according to Theorem 10. What's more, we can show the theorem holds from Theorem 4.2.

Theorem 4.12: The following conditions are equivalent for an $L-$ fuzzy regular language f:

(1) $1-f$ is an $L-$fuzzy regular language;

(2) $Im(f)$ is finite;

(3) f can be recognized by an $L-DFA\ \mathscr{A}$.

Proof: By Theorem 4.11, it suffices to show that the condition (2) and (3) are equivalent.

$(3) \Rightarrow (1)$: Because $f \in \overline{L-DFA}$, so we can assume $\mathscr{A}$ can be accepted by an $L-DFA\ \mathscr{A} = (Q, \Sigma, \delta, q_0, \sigma_1)$, then $|\mathscr{A}| = \sigma_1(\delta^*(q_0, \theta))$. Constructing an $L-NFA\ \mathscr{B} = (Q, \Sigma, \delta_1, \{1/q_0\}, 1-\sigma_1)$ such that:

$$\delta_1(q, \mu, p) = \begin{cases} 1, & \delta(q, \mu) = p; \\ 0, & \delta(q, \mu) \neq p. \end{cases}$$

and $|\mathscr{B}|(\theta) = 1 - |\mathscr{A}|(\theta) = 1 - f$.

$(1) \Rightarrow (2)$: By Theorem 4.10, $Im(f \vee \alpha)$ and $Im((1-f) \vee (1-\alpha))$ are finite. $Im(f \vee \alpha) = Im(f) \vee \alpha, Im((1-f) \vee (1-\alpha)) = Im(1-f) \vee (1-\alpha)$, so the sets $D_1 = \{f(\theta)|f(\theta) > \alpha\}, D_2 = \{(1-f)(\theta)|(1-f)(\theta) > (1-\alpha)\}$ are finite. It is reality to see D_2 is equivalent to $D_3 = \{f(\theta)|f(\theta) < \alpha\}$, so D_3 is finite. In this case, $Im(f) = D_1 \cup D_3$ is finite.

V. The decidability of fuzzy regular languages

In this section, we assume that L is an $l-$monoid, Σ is an alphabet. Let us consider the four problems of equality, inequality, local inequality and local equality for $L-Reg(\Sigma)$:

$f, g \in L-Reg(\Sigma), f = g?(Eq)$

$f, g \in L-Reg(\Sigma), f \leq g?(Ineq)$

$f, g \in L-Reg(\Sigma), \exists\omega \in \Sigma^*, f(\omega) \leq g(\omega)?(LocalIneq)$

$f, g \in L-Reg(\Sigma), \exists\omega \in \Sigma^*, f(\omega) = g(\omega)?(LocalEq)$

Lemma 5.1: [5] Let $L = \langle N \cup \{+\infty\}, min, +, +\infty, 0\rangle$ be a tropical semiring, Σ is an alphabet, then $Eq, Ineq, LocalIneq, LocalEq$ are undecidable when $|\Sigma| \geq 2$.

Lemma 5.2: [5] $L = \langle N \cup \{+\infty\}, min, +, +\infty, 0\rangle$ is a tropical semiring, Σ is an alphabet, then $Eq, Ineq, LocalIneq, LocalEq$ are decidable when $|\Sigma| = 1$.

In the following, we assume that L is commutative but not locally finite.

Theorem 5.3: If L is not locally finite, then there exists an injective homomorphism $\varphi : (N \cup \{+\infty\}) \to L$ defined by $\varphi(i) = a^i$, where $a \in L$ satisfies $a^i \neq a^j$ whenever $i \neq j$.

Proof: Since $a^i \neq a^j$ whenever $i \neq j$, then $\varphi(i \cdot j) = \varphi(i+j) = a^{i+j} = a^i \cdot a^j = \varphi(i) \cdot \varphi(j)$ for any $i, j \in N^\infty$.

Theorem 5.4: Eq is decidable in $L-Reg(\Sigma)$ for $|\Sigma| = 1$.

Corollary 5.5: $Eq, Ineq, LocalIneq, LocalEq$ is undecidable in $L-Reg(\Sigma)$ for $|\Sigma| \geq 2$.

Theorem 5.6: Let L be an $l-$monoid, and for any $a, b, c \in L, k \geq 2$ if $ab = ac \Rightarrow ab^k = ac^k$ is decidable, then for any $f, g \in L - Reg, f = g$ is decidable when $|\Sigma| = 1$.

To prove theorem 5.6 we need the following lemma.

Lemma 5.7: If $\mathscr{A}$ is an automata over $\Sigma = \{\sigma\}$, then $|\mathscr{A}|$ can be accepted by a $L - Seq$.

Proof: Let $\mathscr{A} = (Q, \{\sigma\}, \delta, q_0, F)$, so $|\mathscr{A}|(\theta) = \vee_{q_1, q_2 \cdots q_k} \delta(q_0, \sigma, q_1) \cdot \delta(q_1, \sigma, q_2) \cdot \cdots \cdot \delta(q_{k-1}, \sigma, q_k) \cdot F(q_k) = \delta(q_0, \sigma, q_{i_1}) \cdot \delta(q_{i_1}, \sigma, q_{i_2}) \cdot \cdots \cdot \delta(q_{i_{k-1}}, \sigma, q_{i_k}) \cdot F(q_{i_k})$ when $\theta = \sigma^k$, where $q_0 q_{i_1} \cdots q_{i_k}$ is a successful path of σ^k. Then we construct a $L - Seq$ $\mathscr{B} = (Q, \Sigma, \delta_1, q_0, F_1)$. Then classify Q into $k+1$ categories, in which $Z_0 = \{q_0\}, Z_j = \{q_j | \delta(q_{j-1}, \sigma, q_j) \neq 0\}, Z_{k+1} = \{q | F(q) \neq 0\}, 1 \leq j \leq k$, $\delta_1 : 2^Q \times \Sigma \times 2^Q \to L, F_1 : 2^Q \to L$. $\delta_1(Z_j, \sigma, Z_{j+1}) = \delta(q_{i_j}, \sigma, q_{i_{j+1}}), F_1 = F(q_{i_j})$, then we can show $|\mathscr{B}|(\theta) = |\mathscr{A}|(\theta)$.

If f can be accepted by an $L - Seq\mathscr{A}$, then there exists $a, b \in L, k, d \in N$, for any $m \in N, |\mathscr{A}|(\sigma^{d+km}) = ab^m$. Where $\mathscr{A} = (Q, \{\sigma\}, \delta, q_0, F), a = \delta(q_0, \sigma, q_1) \cdot \ldots \cdot \delta(q_{d-1}, \sigma, q_d) \cdot F(q_{t+1}), b = \delta(q_d, \sigma, q_{d+1}) \cdot \cdots \cdot \delta(q_t, \sigma, q_{t+1})$, $t = d + km, d + k = |Q|$. So, $\mathscr{A}$ must have the following form:

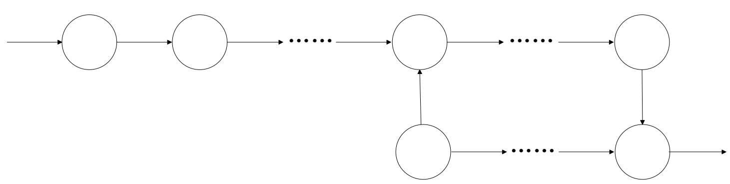

Fig. 5. The fuzzy automaton in Lemma 5.7

Now the proof process of Theorem 5.6 is as follows:

Proof: Assume $f, g \in L - Reg$, then f, g can be accepted by $L - Seq$ specially when $|\Sigma| = 1$. Note that $f(\sigma^{d+km}) = ab^m, g(\sigma^{d+km}) = ch^m$, hence $f = g$ iff $ab^m = ch^m$, i.e. $a = c, ab = ch, ab^2 = ch^2, \cdots, ab^k = ch^k$. Thus, $f = g$ is decidable when $ab = ac \Rightarrow ab^k = ac^k$ for any $k \geq 2$ is decidable.

Here we use $\prec$ to represent two kinds of relations of the fuzzy language $\leq, =$.

Theorem 5.8: L is locally finite, and $f, g \in L - Reg(\Sigma)$, then $f \prec g$ iff for any $r \in Im(f) \cup Im(g), f_r \prec g_r$.

Proof: We give the proof of the case$\prec = \leq$, the proof of the case$=$ is similar.

$\Rightarrow$ Since $f \prec g$, then $f(\theta) \leq g(\theta)$ for any $\theta \in \Sigma^*$, hence it is obviously holds for any $r \in Im(f) \cup Im(g), f_r \prec g_r$.

$\Leftarrow$ Because $f_r \subseteq g_r$, then $rf_r \subseteq rg_r$, which implies $rf_r \subseteq \cup_{r \in L} rg_r$, the last implies $\cup_{r \in L} rf_r \subseteq \cup_{r \in L} rg_r$. Besides, for any $f \in L - Reg, f = \vee_{r \in Im(f)} rf_r$ when L is locally finite, so we can get $f \prec g$.

For regular language L_1, L_2, it is decidable for $L_1 = L_2, L_1 \subseteq L_2$. And, $Im(f), Im(g)$ is finite when L is locally finite, So we can get the following corollary.

Corollary 5.9: If L is locally finite, then for any $f, g \in L - Reg(\Sigma)$, it is decidable for $f \prec g$.

Corollary 5.10: If f, g can be accepted by $L - DFA$, then it is decidable for $f \prec g$.

If L is locally finite, then for any $f, g \in L - Reg(\Sigma)$, there must be $L - DFA$ $\mathscr{A}_f, \mathscr{A}_g$ such that $|\mathscr{A}_f| = f, |\mathscr{A}_g| = g$. According to [5], we can let the two automata have a minimum number of states, so whether $f = g$ depends on whether $\mathscr{A}_f = \mathscr{A}_g$, and there is a algorithm to solve the latter question.

For constant function, a will present the lattice fuzzy regular language with constant$a$ for any input. We have the following results.

Corollary 5.11: If L is locally finite, $f \in L - Reg(\Sigma), a \in L$, then $f = a$ is decidable.

For a lattice fuzzy regular language f, if L is locally finite, then $f_a, f_{[a]}$ are regular languages for any $a \in L$. Then we have the following results.

Theorem 5.12: If L is locally finite, for any $f \in L - Reg(\Sigma), a \in L$, whether there exists $\theta \in \Sigma^*$ satisfies $a \prec f(\theta)$ is decidable.

Theorem 5.13: If L is commutative and locally finite, $\mathscr{A} = (Q, \Sigma, \delta, I, F)$ is an $L - NFA$, for any $\theta \in \Sigma^*, |\theta| < n, n = |Q|$, if $|\mathscr{A}|(\theta) = 0$, then $|\mathscr{A}| = 0$.

Proof: By hypothesis, if $|\theta| < n, |\mathscr{A}|(\theta) = 0$; If $|\theta| \geq n$, $|\mathscr{A}|(\theta) = \vee_{q_0, q \in Q} I(q_0) \cdot \delta^*(q_0, \theta, q) \cdot F(q)$. Assume $q_0 q_1 \cdots q_k$ is a successful path of $\theta = \sigma_1 \sigma_2 \cdots \sigma_k$ when $k \geq n$, so there exists $q_i, q_j \in Q$ satisfies $q_i = q_j$ when $i < j$. Let $x = \sigma_1 \cdots \sigma_i, y = \sigma_{i+1} \cdots \sigma_j, z = \sigma_{j+1} \cdots \sigma_n$, and $\theta_1 = xz, |\theta_1| < |\theta|$, then $q_0 \cdots q_i q_{j+1} \cdots q_k$ is a successful path for θ_1. If $|\theta_1| < n$, then $I(q_0) \cdot \delta^*(q_0, x, q_i) \cdot \delta^*(q_j, z, q_k) \cdot F(q_k) = 0$, which implies $I(q_0) \cdot \delta^*(q_0, x, q_i) \cdot \delta^*(q_i, y, q_j) \cdot \delta^*(q_j, z, q_k) \cdot F(q_k) = I(q_0) \cdot \delta^*(q_0, x, q_i) \cdot \delta^*(q_j, z, q_k) \cdot F(q_k) \cdot \delta^*(q_j, z, q_k) = 0$; If $|\theta_1| \geq n$, we can repeat the above steps to segment θ_1, because the length of θ_1 is finite, so $|\mathscr{A}| = 0$.

Corollary 5.14: If L is commutative and locally finite, then for any $f \in L - Reg, f(\theta) = 0$ is decidable.

VI. Conclusion

In this paper, the lattice-valued fuzzy automata are divided into five types, that is, deterministic, sequential, unambiguous, finitely ambiguous and infinitely ambiguous respectively. And we discuss the relationship between the fuzzy languages accepted by these fuzzy automata. We get the following conclusions: $\overline{L - DFA} \subseteq \overline{L - Seq} \subseteq \overline{L - NAmb} \subseteq \overline{L - FAmb} \subseteq L - Reg$. What's more, when the lattice is locally finite, the contains relation are completely equal; when the lattice is not locally finite, we can get $\overline{L - DFA} \subsetneqq \overline{L - Seq} \subsetneqq \overline{L - NAmb} \subsetneqq \overline{L - FAmb} \subsetneqq L - Reg$. In section 5, the decidability of lattice valued fuzzy language is discussed, the decidability is related to the locally finite monoid and the number of characters in the alphabet.

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