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Type-1 or Interval Type-2 Fuzzy Logic Systems - On the Relationship of the Amount of Uncertainty and FOU Size

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***Abstract*— A recurring theme in research employing type-2 fuzzy sets is the question of how much uncertainty in a given context warrants the application of type-2 fuzzy sets and systems over their type-1 counterparts. In this paper we provide insight into this challenging question through a detailed investigation into the ability of both types of Fuzzy Logic Systems (FLSs) to capture and model different levels of uncertainty/noise through varying the size of the Footprint Of Uncertainty (FOU) of the underlying fuzzy sets from type-1 fuzzy sets to very "wide" interval type-2 fuzzy sets. By applying the study in the well-controlled context of chaotic time-series prediction, we show how, as uncertainty/noise increases, type-2 FLSs with fuzzy sets with FOUs of increasing size become more and more viable. While the work in this paper is focused on a specific application, we believe it provides crucial insight into the challenging question of the viability of interval type-2 over type-1 FLSs.**

Keywords—fuzzy set; interval type-2; uncertainty; footprint of uncertainty; quantification of uncertainty; noise

I. INTRODUCTION

Fuzzy set theory was first introduced by Zadeh in 1965[1]. Fuzzy sets and systems have evolved for more than 45 years and have been accepted as a methodology for building systems that can deliver satisfactory performance in the face of uncertainty and imprecision [2]. Hence, Fuzzy Logic Systems (FLSs) have been successfully implemented in many real world applications, including modelling and control [3],[4], forecasting of time series [5]-[7] and data mining [8],[9].

Type-1 FLSs (T1 FLSs) are the most known and widely used type of FLS. In spite of this, recent years have shown a significant increase in research toward more complex forms of fuzzy logic such as interval type-2 fuzzy logic systems (IT2 FLSs) [10],[11] and more recently, general type-2 FLSs (T2 FLSs) [2],[12]-[20]. This transition was motivated by the realization that type-1 fuzzy sets (T1 FSs) can only handle a limited level of uncertainty whereas real-world applications are often faced with multiple sources and high levels of uncertainty [21]. In 1975, Zadeh [22] recognized this potential limitation and introduced the concept of (general) type-2 fuzzy sets which are an extension to T1 FSs. As more complex models, T2 FSs are considered to be potentially better suitable for modelling uncertainty. The additional complexity arises from the inclusion of a Footprint Of Uncertainty (FOU) and a third dimension, offering extra degrees of freedom to T2 FSs in comparison to T1 FSs [21],[23]. It is the same complexity however, which makes FLSs which employ T2 FSs computationally very demanding in comparison to those that employ T1 FSs.

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The computational complexities of using T2 FLSs led to the introduction of the simplified IT2 FLSs which today are the most commonly used kind of T2 FLS. IT2 FLSs employ IT2 FSs, which are a special case of a general T2 FSs where all the secondary membership grades are equal to one. Many researchers argue in favour of IT2 FLSs over T1 FLSs because of their potential to model and mitigate the effects of uncertainty [24]-[26]. Table I provides a sample of some articles which directly cite the presence of "*large amounts of uncertainty*" in their given applications as the reason for employing type-2 FSs. From this, it is important to note that many of the papers considering T2 FSs in their study were expecting to design a system that will perform well in the face of "*high levels of uncertainty*", without however quantifying what "*high*" means in general or in the case of their specific application.

Thus, while a main issue in the application of FLSs is the estimation of parameters such as the type of fuzzy sets and their parameters as well as the number of rules, an even more fundamental question is generally whether T1 or T2 FSs should be used. Although there is a record of experimental evidence showing improvements in terms of uncertainty handling of IT2 over their T1 counterparts [30],[31], no systematic way of determining the potential advantage of employing T2 FSs over T1 FSs has been developed.

In this paper, we describe an application driven investigation into the relationship between the FOU size of FSs and the level of uncertainty in an application by using Time Series Prediction (TSP) as a well-defined sample application. We design T1 FLSs for TSP and proceed by generating different FLSs with continually increasing FOU sizes over a number of steps. In parallel, we inject increasing levels of noise to provide an a priori known and well defined/understood source of uncertainty. Thus, the time series prediction application provides a platform to explore the behaviour of the FLSs with different FOU sizes in respect to different levels of noise/uncertainty. The main objective of this work is not to achieve an optimal performance in the prediction, but to shed light on the appropriate size of the FOUs for given levels of uncertainty/noise. While the results are clearly application dependent and not generalizable, we feel they provide significant insight into the relationship of FOU size and levels/amount of uncertainty in a given setting.

In summary, the main contributions of this paper can be

TABLE I SAMPLE ARTICLES CITING TO THE PRESENCE OF "*LARGE AMOUNTS OF UNCERTAINTY*" IN THEIR APPLICATIONS AS THE REASON FOR EMPLOYING TYPE-2 FSs

Author(s)	Ref. no.	Direct Quote
Lin et al.	[27]	*"Due to the rule uncertainties and the training data corrupted by noise, the circumstances are* ***too uncertain*** *to determine exact membership grades. A new direct adaptive interval type-2 fuzzy controller is developed to handle such uncertainties for a class of multivariable nonlinear dynamical systems involving external disturbances"*
Wagner and Hagras	[28]	*"The type-2 Fuzzy Logic Controller (FLC) has started to emerge as a promising control mechanism for autonomous mobile robots navigating in real world environments. This is because such robots need control mechanisms such as type-2 FLCs which can* ***handle the large amounts of uncertainties*** *present in real world environments".*
Baklouti and Alimi	[29]	*"Motion planning of mobile robots in unknown and dynamic environments is faced with a* ***large amount of uncertainties****."*

listed as: (1) Systematic design and analysis of the performance and uncertainty capturing ability of IT2 FSs. (2) Introspection into how the level of uncertainty relates to the size of the FOUs of IT2 FSs or conversely, what size of FOU works best for a given level of uncertainty in the given application. (3) The proposed analytical method provides insight which can support the selection of the FOU size at design time in response to known quantifications of uncertainty.

This paper is organized as follows. In Section II a review of type-2 FLSs, rules creation and time series prediction is provided. In Section III, the proposed method of FOU construction is described. Section IV discusses the FLSs' design and evaluation. In Section V, we apply our method to the Mackey-Glass time-series prediction and the results are presented. Section VI provides analysis and discussion. Finally, we provide some conclusions and outline future work in Section VII.

II. BACKGROUND

This section provides a review of the underlying areas built on in this paper, namely type-2 fuzzy systems, automatic generation of rules for FLSs from data and finally time series prediction as an application area.

A. Type-2 Fuzzy Logic Systems

A type-2 FLS consists of five components: fuzzifier, rule base, inference engine, type reducer, and defuzzifier. It is very similar in structure to a type-1 FLS with the only difference being the introduction of a type-reducer component [21]. A type-2 FLS operates on type-2 fuzzy sets, which are used to represent the inputs and outputs of the FLS. An FLS that uses at least one T2 FS (IT2 FS) is referred to as a T2 FLS (IT2 FLS).

In a type-2 FLS, crisp inputs are first fuzzified, usually into input T2 FSs. These activate the inference engine and the rule base to produce output type-2 fuzzy sets. They are then processed by the type-reducer, which combines the output sets and performs a centroid calculation, leading to type-1 fuzzy sets known as type-reduced set(s) [21]. The defuzzifier can then defuzzify the type-reduced type-1 fuzzy outputs to produce crisp outputs. Further detail on T2 FLSs can be found for example in [21].

B. Rule Creation for FLSs

As mentioned, a core part of designing FLSs beyond the specification of the types and parameters of the FSs employed is the specification of a rule base. A number of different approaches are commonly taken to achieve this, including the design of the rule base by an expert, the learning of the rule base over time (e.g., using a genetic algorithm) or the learning of the rule base from existing data. In this paper we apply the latter approach by using the Wang-Mendel method (WM-method) [7] to generate a fuzzy rule base from a number of input-output data pairs. The WM-method generates a rule with an associated weight for each training data pair; then the resulting rule set is truncated by removing the conflicting rules and by using the calculated weights in order to obtain the final rule base. For a more detailed view of the approach, we refer the reader to [7] and [21].

In the current paper, T1 FSs are generated before the WM-method is applied in order to produce a rule base. The same rule base is employed for all FLSs in order to enable the comparison of all FLSs with a sole focus on their FSs (rather than differences in the rule bases). As further detailed in Section III.A, we acknowledge that this approach does not guarantee the best rule base for each FLS, however, as the aim of the paper is the comparison based on FSs rather than achieving best performance per se, we believe this approach is suitable to support the aim.

C. Time Series Prediction

Time series prediction is an important application that is frequently addressed in the literature, e.g., [21], [32]-[34]. It is valuable in many research areas such as weather forecasting, signal processing, economics and production control.

Consider a time series $x(k)$, where $k = \{1,2,3, \dots, N\}$. We have p known data points of $x(k)$ to predict the future value of x such that $k - p > 0$. For example, we are given

$$x(k-p+1), x(k-p+2), x(k-p+3), \dots, x(k)$$

past measurements of $x(k)$ to predict the future value of x, $x(k+1)$ in case of considering a single stage prediction for x. Further, if these measurements are not perfect, e.g., contain noise, we refer to a given value of the time series $x(k)$ as $s(k)$, where $s(k) = x(k) + n(k)$, and $n(k)$ is the measurement error (noise) [21].

The Mackey-Glass time series is a chaotic time series proposed in [35] containing a first-order differential-delay equation to model a physiological systems (Equation (4b) in [35]). It is obtained from the non-linear equation:

$$\frac{dx(t)}{dt} = \frac{a * x(t-\tau)}{1 + x^n(t-\tau)} - b * x(t), \tag{1}$$

where a, b and n are constant real numbers, t is the current time and τ is the difference between the current time and the previous time $(t-\tau)$. For $\tau \leq 17$, the system is known to exhibit a deterministic/periodic behaviour which turns chaotic with τ >17. To obtain simulation data, (1) can be discretised using Euler's method [36] with a step size equal 1.0 and the initial values of $x(t)$ for all values of $t \leq \tau$ are set randomly. To make the prediction more challenging, noise can be added to the time series. The level of noise is commonly measured by the signal-to-noise ratio (SNR) where a high SNR refers to a clear signal (low noise) and a low SNR refers to a noisy signal (high amounts of noise). In this paper, we will use single-stage prediction for the Mackey-Glass chaotic time series. Different levels of uniform noise will be introduced into a testing set of existing samples of the time series, resulting in a series of testing sets (for different noise levels). Each testing data set is used in order to test and evaluate FLS performance.

III. From Type-1 to Interval Type-2 Fuzzy Sets and Systems

A. Overview

Commonly, there are two different approaches to designing IT2 FLSs [21]: a partially dependent approach and a totally independent approach. The former approach starts with the design of a T1 FLS the parameters of which are then used as a basis for the design of the IT2 FLSs. An advantage of this approach is the potentially faster design of the IT2 FLS (in particular when a learning approach is employed) as well as a good comparability between the T1 and IT2 FLSs. The latter approach is used to design IT2 FLS directly without relying on an intermediate T1 FLS and thus avoids the potential shortcoming of the former approach that the "best" IT2 FLS may in fact be very different from the "best" T1 FLS, i.e. using a T1 FLS may in fact be detrimental to the construction of a well-performing IT2 FLS.

In this paper, we adopt the partially dependent approach as the preservation of the basic structure (number of membership functions, rule base) is essential to our approach of comparing a series of FLSs that range from a T1 to IT2 FLSs with increasing FOUs. We acknowledge the risk that the best performance for the IT2 FLSs may be achieved without relying on a T1 implementation but we feel that for the proposed investigation (which is not about finding the best prediction performance), the use of the partially dependent approach is warranted and suitable.

Following a partially dependent approach, the main question after a type-1 system has been designed is the transition from the type-1 fuzzy membership functions to interval type-2 membership functions, in other words, the introduction of an FOU. In the following subsection, we discuss an approach to relate an FOU's size to the amount of uncertainty in order to establish a direct relationship between FOU size and amount of uncertainty present.

B. FOU Construction Approach

There exists a number of common IT2 FSs in the literature, i.e., triangular, Gaussian, trapezoidal, sigmoidal, pi-shaped, etc. In most of the literature two types of IT2 FSs are used: fuzzy sets with uncertain mean (centre for triangular case) and with uncertain standard deviation (spread for triangular case). In this paper, largely because of space limitations, only triangular MFs are considered. In order to study the uncertainty modelling of IT2 FSs, an uncertainty indicator is introduced here. The uncertainty indicator is intended to show the amount of uncertainty captured by the FSs and modelled by their FOUs. Consider an IT2 FS $\tilde{A}$ described by its FOU using the upper and the lower membership functions $\bar{\mu}_{\tilde{A}}(x)$ and $\underline{\mu}_{\tilde{A}}(x)$ [21] as:

$$\mathrm{FOU}(\tilde{A}) = \bigcup_{\forall x \in X} \left(\bar{\mu}_{\tilde{A}}(x), \underline{\mu}_{\tilde{A}}(x) \right) \tag{2}$$

The Uncertainty Indicator (UI), $U_{\tilde{A}}(x)$ associated with the input vector x by fuzzy set $\tilde{A}$ can be expressed as:

$$U_{\tilde{A}}(x) = \bar{\mu}_{\tilde{A}}(x) - \underline{\mu}_{\tilde{A}}(x) \tag{3}$$

Fig. 1(a) and Fig. 1(c) show IT2 FSs with uncertain spread and uncertain centre, respectively. The Uncertainty Indicator (UI), $U_{\tilde{A}}(x)$for IT2 FSs with uncertain spread is plotted in Fig 1(b) and for IT2 FS with uncertain centre is shown in Fig. 1(d). From both cases, it can be seen that the UI value is not constant over the respective supports of the lower MFs ($\underline{\mu}_{\tilde{A}}(x) > 0$).This in turn highlights that the FOU is not uniform in this region of the FS. With an assumption that the noise/uncertainty is uniform, an FOU construction method should give rise to an equal amount of uncertainty in membership at least within the core of the FS (which we consider to be delineated by the lower MF).

In an effort to obtain an IT2 FSs with a uniform FOU all over the core of the fuzzy set, in this paper, we propose an FOU construction method based on a fixed parameter that is used to create an FOU of a given size around a principal (type-1) MF. As an input, we employ T1 FSs (in our case designed based on expert knowledge).

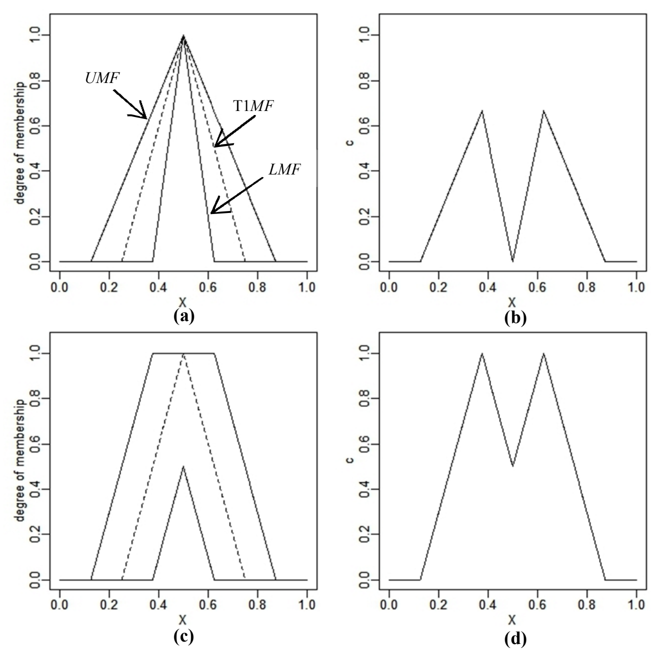

Fig. 1. Interval type-2 fuzzy sets and their Uncertainty Indicators (UI) (a) IT2 FS with uncertain spread and (b) its Uncertainty Indicator , (c) IT2 FS with uncertain centre and (d) its Uncertainty Indicator. The dotted lines in both (a) and (c) are T1 FS.

Then, we start the design of the interval type-2 fuzzy sets by including the FOU size parameter to form the IT2 FSs of the system. In order to create the IT2 version of each initial T1 set, the FOU size is specified using the parameter $c \in [0,1]$. Note that $c = 0$ results in a type-1 FS with the original membership function while $c = 1$ results in an IT2 set with a very wide FOU (as detailed further below).

In order to create the IT2 FSs based on the uncertainty parameter c and the T1 MF, we employ (4) and (5) shown below to create the resulting upper and lower MFs respectively. Note that, the minimum operation in (4) and the maximum operation in (5) ensure that the values of both upper and lower membership functions are bounded in the interval [0, 1]. In (5), note that the minimum operation prevents the lower membership function not exceeding the value of 1.0-c. Both (4) and (5) have been designed to ensure that the resulting FOU is uniform within the core (i.e. the support of the lower MF) of the IT2 FSs. The upper MF, $\bar{\mu}_{\tilde{A}}(x)$ and the lower MF, $\underline{\mu}_{\tilde{A}}(x)$ of IT2 FS $\tilde{A}$ can then be obtained as follows:

$$\bar{\mu}_{\tilde{A}}(x) = \min\left[\mu_A(x) + \frac{c}{2}, 1.0\right] \tag{4}$$

$$\underline{\mu}_{\tilde{A}}(x) = \min\left[\max\left[\mu_A(x) - \frac{c}{2}, 0\right], 1.0 - c\right] \tag{5}$$

An illustrative example is depicted in Fig. 2 for the case of triangular membership functions designed with the FOU sizes parameter c =0.5 and c = 0.80 using equation (4, 5). From the T1 FS shown in Fig. 2(a), two IT2 FSs are created by including the pre-specified FOU size parameters c = 0.5 and c = 0.8 in Fig. 2(b) and Fig. 2(c) respectively. The upper and lower MFs are obtained using equations (4) and (5). By using equation (3), the UI for both IT2 FSs are obtained and are depicted in Fig. 2(d) and 2(e). From both cases, it can be seen that, the UI value is constant within the supports of the lower MFs ($\underline{\mu}_{\tilde{A}}(x) > 0$) with a value equal to c (note that, in Fig. 2(d) c= 0.5 and in Fig. 2(e) c= 0.8), hence, the value of the chosen FOU sizes. This result makes it clear that the FOU is uniform in this region of the FS.

After considering the transition process from type-1 to type-2 fuzzy sets, we proceed to the FLSs' design and evaluation process .

IV. The FLSs' Design and Evaluation Process

This section describes the initial design of the T1 FLS for a given application and its subsequent transformation to one or more IT2 FLSs. Specifically, the design of multiple IT2 FLSs by creating different size FOUs with respect to different noise levels is explained. The design methodology can be summarized in four steps as follows:

Step 1: Generate training and testing data from the system under study (e.g., the time series).The training data is kept noise free, while later testing data will be injected with different noise levels. The training data is used to train the system (generating rules) under "ideal conditions". Whereas, the testing data is corrupted by noise (as is expected in real world situations) to test the performance of a designed system in the face of the given level of noise (uncertainty).

Step 2: Design an initial T1 FLS. First, T1 FSs are created, either by an expert (as is the case in our case as further detailed in Section V) or through an automatic method (e.g., a genetic algorithm).

Second, the training data is used to generate the rules using for example the WM-method explained in Section II. The created rule base will be used for all subsequent FLSs.

Step 3: Extend the T1 FLS into an IT2 FLS using the partially dependent design introduced in Section II. The FOU size $c \in [0,1]$ is discretized to a set of k values, where k also defines the number of noise/uncertainty levels that will be investigated. The upper and lower MFs will be constructed using equations (4) and (5). The result is a number of k FLSs that each one of them needs to be tested over a number of k noise levels.

Step 4: Performance Testing and Evaluation.

After finishing the design of each of the IT2 FLS with the chosen FOU size parameter, we test its performance using the pre-generated testing data at each of the k noise/uncertainty levels. At each noise level, the performance testing is repeated a number of times (in our case: 30 times) to account for the random character of the noise injection.

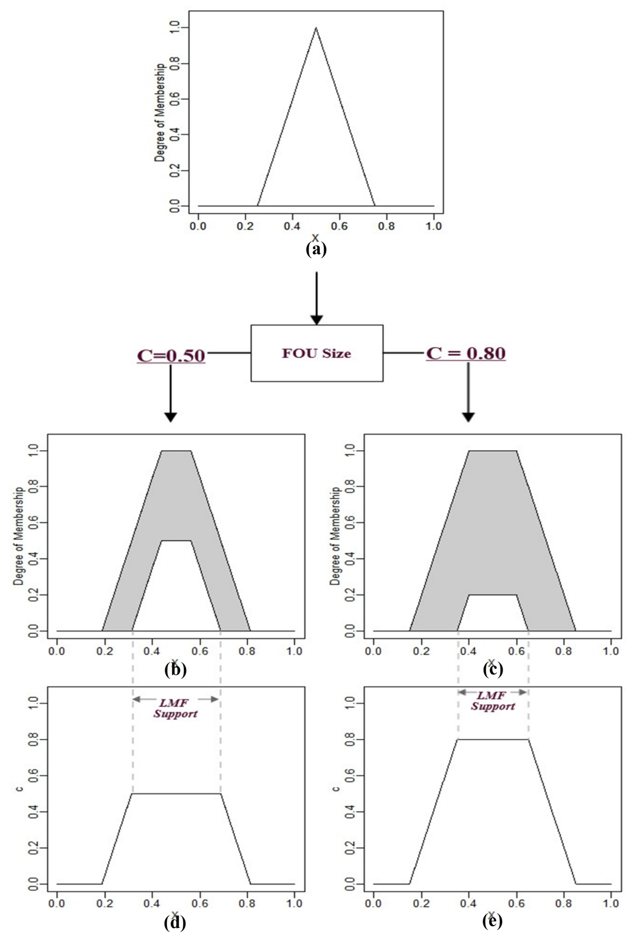

Fig. 2. Illustration of the uniform design of IT2 membership functions. (a) Initial T1 FS, (b) IT2 obtained using FOU size parameter c= 0.5 and (c) IT2 FS obtained using FOU size parameter c= 0.8. (d) The Uncertainty Indicator (UI) obtained from IT2 FS with FOU size parameter c= 0.5 and (e) UI obtained from IT2 FS with FOU size parameter c= 0.8

The performance of the design(s) is evaluated, for example using the root mean-squared error (RMSE) as shown in (6):

$$RMSE = \sqrt{\frac{1}{N}\sum_{i=1}^{N}[Y_i - \tilde{Y}_i]^2} \tag{6}$$

where N is the total number of data points (testing data), Y is the actual output (from the testing data) and $\tilde{Y}$ is the crisp output of the FLS (predicted). The average of the RMSEs is then calculated of all of the iterations for each FLS at each noise/uncertainty level.

The complete process is illustrated by flowchart in Fig. 3.

In the following section, we provide some results and analysis based on an initial application of the described process in the context of M-G time series prediction.

V. Experiments and Results

In order to illustrate the methodology proposed in Section IV, we conduct some experiments for the forecasting of the Mackey-Glass time series [35] described in Section II.

In summary, we generate a data set (both training and testing data) from the Mackey-Glass time series. Then, we design type-1 FSs (based on expert insight) and create the rule base using the Wang-Mendel [7] approach using the training data set. Next, we start the design of the interval type-2 fuzzy sets by including the FOU size parameter as described earlier to form the IT2 MFs of the system. The actual number of FSs and the rules are maintained from the T1 system. Different levels of noise as a source of uncertainty are added to the testing data. Different variations of the designed FLSs are created by introducing different size FOUs. Finally, the performance of the IT2 FLSs is compared at the given uncertainty/noise levels.

The actual detail of the steps is given below.

Step 1: Noise-free time series is generated using (1) with the following parameters: a = 0.2, b = 0.1, τ = 30 and n = 10. Euler's method is used to obtain the values of $x(t)$ at each time point with a time step of 1.0 and the initial condition $x(0)$ = 0.1. Based on this data we proceed to design a four-input, one-output, T1 FLS for single-stage prediction of the Mackey-Glass time series. Specifically, we extract 700 input-output data pairs as described in Section II.C. The first 500 pairs (the training dataset) were used for training the FLS (generating the rules), while the remaining 200 pairs (the data testing set) were used for testing the system. Training was performed with the 500 input-output pairs from $x(1001)$ to $x(1504)$ and testing was done with 200 input-output pairs from $x(1505)$ to $x(1708)$. The testing data is corrupted with zero-mean uniform noise when different levels of signal-to-noise ratios (SNRs) are used. The number of noise/uncertainty levels, k chosen to be equal 11 levels. This number is discretized from 0dB to 20dB with increments of 2. In Fig. 4, the training data and four samples of testing data using four SNRs levels are shown. One noise free (NF) data set is also used for testing the designed FLSs.

Step 2: We used triangular MFs for both the inputs and the output. The number of membership functions assigned to each input and output of the FLS was chosen arbitrarily to be 7. While, a higher number of MFs would enable better performance, 7 proved a good compromise for readability (in figures) and reasonable performance – in particular as optimal prediction performance is not a primary aim in this paper. First, we defined the fuzzy sets to evenly cover the input and output spaces. The type-1 membership functions that are used for the inputs and the outputs are shown in Fig. 5. The MFs are labelled using numbers (e.g. F11 represents FS 1 of input 1 and for the output F1 represent FS 1). In our simulation, the T1 FLS uses singleton fuzzification, product t-norm, product inference and centroid defuzzification. Then, we apply the Wang-Mendel (WM) method (see Section II.B) in order to generate the rules from the given input-output pairs (training data). The resulting rules are used for all FLSs used in our experiments in order to enable a comparison which focusses on the FSs.

Step 3: Following Section IV, we extend the T1 FLS into an IT2 FLS using the partially dependent design. The construction of an FOU around the T1 FSs is accomplished as follows. First, the FOU size parameter $c \in [0,1]$ is discretized to a set of k=11 values. We use eleven different values for each FOU size, starting at 0 and increasing to a maximum of 1.0 in increments of 0.1. If the FOU size parameter equals 0.0, the interval type-2 fuzzy sets reduce to the original type-1 fuzzy sets, whereas in case of using 1.0 the interval type-2 fuzzy set reach the maximum amount of their width. In our application, we design k =11 IT2 FLSs where each system was designed using specific IT2 FSs with the given FOU size.

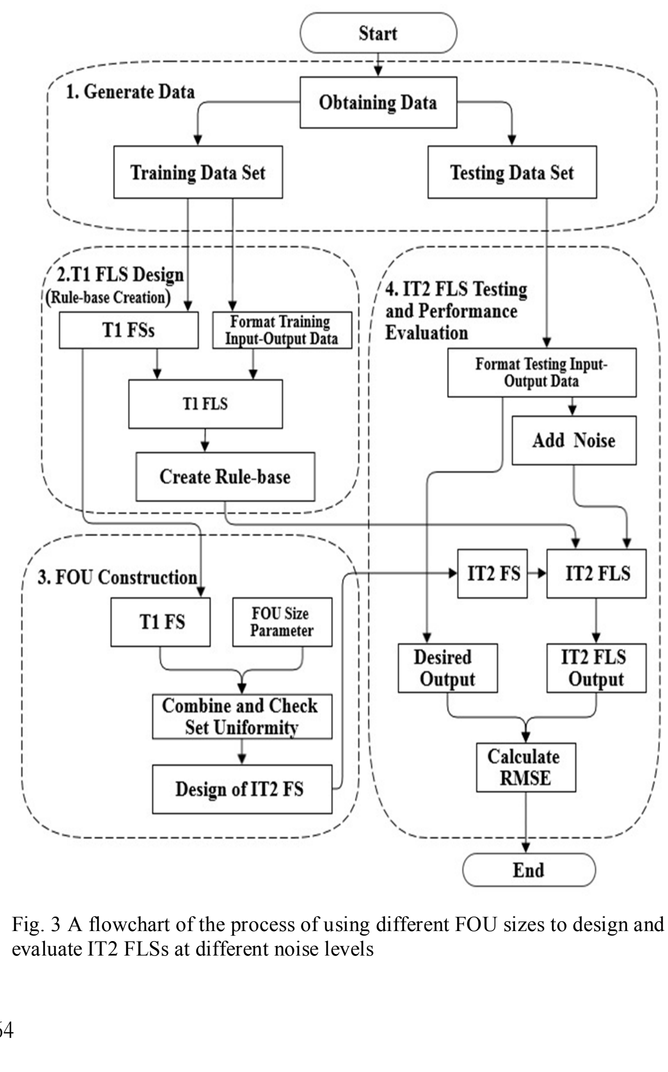

Fig. 3 A flowchart of the process of using different FOU sizes to design and evaluate IT2 FLSs at different noise levels

To construct the upper and lower membership functions of the interval type-2 fuzzy sets, we use the method detailed in Section IV and apply equations (4, 5) by combining the T1 FSs with the chosen FOU size represented by parameter c. An example of FOU construction for IT2 FSs of two IT2 FLSs using different FOU sizes (0.30, and 0.80) is shown in Fig. 6.

Step 4: After finishing the design of the IT2 FLS with the chosen FOU size, the testing data is used to test the performance of the individual IT2 FLSs when faced with the different uncertainty/noise levels. The k=11 FLSs are tested on 200 points from $x(1505)$ to $x(1708)$ corrupted by zero-mean uniform noise when 11 SNRs values are used (see Step1). In addition to the eleven different signal-to-noise ratios (SNRs), one noise free (NF) data set is also used for testing the designed FLSs and each test is repeated 30 times.

The performances of all the designs were evaluated using RMSE based equation shown in (7). I.e.,

$$RMSE = \sqrt{\frac{1}{200}\sum_{k=1508}^{1707}[x(k+1) - f(x^{(k)})]^2} \qquad (7)$$

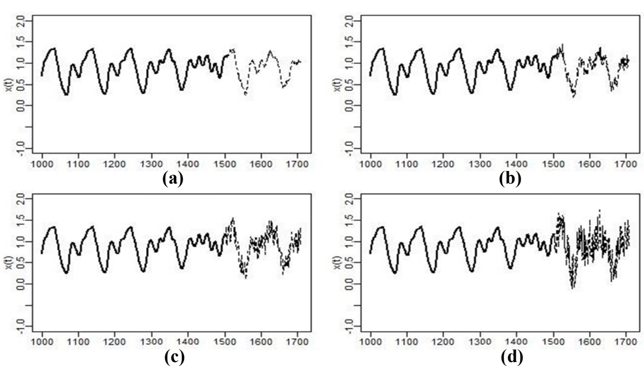

Fig. 4. Noise free training data (solid lines) and four samples of noisy testing data (dashed lines) at SNR levels (a) 20, (b) 10, (c) 6 and (d) 0. Training is performed with the 500 input-output pairs in $x(1001)$, $x(1002)$,..., $x(1504)$ and testing is done with 200 input-output pairs $x(1505)$, $x(1506)$,..., $x(1708)$.

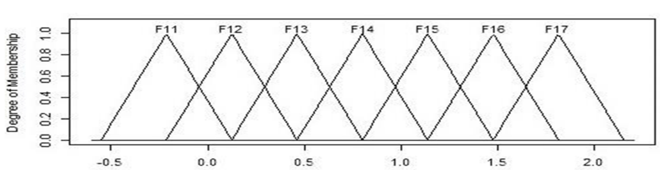

Fig. 5. Type-1 Membership functions for the inputs and the output.

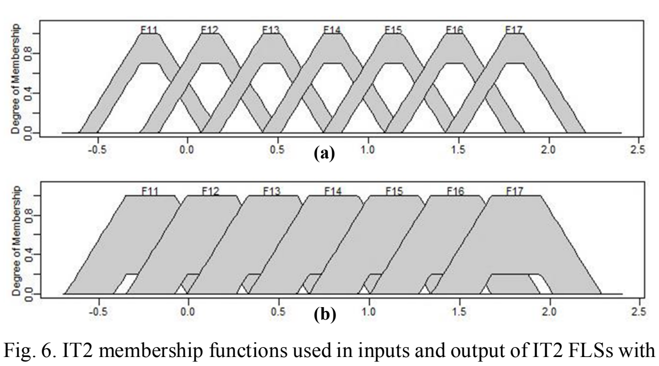

Fig. 6. IT2 membership functions used in inputs and output of IT2 FLSs with different FOU size c. (a) c=0.30 , and (b) c = 0.80 .

where, N = 200 testing points, $x(k+1)$ is the output of the testing data and $f(x^{(k)})$ where, $x^{(k)} = [x(k-3), x(k-2), x(k-1), x(k)]^T$ is the crisp output of the FLS. The RMSE results are averaged over 30 runs and are depicted in Table II. In Table II, the average RMSE for 30 runs is shown. Each column represents an IT2 FLS design with a given FOU size parameter and the rows show the average RMSE value at the different SNR values. The shaded values are the minimum of each row representing the best FOU at a particular SNR level.

VI. Analysis and Discussion

The presented experiments investigate the relationships between FOU sizes and the uncertainty/noise levels applied to the Mackey-Glass time series prediction. From Table II, a direct relationship between the FOU size, the SNR and the performance can be seen. As the uncertainty/noise level increases (SNR decreases), the FOU size of the FLS with best performance (i.e. giving the minimum RMSE value) increases.

The first column of Table II contains the performance results of the IT2 FLS designed using FOU size c = 0.0 which reduces to the original T1 FLS. From this result, it is clear that IT2 FLSs outperform their counterpart T1 FLS in all cases presented in these experiments even in the case of noise free testing data. We believe that the reason for this is the use of a small number of MFs (7 MFs) leading to such a limited number of parameters that the type-1 FSs cannot account for the complexity in the time series. This agrees with the finding by Wu and Tan [37] showing that the extra degrees of freedom provided by the footprint of uncertainty enables a IT2 FLSs to outperform T1 FLSs with the same number of MFs. If we were to add more labels (MFs) to the design of the T1 FLSs this would increase the chance of the T1 FLS to outperform the IT2 FLSs in noise-free or very low noise conditions. Mendel [21] and Hagras [38] have mentioned this case (from another perspective) when using T2 FSs; this will reduce the rule base of the FLS where FOU make it possible to cover the same range as T1 FSs with a smaller number of MFs.

This conjecture is supported by the performance analysis of the systems shown in Table II (e.g. column 3) where the system starts to perform well at NF and two SNRs (20 and 18 dBs) but we will investigate it further as part of a future publication.

The next transition is shown at the next three SNRs (16, 14, and 12 dBs) levels for the same FLS (FOU size 0.30). As the noise level increases (moving down in the table), the best results occur as the FOU size increases (moving to the right in the table). However, performance degradation is recorded at higher levels of noise (starting from 4 dBs) and this is clearly shown in Fig. 8 where the RMSE starts to exceed 0.2 in the first 2 FLSs (i.e. at FOU sizes, 0 and 0.1) and then decreases again until reaching its minimum at c =0.80. In spite of this, and with expanding the FOU size, the performance becomes better until some higher noise level (e.g. 6dBs). To explain this observation, from Fig.7, the output results of three different IT FLS examples (Fig. 7(a)-7(c)) show a degradation of performance as the level of noise increases. In Fig. 7(a), the FOU size parameter was set to c =0.3. This IT2 FLS, tested over 3 different SNRs: Noise Free (NF), 10 and 0 dBs). At the noise level (0dB), the system delivers lower performance than at (10dB). (Note, 0dB is the highest noise level possible). In comparison, Fig. 7(b) (FOU size *c*=0.5) shows an improvement in the performance at noise level (0dB) whereas, Fig.7(c), where c=0.80, shows much better performance at noise level (0dB) comparing to the other two FLSs (Fig.7 (a) and Fig.7 (b)).

In an effort to analyse the general behaviour of the results, we used the data from Table II and have visualised it as bar charts in Fig. 8, representing the average RMSE values of the 11 IT2 FLS (11 FOU sizes) over different noise levels (NF, 20, 18, 16,…, 0).

Thus, Fig. 8 illustrates RMSE values of the group of 11 IT2 FLSs over different noise levels starting from noise free data (NF) and ending with the highest noise level (0dB). The eleven chosen FOU sizes in these experiments appear in the graph at each noise level from left to right starting from 0 up to 1.0 with different shades of grey. From Fig. 8, it is clear that there is a direct relationship between the FOU size and noise level in relation to achieved performance. The result from this relationship may lead to a better choice of the FOU size and hence the FLS in this application and provides general insight to the selection of FOU size (and the fundamental choice between T1 or T2 FSs) in applications where an assessment of the noise level (SNR) can be made or is available.

VII. Conclusion

In this paper, we have made an initial step to investigate the challenging question of “When are T2 FLSs viable in comparison to T1 FLSs?” and in particular, how much uncertainty warrants the use of T2 FLSs and exactly how “wide” should the FOUs of the respective T2 FSs be. Incidentally, we have highlighted that the choice between T1 FLS and IT2 FLS is not a binary one but a matter of degree, i.e. from no FOU to very wide FOUs. In other words, as the amount of uncertainty increases, the FOU of the FSs grows from initially T1 FSs to wider and wider IT2 FSs.

We presented a method for designing interval T2 FSs so that their FOUs can capture the faced levels of noise/uncertainty. Using the proposed method, we investigated the relationship between the FOU size of T2 FSs and the level of the presented uncertainty. Initially, we designed T1 FLSs and from there we moved to the design of IT2 FLSs by incorporating a certain amount/size of FOU to the fuzzy sets. We demonstrated and analyzed the performance of the resulting FLS in the context of Mackey Glass chaotic time-series prediction. We have found a direct relationship between the FOU sizes of the FSs and the noise levels. As the noise level increases, the FOU that gives the minimum RMSE value increases as well. While the performance analysis of the systems shows an expected degradation due to a large/excessive injection of noise to the data, with expanding FOU size, the performance improves for cases with higher noise level. From this, a strong relationship between the FLSs performance and the FOU size is apparent.

It has been observed that the IT2 FLSs outperform their counterpart T1 FLS in all cases presented in the experiments even in the case of noise free data – an effect which we attribute to the very limited number of MFs/parameters in the given experiment.

In future work, we will focus on further analyzing the relationship of increasing SNR levels and FOU size in regards to performance, in particular by considering different applications and performing a more detailed statistical analysis of the results. Further, more effective uncertainty identification and capturing methods for designing IT2 FLSs will be considered.

Table II The average RMSE values (for 30 test runs) of each FLS designed with different FOU sizes at different noise levels

	IT2 FS → Sample											
Parameter *c* (FOU size)		**0**	**0.10**	**0.20**	**0.30**	**0.40**	**0.50**	**0.60**	**0.70**	**0.80**	**0.90**	**1.0**
SNR (dBs)	**NF***	0.03032	0.02973	**0.02935**	0.03101	0.03403	0.03552	0.03514	0.03971	0.05634	0.08190	0.10808
	20	0.04148	0.04017	**0.03922**	0.03988	0.04208	0.04314	0.04357	0.04837	0.06365	0.08708	0.10913
	18	0.04683	0.04656	**0.04519**	0.04520	0.04666	0.04732	0.04800	0.05425	0.06975	0.09207	0.11125
	16	0.05415	0.05329	0.05168	**0.05129**	0.05239	0.05322	0.05409	0.05969	0.07382	0.09412	0.11222
	14	0.06391	0.06271	0.06109	**0.06041**	0.06114	0.06185	0.06312	0.06844	0.08054	0.09887	0.11601
	12	0.07673	0.07603	0.07425	**0.07302**	0.07303	0.07377	0.07551	0.08000	0.09075	0.10597	0.12076
	10	0.09319	0.09292	0.09103	0.08954	**0.08934**	0.08977	0.09130	0.09505	0.10296	0.11588	0.12915
	8	0.11422	0.11434	0.11238	0.11085	0.10984	**0.10959**	0.10984	0.11132	0.11665	0.12748	0.13964
	6	0.14788	0.14337	0.13869	0.13461	0.13305	0.13103	0.12981	**0.12905**	0.13065	0.13828	0.14848
	4	0.22032	0.21841	0.19130	0.17781	0.16168	0.15487	0.15128	0.14904	**0.14808**	0.15265	0.15915
	2	0.33157	0.31884	0.28397	0.25829	0.23738	0.21375	0.19765	0.18431	0.17378	**0.17016**	0.17448
	0	0.49397	0.44125	0.40718	0.37677	0.34288	0.31175	0.28386	0.25518	0.23685	0.22305	**0.21278**

*NF Noise Free Data

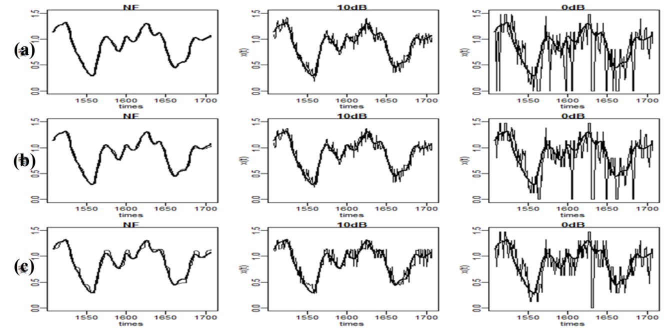

Fig. 7. The result of Mackey-Glass time-series prediction using IT2 FLS at different SNR levels (Noise free NF (left), 10 (middle) and 0 dB (right)). The thick solid line indicates the true time series; the thin line indicates interval type-2 crisp output. (a). the result of using FOU size c= 0.30, (b) c=0.50 and (c) c=0.80.

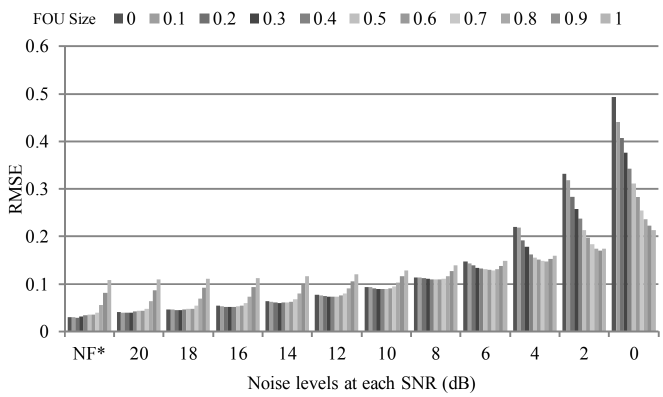

Fig. 8. RMSE value of 21 FLSs using different noise levels at different SNR values of testing data.

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