2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)
July 6-11, 2014, Beijing, China

Can Indices of Ecological Evenness Be Used to Measure Consensus?

Gleb Beliakov, Simon James
School of Information Technology
Deakin University
Burwood, Victoria 3125
Australia
Email: gleb@deakin.edu.au, sjames@deakin.edu.au

Dale Nimmo
School of Life and Environmental Sciences
Deakin University
Burwood, Victoria 3125
Australia
Email: dale.nimmo@deakin.edu.au

***Abstract*—In the context of group decision making with fuzzy preferences, consensus measures are employed to provide feedback and help guide automatic or semi-automatic decision reaching processes. These measures attempt to capture the intuitive notion of how much inputs, individuals or groups agree with one another. Meanwhile, in ecological studies there has been an ongoing research effort to define measures of community evenness based on how evenly the proportional abundances of species are distributed. The question hence arises as to whether there can be any cross-fertilization from developments in these fields given their intuitive similarity. Here we investigate some of the models used in ecology toward their potential use in measuring consensus. We found that although many consensus characteristics are exhibited by evenness indices, lack of reciprocity and a tendency towards a minimum when a single input is non-zero would make them undesirable for inputs expressed on an interval scale. On the other hand, we note that some of the general frameworks could still be useful for other types of inputs like ranking profiles and that in the opposite direction consensus measures have the potential to provide new insights in ecology.**

***Index Terms*—Consensus measures, decision making, aggregation functions, ecological evenness.**

I. Introduction

The aim in group decision making is to reach a decision which best reflects the preferences or evaluations of the group. However where there are coalitions, decision makers with extreme opinions or general heterogeneity amongst the individuals, an aggregation of raw evaluations may not adequately represent the overall view. It is therefore useful to be able to measure the extent to which the individuals agree with one another and how much consensus there is. Of course, the concept of what consensus means in group decisions is debatable, i.e. if there is one person that disagrees with the rest of the group strongly enough, some would argue that there is no consensus, while in some cases we would say that more than 50% of the group holding the same opinion gives us at least some level of consensus.

In [2], we summarized a number of the properties proposed for consensus in the literature and adapted their definitions to the case of real inputs on the unit interval. We saw that, in general, consensus models either aggregated pairwise agreement between the inputs or agreement between each value and the average of the inputs overall.

In the field of ecology, the concept of evenness was first proposed in [17] to describe biological communities based on the distribution of abundance among species. Various measures have since been proposed [7], [12], [14], [16], [18]. These models (along with others) and their properties have been reviewed in [25] and [30]. As well as being a conceptually fundamental component of species biodiversity, the way evenness is measured has implications for environmental decision making. For example, a finding in [32] suggests that reductions in plant species evenness lead to indirect reductions in total plant productivity, which in turn has implications for the management of plant communities.

As the term suggests, an evenness measure attempts to capture how evenly species populations[1] are distributed over a given geographical region, section of forest or grassland etc.

So if there were five species of bird in a given forest, and for each species there were 100 individuals, we would consider this forest to exhibit perfect evenness. On the other hand, if there were two species with 100 individuals for one of the species but only 1 for the other, we may consider the evenness to be very low or almost zero.

We see here then that while decision maker evaluations and species abundances certainly are different concepts, at a basic level evenness and consensus seem to characterize the same features of a dataset, i.e. when all inputs are the same we have perfect evenness and perfect consensus, while the more inputs differ, the less evenness and less consensus. Intuitively it therefore seems that it could be possible to use indices developed in ecology for the purposes of measuring consensus and vice versa.

In this paper, we investigate the relationships between the properties proposed for consensus and for ecological evenness. We also investigate the general frameworks and some specific evenness models and evaluate their appropriateness to the consensus setting. Our main finding is that many evenness measures model evenness as a reciprocal notion to dominance or specificity, i.e. the degree to which one input dominates the rest. This means that they tend towards a minimum when one

[1] Other units such as biomass can also be used.

978-1-4799-2072-3/14/$31.00 ©2014 IEEE

input is high and the rest are low, rather than when inputs are split equally between values at either end of the scale. We note that while this may not be desirable for real inputs, evenness indices could be useful for modeling consensus in more general settings where inputs are provided as preferences or rankings.

The paper will be structured as follows. In the Preliminaries section, we give an overview of aggregation functions and consensus models, which form the basis for our investigations. In Section III, we look at the quantification of ecological evenness and how this can be related to consensus models. In Section IV, we use some example datasets in order to illustrate the key behavior of some of the evenness indices in the consensus setting before concluding with discussion in Section V.

II. Preliminaries

Here we present some basic definitions and properties relating to consensus measures that will be used throughout the rest of the paper.

A. Aggregation functions

Aggregation functions are multivariate functions which combine the inputs into a single representative value. For a broad overview of their behavior and properties we refer the reader to [4], [11], [29]. We will adopt the following definition throughout this paper, which assumes the inputs are provided over the unit interval.

Definition 1: A function $A : [0,1]^n \to [0,1]$ is called an aggregation function if it is non-decreasing in each argument and satisfies $A(0,\ldots,0) = 0$ and $f(1,\ldots,1) = 1$.

In particular, we will be interested in aggregation functions which exhibit *averaging* behavior, i.e. for all $\mathbf{x} \in [0,1]^n$ it holds that $\min(\mathbf{x}) \le A(\mathbf{x}) \le \max(\mathbf{x})$. We note that averaging aggregation functions are idempotent with $A(t,t,\ldots,t) = t$.

A broad family of aggregation functions that will be useful for us are the weighted quasi-arithmetic means.

Definition 2: For a strictly monotone continuous generating function $g : [0,1] \to [-\infty,\infty]$ and weighting vector $\mathbf{w}$, the weighted quasi-arithmetic mean is given by,

$$QAM_{\mathbf{w}}(\mathbf{x}) = g^{-1}\left(\sum_{i=1}^{n} w_i g(x_i)\right). \tag{1}$$

Special cases include weighted arithmetic means, $WAM(\mathbf{x}) = \sum_{i=1}^{n} w_i x_i$ where $g(t) = t$, weighted power means $PM_q(\mathbf{x}) = \left(\sum_{i=1}^{n} w_i x_i^q\right)^{1/q}$ where $g(t) = t^q$ and weighted geometric means $G(\mathbf{x}) = \prod_{i=1}^{n} x_i^{w_i}$ if $g(t) = -\ln t$. The weights w_i are usually seen to indicate the relative importance of a given input source, are non-negative and sum to one.

As well as playing a role in decision making for obtaining overall evaluations from individual inputs pertaining to multiple experts or criteria, aggregation functions are also used in consensus models.

B. Consensus measures

In [2], [3] we reviewed some of the properties and definitions relating to consensus measures. We will use the following definitions.

Firstly we require the concept of a negation.

Definition 3: A negation is a decreasing function $N : [0,1] \to [0,1]$ such that $N(0) = 1$ and $N(1) = 0$. A negation is called:

1) *frontier* if $N(x) \in \{0,1\}$ if and only if $x \in \{0,1\}$;
2) *strict* if N is continuous and strictly decreasing;
3) *strong* if N is involutive, that is, $N(N(x)) = x$ for every $x \in [0,1]$.

Note 1: The interval over which values are considered and the negation used can play a very important role. In decision making contexts, we can generally assume that evaluations will be given over some interval with a pre-determined maximum and minimum and further that in many cases scaling these values to the unit interval will not create too many problems. If the values represent species populations however, it will not always make sense or be possible to consider these over an *a priori* scale. This especially makes interpretations of what constitutes a *high* or *low* input quite difficult.

In [2] we proposed the following definition for the minimum requirements of a consensus measure.

Definition 4: A multivariate function $C : [0,1]^n \to [0,1]$ is said to be a consensus measure if it satisfies the following properties,

- C1 (Unanimity) It holds that $C(a,a,\ldots,a) = 1$, for all $a \in [0,1]$;
- C2 (Minimum consensus for $n = 2$) It holds that $C(0,1) = C(1,0) = 0$.

The following properties are also often considered desirable when trying to obtain an overall level of consensus.

Definition 5: A consensus measure C is said to satisfy

- C3 (Symmetry) when it holds that $C(x_1,x_2,\ldots,x_n) = C(x_{\pi(1)},x_{\pi(2)},\ldots,x_{\pi(n)})$ for all permutations $\pi(i)$ on $\{1,\ldots,n\}$ and $\mathbf{x} \in [0,1]^n$;
- C4 (Maximum dissension) when $C(\underbrace{0,\ldots,0}_{k \text{ times}},\underbrace{1,\ldots,1}_{k \text{ times}}) = 0$ for all permutations of the input vector and $n = 2k$;
- C5 (Reciprocity) if for a strong negation N, it holds that

$$C(x_1,x_2,\ldots,x_n) = C(N(x_1),N(x_2),\ldots,N(x_n));$$

C6 (Replication invariance) when for any input vector $\mathbf{x} \in [0,1]^n$, duplicating the inputs does not alter the level of consensus, i.e. $C(\mathbf{x}) = C(\mathbf{x},\mathbf{x}) = C(\mathbf{x},\mathbf{x},\mathbf{x})$ and so on;

C7 (Monotonicity with respect to the majority) when $|a - x_j| \leq |a - y_j| \implies C(\mathbf{a}, x_1, x_2, \ldots, x_k) \geq C(\mathbf{a}, y_1, y_2, \ldots, y_k)$ where $\mathbf{a} = (a, a, \ldots, a)$ is a set of k equal inputs and $n = 2k$.

Existing consensus measures used in group decision making processes can be considered as special cases of the following general models:

$$C_{\langle M,f \rangle}(x_1, x_2, \ldots x_n) = \underset{i,j=1, i \neq j}{\overset{n}{M}} f(x_i, x_j) \qquad (2)$$

where M is an averaging aggregation function and f is a similarity function; and

$$C_{\langle M_1,M_2,f \rangle}(x_1, x_2, \ldots x_n) = \underset{i=1}{\overset{n}{M_1}} f\left(x_i, \underset{j=1}{\overset{n}{M_2}} x_j\right) \qquad (3)$$

where M_1, M_2 are means and f is a similarity function.

In Eq. (2), the overall consensus is the average of the similarity between each pair of inputs (measures used in [6], [26] are of this form) while in Eq. (2) we take the similarity between each input and some mean or central value (e.g. the consensus measures proposed in [28] and the standard deviation based function given in [3]).

We note that similarities can be obtained from distance or dissimilarity functions (see [8], [9]) and as these concepts are perhaps more well defined, many consensus measures take the form of aggregating dissimilarity or differences and then subtracting this value from 1. In [2], [3] we also looked at using implication functions.

In both Eqs. (2) and (3), the output would need to be scaled if the values are to use the entire $[0,1]$ range (and to formally satisfy property (C4), although in some contexts it could be acceptable simply that the minimum consensus be reached for inputs split between two extremes). For example, for a similarity function $f : [0,1]^2 \to [0,1]$, a minimum value $L = \min_{\mathbf{x} \in [0,1]^n} C(\mathbf{x})$ and an output y, we can use $\frac{y-L}{1-L}$.

III. Ecological evenness

There have been a number of indices proposed in order to capture the evenness of ecological communities. In [25], Smith and Wilson identify 14 indices used throughout the literature and evaluate them in terms of desirable properties, many of which are mathematically equivalent to (C1)-(C7). Other authors have argued that more important than these properties is the semantical interpretations relating to evenness, in particular that it should be derived directly from the relationship $D = E \times R$ (diversity is equal to evenness times species richness)[2] [30].

In the following we will briefly give an overview of some of the properties desired of evenness measures and discuss their relationship to those given for consensus measures. We will then present some of the evenness measures and frameworks used.

A. Relationship between evenness and consensus properties

In [25], 4 requirements and 10 'desirable' properties for evenness measures were outlined with a number of existing evenness measures tested. Table I summarizes these properties and notes the related consensus properties.

Firstly, we note that (C3) is not present, however all evenness indices are considered symmetric.

Evenness properties 5-10 essentially relate to whether the indices range between 0 and 1 and whether the minimum and maximum values are achievable for any number of species. For consensus, the minimum value is attained where evaluations are split between either end of the scale, i.e. where the property of maximum dissension is satisfied (C4) which necessarily follows if both (C2) and (C6) hold.

The emphasis on replication invariance (or independence of species richness) in ecology would seem to necessitate the property of maximum dissension, however there is also the alternate viewpoint that minimum evenness should correspond with the notion of *species dominance*, i.e. that there is just one species present in which case a minimum value would correspond with proportional populations approaching $(0, 0, \ldots, 0, 1)$. This view is incompatible with reciprocity (C5) since clearly the negation $(1, 1, \ldots, 1, 0)$ would not give the same result.

Note 2: Whether minimum evenness should be obtained for these inputs or whether maximum dissension and replication invariance should be satisfied is certainly debatable. Many of the proposed evenness measures were derived from the analogous ideas in economics when considering the equitable distribution of wealth [22], [27]. It could be argued that whilst an input set such as $(0.01, 0.01, 0.01, 0.97)$ represents an inequitable distribution of species proportions, it is actually quite even, and minimum evenness should occur when the species abundances are varied as much as possible with an equal number of species with very high and very low numbers as is the case with maximum dissension in consensus. It hence will come down to a consideration of the context and what behavior is needed.

Another interesting property that has been proposed in ecology is compatibility with the Lorenz curve [10], [20], [21], [27], [30]. The Lorenz curve is obtained from the accumulative proportions when the species abundances are arranged in order. A perfectly even community hence corresponds with a straight

[2] This is argued from there being some agreement amongst ecologists that the concept of diversity combines the notions of species richness (how many species there are altogether) and species evenness [30].

TABLE I
REQUIRED (1-4) AND DESIRED (5-14) PROPERTIES FOR EVENNESS INDICES FROM [25] AND THE RELATED CONSENSUS PROPERTIES

	Evenness Property	Related Consensus property
1	Independence of Species Richness	(C6) Replication invariance
2	Decreased by reducing marginally the abundance of most minor species	(C7) Monotonicity*
3	Decreased by the addition of a very minor species	
4	Unaffected by units used	
5	Maximal when the species are equal	(C1) Unanimity
6	Maximum value 1.0	
7	Minimal, for any number of species, when the species abundances are as unequal as possible	(C2) Minimum consensus for $n = 2$
8	A value close to its minimum when the community is as uneven as we would be likely to meet	
9	The minimum possible index value (not necessarily with a particular number of species) should be 0.	
10	Minimum attainable with any number of species	
11	A value in the middle of the scale for communities that we would intuitively consider intermediate	
12	Respond in a reasonable way to a series of communities that intuitively changes in evenness	
13	Symmetric with regard to minor and abundant species	(C5) Reciprocity
14	Skewed distributions should give a lower value	

* This property is not directly equivalent, but would be ensured by functions satisfying C7

line and hence a Lorenz curve closer to this straight line corresponds with higher evenness. Once again, this idea naturally leads to the point of minimum evenness corresponding with species dominance, rather than the maximum dissension view.

It is worth making mention of the (rather informal) evenness properties 11 and 12. Whilst many of the consensus and evenness properties emphasize behavior and ideal outputs towards the limits of the interval, it is definitely useful that such measures give reasonable intermediate outputs so that two sets of inputs may be compared.

We will now turn to some evenness measures presented in the literature and their compatibility as consensus measures.

B. Evenness measures

The following evenness measures were presented in the review undertaken in [25]. We will use their notation unless specified otherwise.

1) Evenness measures similar to those in consensus: Of the measures considered, we note that only Smith and Wilson's E_{var}, which aggregates the squared difference between each abundance and the average abundance, and E' from [7], which aggregates the differences between each of the population proportions, fall into the frameworks usually employed for consensus (i.e. Eqs. (2) and (3)). They are given as follows:

$$E_{var} = 1 - \frac{2}{\pi}\arctan\left\{\frac{1}{n}\sum_{i=1}^{n}\left(\ln(x_i) - \frac{1}{n}\sum_{j=1}^{n}\ln(x_j)\right)^2\right\} \tag{4}$$

$$E' = 1 - \frac{\sum_{i,j=1,i<j}^{n}|p_i - p_j|}{n} \tag{5}$$

2) Evenness measures based on diversity: Some measures of evenness can be obtained based on the indices used to quantify diversity, which has traditionally been calculated as the reciprocal of Simpson's dominance index [24],

$$D = \sum_{i=1}^{n} p_i^2 \tag{6}$$

or using the Shannon entropy[3] [23],

$$H' = -\sum_{i=1}^{n} p_i \ln p_i \tag{7}$$

In both cases, p_i is the proportional population or biomass of the i-th species, with $\sum_{i=1}^{n} p_i = 1$.

As pointed out in [30], both can be incorporated into a more general framework where diversity is given by

$$\frac{1}{D} = \frac{1}{\left(\sum_{i=1}^{n} p_i p_i^q\right)^{\frac{1}{q}}} \tag{8}$$

We recognize the denominator as the weighted power mean with power q and weights p_i. The entropy based calculation is then $H = \ln(1/D)$ with $q = 0$. Evenness can then be calculated by taking D/n as it is for the measure $E_{1/D}$ (from [31]), while $E_{-\ln D}$ and J' (from [18], [19]) take log transformations of the denominator and numerator. A nice interpretation with this calculation is the number of effective species as a proportion of the total number of species. This implies that the minimum attainable for any number of species n is $\frac{1}{n}$ and that the absolute minimum occurs when the number of species grows infinitely large with a single species dominating.

3) Evenness measures based on ratios: The indices E_{Heip}, E_{1-D} and E_{McI} also can be considered in this framework, however with alternative transformations so that they range from zero to one for all n. For example, E_{1-D} from [14] is given by,

$$E_{1-D} = \frac{1 - D}{1 - 1/n} = \frac{1 - \sum_{i=1}^{n} p_i p_i}{1 - 1/n} \tag{9}$$

We note that if we were to use symmetric weights rather than the p_i values for D, we would obtain $\frac{1}{n}$, so this index

[3] Shannon and Weaver's original entropy equation was in base 2 since its output was bits, however generally in ecology the natural logarithm is used.

can further be considered in a more general framework where the ratio of two dominance (or diversity, as is the case with $F_{2,1}$ and $G_{2,1}$ from [1] and [15]) indices is used.

Another interesting approach which seems to intuitively capture our concept of evenness is to use the slope of the regression curve to the population abundances arranged in descending order.

The equation for slope of least squares fit can be expressed as

$$\beta = \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n}(x_i^2) - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2} \tag{10}$$

For evenness index E_Q, Smith and Wilson let y_i values denote the scaled i-th rank, while the values for x_i are the log of abundance in ascending order. We hence have,

$$\beta = \frac{\sum_{i=1}^{n} \ln(x_{(i)})\frac{i}{n} - \frac{1}{n}\sum_{i=1}^{n} \ln(x_i) \sum_{i=1}^{n} \frac{i}{n}}{\sum_{i=1}^{n}((\ln(x_i))^2) - \frac{1}{n}\left(\sum_{i=1}^{n} \ln(x_i)\right)^2} \tag{11}$$

where the notation $x_{(i)}$ denotes the x_i values arranged in ascending order so that the least abundant species $x_{(1)}$ is associated with $y_1 = \frac{1}{n}$. Then, since $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, we can express the slope as given by

$$\beta = \frac{\sum_{i=1}^{n} \frac{i}{n^2}\ln(x_{(i)}) - \frac{n+1}{2n}\frac{1}{n}\sum_{i=1}^{n} \ln(x_i)}{\sum_{i=1}^{n}(\frac{\ln(x_i)}{n}\ln(x_i)) - \left(\frac{1}{n}\sum_{i=1}^{n} \ln(x_i)\right)^2} \tag{12}$$

Rather than being a different kind of index, such a calculation can be interpreted similarly to E_{1-D} above, where we look at the ratio between two means.

4) Other evenness measures: Evenness measures such as those introduced in [5] on the other hand are not derived from the power mean. Instead, evenness is derived from a measure of community similarity, given by

$$O = \sum_{i=1}^{n} \min(p_i, 1/n) \tag{13}$$

As an aggregation function, O can be framed as a special case of a universal integral [13] with the minimum used as the conjunctive function. We again note that this measure emphasizes the concept of evenness as reciprocal to dominance, reaching a minimum when the population of one species is very high and the remainder are very low.

We also should make note of the evenness index proposed in [20] which is based on Yager's concept of specificity for possibility distributions on fuzzy sets [33]. Since Yager's measure calculates the extent to which a possibility distribution represents a single value, we are able to use this to quantify the notion of dominance and hence the evenness as a reciprocal value.

IV. Examples

In the previous section, we saw that most of the evenness measures tended toward a minimum when one of the inputs is high and the remainder are low, however it is also worth considering how the measures behave for distributions that graduate between the extremes. Here we consider some example sets of inputs and the level of consensus that would be indicated by each of the evenness indices. We then look at their behavior as the inputs graduate from split evaluations at either end of the scale to a slight majority.

A. Consensus level for various distributions

Suppose we have six decision makers, each of whom provides evaluations for six different objects. Evaluations with six different distributions between the experts are shown in Fig. 1 with the values in Table II. The values for objects (b) and (f) are obtained by taking negations $N(x_i)$ from the evaluations of (a) and (e) respectively. We can assume that decision maker or expert A always gives the lowest evaluation or equivalently that the evaluations have been sorted into increasing order (all experts are considered to be equally important for the consensus evaluation).

TABLE II
Decision maker evaluations for Fig.1 (a) - (f)

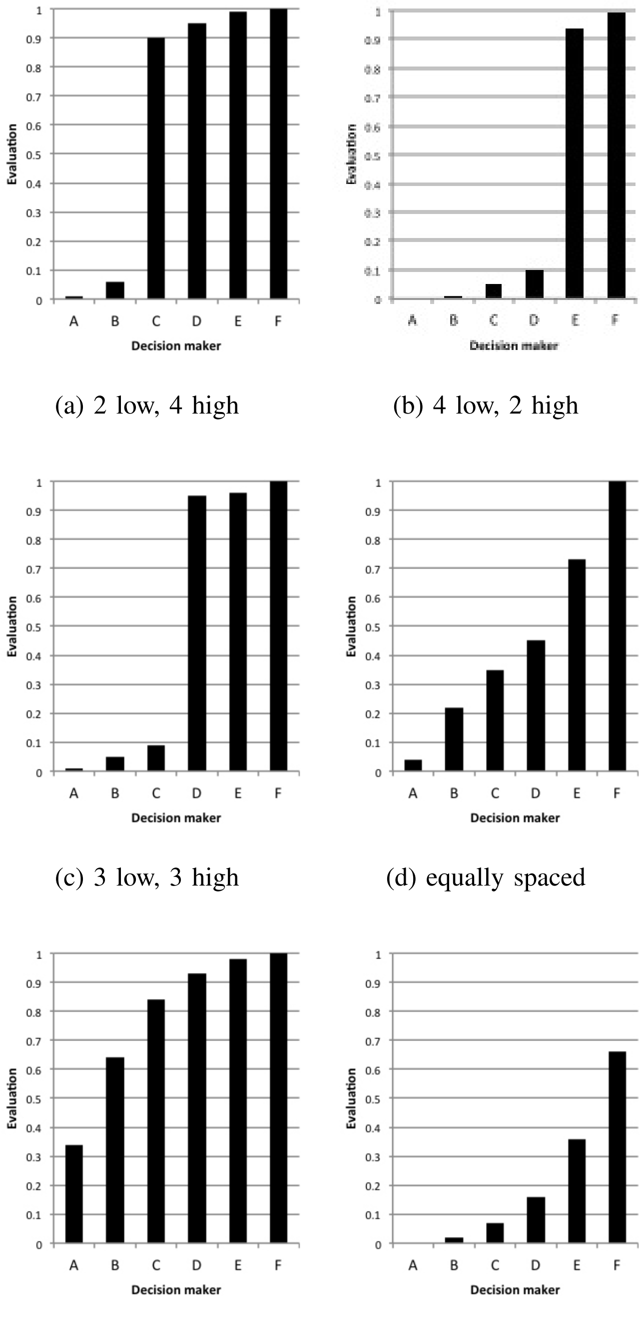
	Decision maker					
Distribution	A	B	C	D	E	F
(a)	0.01	0.06	0.9	0.95	0.99	1
(b)	0	0.01	0.05	0.1	0.94	0.99
(c)	0.01	0.05	0.09	0.95	0.96	1
(d)	0.04	0.22	0.35	0.45	0.73	1
(e)	0.34	0.64	0.84	0.93	0.98	1
(f)	0	0.02	0.07	0.16	0.36	0.66

From the graphs, we can make intuitive assessments about how much the decision makers tend to agree about the evaluations. For Fig. 1 (a) - (c), the decision makers give either very high or very low results. We might say that since in (a) and (b) there is a 4-2 majority, there should be more consensus for these than for (c). For Fig. 1 (d)-(f), the groups are not so polarized, and we may expect higher consensus for (e) and (f), since at least 4 people are close to agreement and the remaining values are not as far away as in (a) and (b). For object (c), there is no particular agreement, however there is also not a strong sense of disagreement between the evaluations.

We calculated the consensus (or evenness values) for these six sets using all the functions we have discussed in this paper so far. For consensus measures of the form given in Eqs. (2) and (3), we used arithmetic means for M and M_1 and both the arithmetic mean and the median for M_2. In M_2, we excluded the i-th input, so that the similarity was calculated between an input and the average of the rest. For similarity functions, we used both $1 - |x_i - x_j|$ and $1 - (x_i - x_j)^2$.

The resulting orderings are shown in Table III.

We note that all non-consensus indices, including E_{var}, E_Q and E' which are replication invariant (and hence reach a



(a) 2 low, 4 high (b) 4 low, 2 high

(c) 3 low, 3 high (d) equally spaced

(e) concave (f) convex

Fig. 1. Column graph representations of evaluations from 6 experts.

minimum for inputs (0,0,0,1,1,1)) order (b) and (f) below (c). In fact E_{var} and E_Q order (c) as the third highest. This is not surprising for many of these indices, especially those which are based on dominance and hence reach their minimum for (0,0,0,. . . ,0,1). Agreement or similarity between higher inputs contributes to the overall consensus evaluation much more than between lower values.

An interesting thing to note is the level of consensus for (d). Both consensus measures and evenness measures resulted

TABLE III
DIFFERENT ORDERS OBTAINED FROM DIFFERENT CONSENSUS AND EVENNESS MEASURES FOR INPUT SETS IN FIG. 1.

Measures used	ordering in terms of consensus evaluation
All consensus measures based on Eqs. (2) and (3)	e = f $\succ$ d $\succ$ a = b $\succ$ c
$E', E_{McI}, E_{-\ln D}, E_{1-D}, E_{1/D}$	e $\succ$ a $\succ$ d $\succ$ c $\succ$ f $\succ$ b
E, O, E_{Heip}, J	e $\succ$ d $\succ$ a $\succ$ c $\succ$ f $\succ$ b
$F_{2,1}$	e $\succ$ a $\succ$ c $\succ$ d $\succ$ b $\succ$ f
E_Q	e $\succ$ d $\succ$ c $\succ$ a $\succ$ b $\succ$ f
E_{var}	e $\succ$ d $\succ$ c $\succ$ a $\succ$ f $\succ$ b

in evaluations which ordered these inputs to have higher agreement than (a) and (b), suggesting that there is more consensus here than in a situation where there is a 4-2 majority. Since it would often be desired that consensus measures take the degree of difference between inputs into account, equally spread measures may seem to disagree less than clusters of inputs at either end of the scale, even though there seems to be no particular agreement. If the concept of majority is more important to the decision makers than the overall differences between the individual evaluations, then other approaches or models could be desirable.

B. Monotonic behavior as one input is increased toward a majority

Here we take a look at the behavior of some of the indices for the set of inputs $\mathbf{x} = (0, 0, x_3, 1, 1, 1)$ as x_3 is gradually increased from 0 to 1. For consensus measures that satisfy the property of maximum dissension, the input starts at zero (or a minimum value) and then gradually increases until $x_3 = 1$ and we have a majority of 4-2. Even for indices which have a minimum at $\mathbf{x} = (0, 0, 0, 0, 0, 1)$, we would still expect that increasing x_3 increases the evenness.

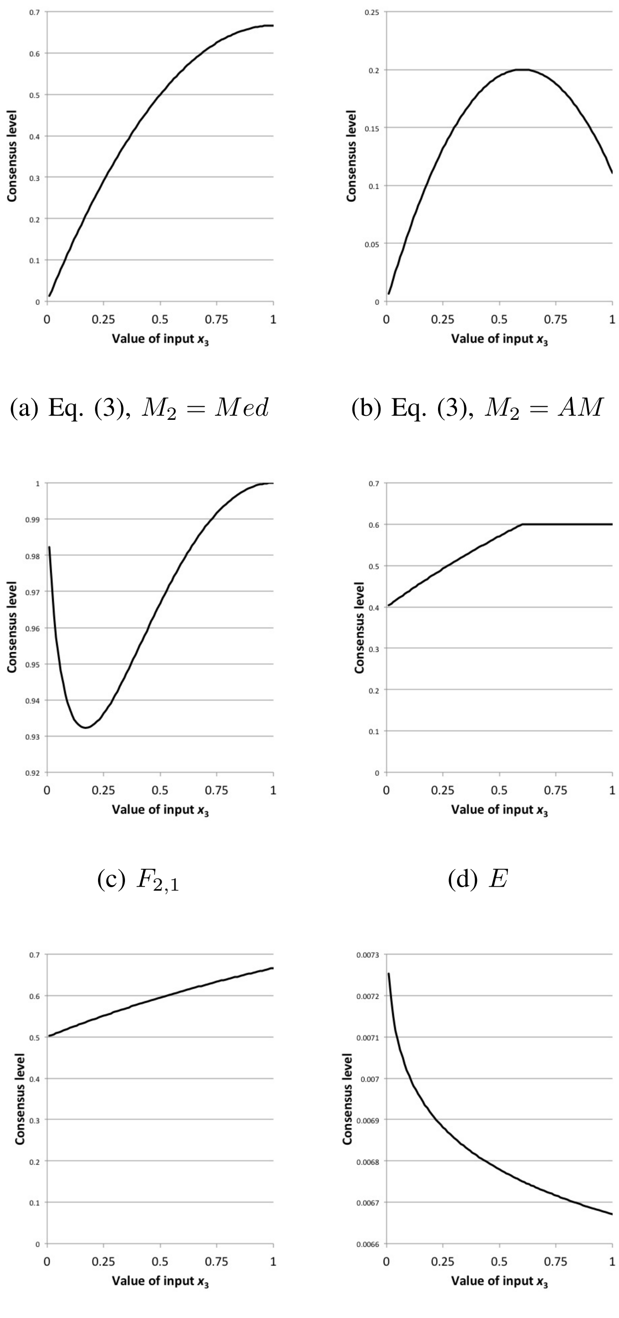
Graphs illustrating the behavior for some of the indices are shown in Fig. 2.

As we have mentioned in our previous work, using the median for M_2 allows Eq. (3) to satisfy monotonicity with respect to the majority, however if the arithmetic mean AM is used there is a point at which the consensus level begins to decline (Fig. 2 (a) and (b)). This happens whether $1-|x_i-x_j|$ is used for the similarity function or the least squares based $1 - (x_i - x_j)^2$ (in these examples we used the latter).

The behavior for $F_{2,1}$ is quite undesirable, as we see the function quickly decreases and then begins to increase (Fig. 2 (c)). Function E reaches a point at which the consensus level reaches a maximum (where the proportion of x_3 becomes greater than $\frac{1}{n}$ ((Fig. 2 (d)).

Since the value of E' depends on the difference in pairwise proportions, increasing x_3 results in monotone behavior.

The decreasing trend of E_{var} is actually due to the unstable behavior of the function when values are close to zero (or the



(a) Eq. (3), $M_2 = Med$ (b) Eq. (3), $M_2 = AM$

(c) $F_{2,1}$ (d) E

(e) E' (f) E_{var}

Fig. 2. Consensus level obtained for the input set $\mathbf{x} = (0, 0, x_3, 1, 1, 1)$ as x_3 increases from 0 (corresponding with maximum disagreement between the inputs) to 1 (where a crisp majority of 4-2 is obtained).

ratio between the smallest and largest input is infinitely large since it is scale independent). Here, rather than $x_1 = x_2 = 0$, we used 10^{-10}. For closer values, the function behaves more reasonably.

V. Discussion and conclusions

In this paper, we have shown some of the relationships that exist between ecological evenness and consensus. We have shown that although at first look these concepts seem to be looking for the same thing numerically, ecological studies tend to emphasize the view that evenness as the opposite of dominance, a view consistent with equitable distributions of wealth in economics. On the other hand, even the evenness indices designed to reach a minimum where high and low evaluations are equally split were not suitable given their unstable behavior for input sets close but not equal to these limits.

The main problem with trying to use evaluations given in a consensus setting is that many of the evenness indices are based on proportional values rather than a scale. However in settings where we are not dealing with real inputs over the unit interval, evenness and equity concepts may be more appropriate than attempting to extend Eqs. (2) and (3). For example, if a set of evaluations represents the number of allocated votes to each candidate, then here we can use a calculation of dominance to denote the degree to which all the votes were allocated to a single candidate. It may therefore be possible to consider extending these evenness measures to situations involving multiple candidates and preferences or transforming the inputs so that each input is interpreted as support for a particular ‘candidate’ or evaluation.

On the other hand, we note that consensus measures have been well defined to exhibit reasonable properties in the context of measuring evenness in ecology - provided the desired view of evenness is one consistent with the minimum coinciding with maximum dissension. One direction of our future work will lie in this direction - the use of consensus measures and aggregation functions in ecological contexts for capturing notions of diversity and evenness.

Acknowledgement

This work was supported by the Deakin University Central Research Grants Scheme, project number RM25299.

References

[1] R. V. Alatalo. Problems in the measurement of evenness in ecology. *Oikos*, 37:199–204, 1981.

[2] G. Beliakov, T. Calvo, and S. James. Consensus measures constructed from aggregation functions and fuzzy implications. *Knowledge-Based Systems*, in press, doi: 10.1016/j.knosys.2013.07.002, 2013.

[3] G. Beliakov, S. James, and T. Calvo. Aggregating fuzzy implications to measure group consensus. In *Proc. Joint IFSA World Congress and NAFIPS Annual Meeting*, pages 1016 – 1021, Edmonton, Canada, 2013.

[4] G. Beliakov, A. Pradera, and T. Calvo. *Aggregation Functions: A Guide for Practitioners*. Springer, Heidelberg, 2007.

[5] L. Bulla. An index of evenness and its associated diversity measure. *Oikos*, 70:167–171, 1994.

[6] F. J. Cabrerizo, S. Alonso, I. J. Pérez, and E. Herrera-Viedma. On consensus measures in fuzzy group decision making. In *Proceedings of MDAI 2008*, pages 86–97, 2008.

[7] J.A. Camargo. Must dominance increase with the number of subordinate species in competitive interactions? *Journal of Theoretical Biology*, 161(4):537 – 542, 1993.

[8] F. Chiclana, J. M. Tapia García, M. J. del Moral, and E. Herrera-Viedma. A statistical comparative study of different similarity measures of consensus in group decision making. *Information Sciences*, 221:110–123, 2013.

[9] J. L. García-Lapresta and D. Pérez-Román. Consensus-based hierarchical agglomerative clustering in the context of weak orders. In *Proceedings of IFSA World Congress*, pages 1010–1015, Edmonton, Canada, 2013.
[10] F. Gosselin. Lorenz partial order: the best known logical framework to define evenness indices. *Community Ecology*, 2(2):197–207, 2001.
[11] M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation Functions*. Cambridge University press, Cambridge, 2009.
[12] C. Heip. A new index measuring evenness. *Journal of Marine Biological Association of the United Kingdom*, 54(3):555 – 557, 1974.
[13] E. Klement, F. Spizzichino, R. Mesiar, and A. Stupňanová. Copula-based universal integrals. In *Proceedings of FUZZIEEE 2013*, pages 1–4, Hyderabad, India, 2013.
[14] C.J. Krebs. *Ecological Methodology*. Harper Collins, New York, 1989.
[15] J. Molinari. A calibrated index for the measurement of evenness. *Oikos*, 56:319–326, 1989.
[16] S. Nee, P.H. Harvey, and P. Cotgreave. Population persistence and the natural relationship between body size and abundance. In *Conservation of biodiversity for sustainable development*, pages 124–136. Scandinavian University Press, Oslo, 1992.
[17] E.C. Pielou. *An Introduction to Mathematical Ecology*. Wiley-Interscience, New York, 1969.
[18] E.C. Pielou. *Ecological Diversity*. Wiley, New York, 1975.
[19] E.C. Pielou. *Mathematical ecology*. Wiley, New York, 1977.
[20] C. Ricotta. A recipe for unconventional evenness measures. *Acta Biotheoretica*, 52:95–104, 2004.
[21] C. Ricotta, E. De Zuliani, A. Pacini, and G. C. Avena. On the mutual relatedness of evenness measures. *Community Ecology*, 2(1):51–56, 2001.
[22] R. D. Routledge. Evenness indices: are any admissible? *Oikos*, 40:149–151, 1983.
[23] C. E. Shannon and W. Weaver. *The mathematical theory of communication*. Univ. of Illinois Press, Urbana, IL., 1949.
[24] E. H. Simpson. Measurement of diversity. *Nature*, 163:688, 1949.
[25] B. Smith and J. B. Wilson. A consumer's guide to evenness indices. *Oikos*, 76:70 – 82, 1996.
[26] E. Szmidt and J. Kacprzyk. Analysis of consensus under intuitionistic fuzzy preferences. In *Proc. Int. Conf. in Fuzzy Logic and Technology*, pages 79–82, De Montfort University, Leicester, UK, 2001.
[27] C. Taillie. Species equitibility: a comparative approach. In *Ecological diversity in theory and practice*, pages 51–62. Int. Coop. Publ. House, Fairland, Maryland, 1979.
[28] J. M. Tastle and W. J. Tastle. Extending the consensus measure: analyzing ordinal data with respect to extrema. In *Proc. of ISECON 2005, v22*, pages 1–5, Columbus OH, USA, 2005.
[29] Y. Torra, V. Narukawa. *Modeling Decisions. Information Fusion and Aggregation Operators*. Springer, Berlin, Heidelberg, 2007.
[30] H. Tuomisto. An updated consumer's guide to evenness and related indices. *Oikos*, 121:1203 – 1218, 2012.
[31] C. B. Williams. *Patterns in the balance of nature*. Academic Press, London, 1964.
[32] B.J. Wilsey and C. Potvin. Biodiversity and ecosystem functioning: importance of species evenness in an old field. *Ecology*, 81(4):887 – 892, 2000.
[33] R. R. Yager. On the specificity of a possibility distribution. *Fuzzy Sets and Systems*, 50:279–292, 1992.

APPENDIX

We provide the following ecological indices (reviewed in [25]) which were not given throughout the main text.

[Williams 1964]

$$E_{1/D} = \frac{1/D}{n} = \frac{1}{n\left(\sum\limits_{i=1}^{n} p_i p_i\right)} \tag{14}$$

[Pielou 1977]

$$E_{-\ln D} = \frac{-\ln D}{\ln n} = \frac{\ln\left(\frac{1}{\sum\limits_{i=1}^{n} p_i p_i}\right)}{\ln n} \tag{15}$$

[Pielou 1975]

$$J' = \frac{H'}{\ln(n)} = \frac{\ln\left(\frac{1}{\prod\limits_{i=1}^{n} p_i^{p_i}}\right)}{\ln n} \tag{16}$$

[Heip 1974]

$$E_{Heip} = \frac{e^{H'} - 1}{n-1} = \frac{\frac{1}{\prod\limits_{i=1}^{n} p_i^{p_i}} - 1}{n-1} \tag{17}$$

[Pielou 1969, McIntosh 1967]

$$E_{McI} \frac{\left(\sum\limits_{i=1}^{n} x_i\right) - \sqrt{\sum\limits_{i=1}^{n} x_i x_i}}{\left(\sum\limits_{i=1}^{n} x_i\right) - \frac{\left(\sum\limits_{i=1}^{n} x_i\right)}{\sqrt{n}}} \tag{18}$$

[Bulla 1994]

$$E = \frac{O - 1/n}{1 - 1/n} \tag{19}$$

[Alatalo 1981]

$$F_{2,1} = \frac{1/D - 1}{e^{H'} - 1} = \frac{\frac{1}{\sum\limits_{i=1}^{n} p_i p_i} - 1}{\frac{1}{\left(\prod\limits_{i=1}^{n} p_i^{p_i}\right)} - 1} \tag{20}$$

[Molinari 1989]

$$G_{2,1} = \begin{cases} F_{2,1} \frac{2}{\pi} \arcsin F_{2,1}, & F_{2,1} > \sqrt{1/2}, \\ (F_{2,1})^3, & \text{otherwise.} \end{cases} \tag{21}$$

[Wilson 1991]

$$E_Q = -\frac{2}{\pi} \arctan(1/\beta) \tag{22}$$

where β is the slope of the least-squares fit regression curve for the scaled log abundance (scaled by dividing by the maximum rank) in decreasing order.