Analytical Solution for the Linguistic Weighted Average Problem

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***Abstract*—The linguistic weighted average (LWA) of interval type-2 fuzzy sets is an extension of the fuzzy weighted average (FWA) of type-1 fuzzy sets. Currently, the commonly used methods of both FWA and LWA are based on α-cuts decomposition of the type-1 of type-2 fuzzy sets, which involves large amount of calculations, and the result is not accurate. In this paper, we propose a new algorithm to obtain the analytical solution methods, which is more accurate and efficient than the current α-cuts methods. Some properties of the algorithms are discussed. A numerical example is used to illustrate our new proposed algorithms.**

Key words: Linguistic weighted average(LWA); fuzzy weighted average (FWA); discrete α -cuts; Karnik-Mendel (KM) algorithm; analytical solution method.

I. INTRODUCTION

The fuzzy weighted average (FWA) is an important topic in fuzzy logic theory and applications[1-4]. It has been used in risk evaluation[5], multi-criteria decision making[6-8], information processing and decision making[9-12], etc. An extensions of FWA which is linguistic weighted average (LWA) was also proposed [13, 14], which is widely used in investment decision making[15], social judgment making[15-17], hierarchical decision making[15, 18, 19] and perceptual computer (Per-C)[14, 15, 20].

LWA was the first proposed by Herrera and Herrera-Viedma [13] who considered LWA as type-1 fuzzy set. However, Mendel[15] said :"Words mean different things to different people and IT2 FS should be used as a FS model of a word". Basing on that, Mendel and Wu[15] put forward the definition of FWA with type-2 fuzzy sets. Let X_i represent interval type-2 function set (IT2 FSs), W_i denote their corresponding TT2 FS weights. The LWA can then be expressed as

$$Y_{FWA}=\frac{\sum_{i=1}^{n} X_i * W_i}{\sum_{i=1}^{n} W_i} \tag{1}$$

Y_{FWA} is also an IT2 FS [17, 21].

A recent important progress about computing the FWA is the use of the Karnik-Mendel (KM) algorithm [4, 22, 23]. The KM algorithm was originally used for computing the generalized centroid of interval type-2 fuzzy sets [24]. Liu and Mendel [4] connected the FWA with centroid computation of type-2 fuzzy set, and proposed a new α-cut based algorithm for solving the FWA problem using the KM algorithm. Liu and Mendel and Wu [22] and Liu and Wang [22] proposed the analytical solutions for the FWA and general FWA operators respectively.

The KM algorithm transforms the fractional programming problem into finding the optimal switch points of the α-cuts, and it converges monotonically and super exponentially fast [25]. Wu and Jerry Mendel [14] compute $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$ by the method of discrete α-cuts. Assuming there are n α-cuts on UMF and m α-cuts on LMF, one has to discretize each X_i and W_i into a set of α levels and then compute each α-cut as an interval weighted average using KM algorithm. For an LWA problem discretized with *m* sample points in the value domain and n sample points in the membership value, we have to determine *2(m+n)* optimal switch points. Each optimal switch points need an iteration calculation with KM algorithms. a large number of α levels for UMF and LMF are need for high accuracy. Consequently, the final result of $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$,which is also a fuzzy set, can only be constructed approximately with many iterative calculations by connecting the solution of each α-cuts.

In addition to the large number of calculations and the inaccurate outcome, the discrete α-cuts algorithm of LWA does not provide closed-form solutions, making it difficult to analyze the properties of the output, where usually a functional form is needed.

In the paper, based on the ideas of [22], we will give an alternative method and format for the solution of LWA in [14, 17], We introduce an analytical method which is new to compute $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$. This method has replaced the KM

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algorithm in switch points with an alternative optimization criterion which can be directly connected with α, leading to the analytical solution for the $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$. Compared with the current α-cuts method, our new approach is more accurate and efficient; besides, the closed-form output is very convenient to express and provides possibility for further analysis.

The rest of this paper is organized as follows. Section 2 reviews the main contents on the FWA, which serve as the basis to deduce the LWA algorithm. In Section 3, we present the main idea of the LWA and explain the discrete α-cuts method for computing LWA, which, however, involves large amount of calculations. Section 4 introduces an analytical method to get LWA. It is much more efficient and accurate as will be shown. A numerical example is given to illustrate the analytical method algorithm in Section 5. In section 6, we do the summarizing job of the whole paper.

II. PRELIMINARIES

In this section, we will give an introduction on the the discrete α-cuts algorithms of FWA.

The FWA [15] is defined as:

$$Y_{FWA} = \frac{\sum_{i=1}^{n} X_i * W_i}{\sum_{i=1}^{n} W_i} \tag{1}$$

Where X_i and W_i are all type-1 fuzzy set (T1 FS).

Currently, the general approach of FWA problem (1) is discrete solution with α-cuts. The first step of the discrete computation method of (3) is to discretize all its fuzzy numbers using α-cuts. For any $\alpha_{\mathrm{j}} \in [0,1]$, the corresponding α-cuts of the T1 FS X_i and W_i, can be expressed as:

$$X_i(\alpha_j) = [X_i(\alpha_j)^L, X_i(\alpha_j)^R]$$

and

$$W_i(\alpha_j) = [W_i(\alpha_j)^L, W_i(\alpha_j)^R]$$

For any $\alpha_{\mathrm{j}} \in [0,1]$, the corresponding α-cuts of the FWA can be recognized as an interval weight average. Consequently, $Y_{FWA}(\alpha_{\mathrm{j}})$, a closed interval, can be expressed as $Y_{FWA}(\alpha_j) = [Y(\alpha_j)^{\mathrm{L}}, Y(\alpha_j)^{\mathrm{R}}]$.

$Y(\alpha_j)^{\mathrm{L}}$ and $Y(\alpha_j)^{\mathrm{R}}$ can be computed with Karnik-Mendel algorithms. For more details, please see [4, 24, 25].

We have n α-cuts, where n can be determined by the tolerance error bound of the problem. For example, if one wants the solution error about α to be no more than 0.01, one chooses n = 100. Generally, the higher precision that is required, the bigger n becomes. In addition to that, we need to calculate 2n switch points in this process.

At last, according to Decomposition Theorem, we connect all left coordinates $(Y(\alpha_j)^{\mathrm{L}}, \alpha_j)$ and all right coordinates $(Y(\alpha_j)^{\mathrm{R}}, \alpha_j)$ to form the Y_{FWA}.

The method of computing FWA can be used to compute the LWA. Linguistic weighted average (LWA) [14, 17] is an extension of the fuzzy weighted average (FWA). X_i and W_i are interval type-2 fuzzy sets (IT2 FSs) in the LWA, while they are type-1 fuzzy set in the FWA. Therefore, LWA consist of numerous FWAs.

The following is the expression of LWA.

$$Y_{LWA} = \frac{\sum_{i=1}^{n} X_i * W_i}{\sum_{i=1}^{n} W_i} \tag{2}$$

Where X_i and W_i are IT2 FSs, and Y_{LWA} is also an IT2 FS.

Y_{FWA} is characterized by its FOU as: $FOU(Y_{LWA}) = [\underline{Y}_{LWA}, \overline{Y}_{LWA}]$. Where $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$ are the LMF and UMF of Y_{LWA} respectively, which are FWAs. Once $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$ are obtained, Y_{LWA} is determined. The FOU of Y_{FWA} is the area between $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$.

The first step, getting $\overline{Y}_{LWA}$ using discrete α-Cuts [26].

For any $\alpha_{\mathrm{j}} \in [0,1]$, the α_j cuts on UMF of $\overline{X}$ and $\overline{W}$. The following intervals can be obtained.

$$\overline{X}_i(\alpha_j) = [X_i(\alpha_j)^{Ll}, X_i(\alpha_j)^{Rr}]$$

$$\overline{W}_i(\alpha_j) = [W_i(\alpha_j)^{Ll}, W_i(\alpha_j)^{Rr}]$$

Where $\alpha_j \in [0,1]$

According discrete α-Cuts method of FWA and the KM algorithm, the result of $y_{Ll}(\alpha_j)$ and $y_{Rl}(\alpha_j)$ are as:

$$y_{Ll}(\alpha_j) = \frac{\sum_{i=1}^{k_{Ll}} X_i(\alpha_j)^{Ll} * W_i(\alpha_j)^{Rr}}{\sum_{i=1}^{k_{Ll}} W_i(\alpha_j)^{Rr} + \sum_{i=k_{Ll}+1}^{n} W_i(\alpha_j)^{Ll}} + \frac{\sum_{i=k_{Ll}+1}^{n} X_i(\alpha_j)^{Ll} * W_i(\alpha_j)^{Ll}}{\sum_{i=1}^{k_{Ll}} W_i(\alpha_j)^{Rr} + \sum_{i=k_{Ll}+1}^{n} W_i(\alpha_j)^{Ll}} \tag{3}$$

$$y_{Rr}(\alpha_j) = \frac{\sum_{i=1}^{k_{Rr}} X_i(\alpha_j)^{Rr} * W_i(\alpha_j)^{Ll}}{\sum_{i=1}^{k_{Rr}} W_i(\alpha_j)^{Ll} + \sum_{i=k_{Rr}+1}^{n} W_i(\alpha_j)^{Rr}} + \frac{\sum_{i=k_{Rr}+1}^{n} X_i(\alpha_j)^{Rr} * W_i(\alpha_{\mathrm{j}})^{Rr}}{\sum_{i=1}^{k_{Rr}} W_i(\alpha_j)^{Ll} + \sum_{i=k_{Rr}+1}^{n} W_i(\alpha_j)^{Rr}} \tag{4}$$

Where $\alpha_j \in [0,1]$.

Then $\overline{Y}_{LWA}(\alpha_{\mathrm{j}})$ can be obtained:

$\overline{Y}_{LWA}(\alpha_j) = (y_{Ll}(\alpha_j), y_{Rr}(\alpha_j))$

Finally, we get $\overline{Y}_{LWA}$.

The second step, getting $\underline{Y}_{LWA}$ using discrete α -Cuts.

For any $\alpha_j \in [0, h_{\underline{Y}_{LWA}}]$, the α_j cuts on LMF of $\underline{X}_i$ $\underline{W}_i$ and $\underline{Y}_{LWA}$, the following intervals can be obtained.

$$\underline{X}_i(\alpha_j) = (X_i(\alpha_j)^{Lr}, X_i(\alpha_j)^{Rl})$$

$$\underline{W}_i(\alpha_j) = (W_i(\alpha_j)^{Lr}, W_i(\alpha_j)^{Rl})$$

$$\underline{Y}_{LWA}(\alpha_j) = (y_{Lr}(\alpha_j), y_{Rl}(\alpha_j))$$

Where $\alpha_j \in [0, h_{\underline{Y}_{LWA}}]$

According discrete α -Cuts method of FWA and the KM algorithm, Dongrui WU and Mendel [14] has the result of $y_{Lr}(\alpha_j)$ $y_{Rr}(\alpha_j)$.

$$y_{Lr}(\alpha_j) = \frac{\sum_{i=1}^{k_{Lr}} X_i(\alpha_j)^{Lr} * W_i(\alpha_j)^{Rl}}{\sum_{i=1}^{k_{Lr}} W_i(\alpha_j)^{Rl} + \sum_{i=k_{Lr}+1}^{n} W_i(\alpha_j)^{Lr}} + \frac{\sum_{i=k_{Lr}+1}^{n} X_i(\alpha_j)^{Lr} * W_i(\alpha_j)^{Lr}}{\sum_{i=1}^{k_{Lr}} W_i(\alpha_j)^{Rl} + \sum_{i=k_{Lr}+1}^{n} W_i(\alpha_j)^{Lr}} \quad (5)$$

$$y_{Rl}(\alpha_j) = \frac{\sum_{i=1}^{k_{Rl}} X_i(\alpha_j)^{Lr} * W_i(\alpha_j)^{Lr}}{\sum_{i=1}^{k_{Rl}} W_i(\alpha_j)^{Lr} + \sum_{i=k_{Rl}+1}^{n} W_i(\alpha_j)^{Rl}} + \frac{\sum_{i=k_{Rl}+1}^{n} X_i(\alpha_j)^{Rl} * W_i(\alpha_j)^{Rl}}{\sum_{i=1}^{k_{Rl}} W_i(\alpha_j)^{Lr} + \sum_{i=k_{Rl}+1}^{n} W_i(\alpha_j)^{Rl}} \quad (6)$$

Where $\alpha_j \in [0, h_{\underline{Y}_{LWA}}]$

Then $\underline{Y}_{LWA}(\alpha_j)$ can be obtained by $\underline{Y}_{LWA}(\alpha_j) = (y_{Lr}(\alpha_j), y_{Rl}(\alpha_j))$. Finally, we get $\underline{Y}_{LWA}$.

The detailed process of computing LWA with α -cut method is given in Table I.

The method of the discrete α-Cuts has the following shortcoming to compute LWA:

1. The computation amount increases very fast. If the precision has an additional digit, such as the error bound limit from 0.01 to 0.001,the model will have to be solved 2000 times, which is a 20 fold increase.

2. There is a large amount of repetitive computation, because the same model is used to solve switch points. We assume n α-Cuts on UMF and m α-Cuts on LMF. The switch points can be computed 2*(m+n) making use of KM algorithm.

3. We can never get the exact values of $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$ regardless of how many α-cuts is taken.

4. Because we do not have mathematical expressions for the final solutions of $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$, it is hard to analyze the properties of the problem. We can only observe these properties with usually very limited numerical simulations.

In the next section, we will present a new method, which can avoid those shortcomings.

III. The Analytical Solution for the Linguistic Weighted Average Problem

In this section, we use the analytical method to compute $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$. The essence of the new method is to connect the points with the same optimal switch points together, so that the final solution can be expressed in an analytical way for $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$.

The advantage of this approach is that we can obtain an accurate analytical solution of the $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$ as functions of α. Instead of connecting the $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$ values at different α-cut levels, we can use the analytical solution to perform further analyses. And the repetitive iteration computation in the discrete algorithms for different α-cut levels for $\underline{Y}_{LWA}$ and $\overline{Y}_{LWA}$ can be avoid. As a result, this method reduces the computation amount and improves computational efficiency. To get Y_{LWA}, we divided into two steps. The first is to get $\overline{Y}_{LWA}$ and second $\underline{Y}_{LWA}$.

The first step----getting $\overline{Y}_{LWA}$ using the analytical method

The α Cuts on UMF of $\overline{X}_i$ and $\overline{W}_i$ get the following intervals, where α is a variable without assigning any values.

$$\overline{X}_i(\alpha) = (X_j(\alpha)^{Ll}, X_j(\alpha)^{Rr}), \overline{W}_i(\alpha) = (W_j(\alpha)^{Ll}, W_j(\alpha)^{Rr})$$

Let

$$\varphi_{Ll}(\alpha,k) = \frac{\sum_{i=1}^{k} X_i(\alpha)^{Ll} * W_i(\alpha)^{Rr}}{\sum_{i=1}^{k} W_i(\alpha)^{Rr} + \sum_{i=k+1}^{n} W_i(\alpha)^{Ll}} + \frac{\sum_{i=k+1}^{n} X_i(\alpha)^{Ll} * W_i(\alpha)^{Ll}}{\sum_{i=1}^{k} W_i(\alpha)^{Rr} + \sum_{i=k+1}^{n} W_i(\alpha)^{Ll}} \quad (7)$$

$$\varphi_{Ll}(\alpha,k+1) = \frac{\sum_{i=1}^{k+1} X_i(\alpha)^{Ll} * W_i(\alpha)^{Rr}}{\sum_{i=1}^{k+1} W_i(\alpha)^{Rr} + \sum_{i=k+2}^{n} W_i(\alpha)^{Ll}} + \frac{\sum_{i=k+2}^{n} X_i(\alpha)^{Ll} * W_i(\alpha)^{Ll}}{\sum_{i=1}^{k+1} W_i(\alpha)^{Rr} + \sum_{i=k+2}^{n} W_i(\alpha)^{Ll}} \quad (8)$$

According equation (7) $y_{Ll}(\alpha)$ can be expressed as

$$y_{Ll}(\alpha) = \min_{k=0,1,\ldots n-1} \varphi_{Ll}(\alpha,k) = \frac{\sum_{i=1}^{k} X_i(\alpha)^{Ll} * W_i(\alpha)^{Rr}}{\sum_{i=1}^{k} W_i(\alpha)^{Rr} + \sum_{i=k+1}^{n} W_i(\alpha)^{Ll}} + \frac{\sum_{i=k+1}^{n} X_i(\alpha)^{Ll} * W_i(\alpha)^{Ll}}{\sum_{i=1}^{k} W_i(\alpha)^{Rr} + \sum_{i=k+1}^{n} W_i(\alpha)^{Ll}} \quad (9)$$

TABLE I THE DISCRETE α -CUTS TO COMPUTE LWA [14, 17]

STEP	LMF		UMF	
1	Select p α -cuts for $\underline{Y}_{LWA}$ $\alpha_j \in [0, h_{\underline{Y}_{LWA}}]$ j=1,2…n		Select m α -cuts for $\overline{Y}_{LWA}$ $\alpha_j \in [0,1]$ j=1.2…n	
2	Set j=1		Set j=1	
3	Sort $X_i(\alpha_j)^{Lr}$ (i = 1, 2, . . . , n) in increasing order.	Sort $X_i(\alpha_j)^{Rl}$ (i = 1, 2, . . . , n) in increasing order.	Sort $X_i(\alpha_j)^{Ll}$ (i = 1, 2, . . . , n) in increasing order	Sort $X_i(\alpha_j)^{Rr}$ (i = 1, 2, . . . , n) in increasing order
4	Match the corresponding $W_i(\alpha_j)^{Lr}$, i = 1, 2 . . . , n accordingly.	Match the corresponding $W_i(\alpha_j)^{Rl}$, i = 1, 2 . . . , n accordingly.	Match the corresponding $W_i(\alpha_j)^{Ll}$, i = 1, 2 . . . , n accordingly.	Match the corresponding $W_i(\alpha_j)^{Rr}$, i = 1, 2 . . . , n accordingly.
5	*Using KM algorithm ,find optimal sw*itch k_{Lr}	Using KM algorithm ,find optimal switch k_{Rl}	Using KM algorithm ,find optimal switch k_{Ll}	Using KM algorithm ,find optimal switch k_{Rr}
6	Compute $y_{Lr}(\alpha_j)$	Compute $y_{Rl}(\alpha_j)$	Compute $y_{Ll}(\alpha_j)$	Compute $y_{Rr}(\alpha_j)$
7	$\underline{Y}_{LWA}(\alpha_j) = [y_{Lr}(\alpha_j), y_{Rl}(\alpha_j)]$		$\overline{Y}_{LWA}(\alpha_j) = [y_{Ll}(\alpha_j), y_{Rr}(\alpha_j)]$	
8	If j=p go to step3			
9	Connect all left coordinates $(y_{Lr}(\alpha_j), \alpha_j)$ $(y_{Rl}(\alpha_j), \alpha_j)$ to form the $\underline{Y}_{LWA}(\alpha_j)$		Connect all left coordinates $(y_{Ll}(\alpha_j), \alpha_j)$ $(y_{Rl}(\alpha_j), \alpha_j)$ to form the $\overline{Y}_{LWA}(\alpha_j)$	
10	$FOU(Y_{LWA}) = 1 \Big/ [\underline{Y}_{LWA}, \overline{Y}_{LWA}]$			

Consequently,

$$\varphi_{Ll}(\alpha,k+1)-\varphi_{Ll}(\alpha,k) = \frac{W_{k+1}(\alpha)^{Rr}-W_{k+1}(\alpha)^{Ll}}{\sum_{i=1}^{k} W_i(\alpha)^{Rr}+\sum_{i=k+1}^{n} W_i(\alpha)^{Ll}} \times \frac{d_{Ll}(\alpha,k)}{\sum_{i=1}^{k+1} W_i(\alpha)^{Rr}+\sum_{i=k+2}^{n} W_i(\alpha)^{Ll}} \quad (10)$$

Where

$$d_{Ll}(\alpha,k)=\sum_{i=1}^{k}(X_{k+1}(\alpha)^{Ll}-X_i(\alpha)^{Ll})*W_i(\alpha)^{Rr} + \sum_{i=k+2}^{n}(X_{k+1}(\alpha)^{Ll}-X_i(\alpha)^{Ll})*W_i(\alpha)^{Ll} \quad (11)$$

Because $W_{k+1}(\alpha)^{Rr} > W_{k+1}(\alpha)^{Ll}$, and all the values of $W_{k+1}(\alpha)^{Rr} > 0$ and $W_{k+1}(\alpha)^{Ll} > 0$, whether $\varphi_{Ll}(\alpha,k+1)-\varphi_{Ll}(\alpha,k) \geq 0$ is determined by $d_{Ll}(\alpha,k)$. And, if $d_{Ll}(\alpha,k)$ is an increasing function with respect to k, then $k = k_{Ll}$, $d_{Ll}(\alpha,k_{Ll}-1) < 0$ and $d_{Ll}(\alpha,k_{Ll}) \geq 0$, which means $\varphi_{Ll}(\alpha,k) < \varphi_{Ll}(\alpha,k-1)$ and $\varphi_{Ll}(\alpha,k+1) \geq \varphi_{Ll}(\alpha,k)$, k_{Ll} must be the optimal switch point of $y_{Ll}(\alpha)$.

Similarly , let

$$\psi_{Rr}(\alpha,k) = \frac{\sum_{i=1}^{k} X_i(\alpha)^{Rr} * W_i(\alpha)^{Ll}}{\sum_{i=1}^{k} W_i(\alpha)^{Ll}+\sum_{i=k+1}^{n} W_i(\alpha)^{Rr}} + \frac{\sum_{i=k+1}^{n} X_i(\alpha)^{Rr} * W_i(\alpha)^{Rr}}{\sum_{i=1}^{k} W_i(\alpha)^{Ll}+\sum_{i=k+1}^{n} W_i(\alpha)^{Rr}} \quad (12)$$

$$\psi_{Rr}(\alpha,k+1) = \frac{\sum_{i=1}^{k+1} X_i(\alpha)^{Rr} * W_i(\alpha)^{Ll}}{\sum_{i=1}^{k+1} W_i(\alpha)^{Ll}+\sum_{i=k+2}^{n} W_i(\alpha)^{Rr}} + \frac{\sum_{i=k+2}^{n} X_i(\alpha)^{Rr} * W_i(\alpha)^{Rr}}{\sum_{i=1}^{k+1} W_i(\alpha)^{Ll}+\sum_{i=k+2}^{n} W_i(\alpha)^{Rr}} \quad (13)$$

According equation (19) $y_{Ll}(\alpha)$ can be expressed

$$y_{Rr}(\alpha)=\max_{k=0,1,\ldots n-1}\varphi_{Rr}(\alpha,k)$$
$$=\max_{k=0,1,\ldots n-1}\left(\frac{\sum_{i=1}^{k}X_i(\alpha_j)^{Rr}*W_i(\alpha_j)^{Ll}}{\sum_{i=1}^{k}W_i(\alpha_j)^{Ll}+\sum_{i=k+1}^{n}W_i(\alpha_j)^{Rr}}+\frac{\sum_{i=k+1}^{n}X_i(\alpha_j)^{Rr}*W_i(\alpha_j)^{Rr}}{\sum_{i=1}^{k}W_i(\alpha_j)^{Ll}+\sum_{i=k+1}^{n}W_i(\alpha_j)^{Rr}}\right) \quad (14)$$

Consequently, we get the result of $\psi_{Rr}(\alpha,k+1)-\psi_{Rr}(\alpha,k)$

$$\psi_{Rr}(\alpha,k+1)-\psi_{Rr}(\alpha,k)$$
$$=\frac{W_{k+1}(\alpha)^{Rr}-W_{k+1}(\alpha)^{Ll}}{\sum_{i=1}^{k}W_i(\alpha)^{Rr}+\sum_{i=k+1}^{n}W_i(\alpha)^{Ll}}\times\frac{d_{Rr}(\alpha,k)}{\sum_{i=1}^{k+1}W_i(\alpha)^{Rr}+\sum_{i=k+2}^{n}W_i(\alpha)^{Ll}} \quad (15)$$

Where

$$d_{Rr}(\alpha,k)=-\sum_{i=1}^{k}(X_{k+1}(\alpha)^{Rr}-X_i(\alpha)^{Rr})*W_i(\alpha)^{Ll}$$
$$-\sum_{i=k+2}^{n}(X_{k+1}(\alpha)^{Rr}-X_i(\alpha)^{Rr})*W_i(\alpha)^{Rr}$$

Because $W_{k+1}(\alpha)^{Rr}>W_{k+1}(\alpha)^{Ll}$, and all the values of $W_{k+1}(\alpha)^{Rr}>0$,whether $\psi_{Rr}(\alpha,k+1)-\psi_{Rr}(\alpha,k)\geq 0$ is determined by $d_{Rr}(\alpha,k)$. If $d_{Rr}(\alpha,k)$ is a decreasing function with respect to k, then when $k=k_{Rr}$, $d_{Rr}(\alpha,k_{Rr}-1)>0$ and $d_{Rr}(\alpha,k_{Rr})<0$, which means $\psi_{Rr}(\alpha,k+1)<\psi_{Rr}(\alpha,k)$ $\psi_{Rr}(\alpha,k-1)<\psi_{Rr}(\alpha,k)$ and, k_{Rr} must be the optimal switch point of $y_{Rr}(\alpha)$.

So, we can get $\overline{Y}_{LWA}(\alpha)$, $\overline{Y}_{LWA}(\alpha)=(y_{Ll}(\alpha),y_{Rr}(\alpha))$ in an analytical way instead of the two step α -cut method. The analytical solution of

$\underline{Y}_{LWA}(\alpha)=(y_{Lr}(\alpha),y_{Rl}(\alpha))$ can be obtained in a similar way.

The following theorem prove the the monotonicity of $d_{Ll}(\alpha,k)$ and $d_{Lr}(\alpha,k)$, so that the analytical solution of LWA can be guaranteed.

Theorem 1. The optimal solutions of $y_{Lr}(\alpha),y_{Lr}(\alpha),y_{Rl}(\alpha),y_{Rr}(\alpha)$ with $k=k_{Ll}$ $k=k_{Lr}$ $k=k_{Rl}$ and $k=k_{Rr}$ can be determined as follows.

(a) Let

$$d_{Ll}(\alpha,k)=\sum_{i=1}^{k}(X_{k+1}(\alpha)^{Ll}-X_i(\alpha)^{Ll})*W_i(\alpha)^{Rr}$$
$$+\sum_{i=k+2}^{n}(X_{k+1}(\alpha)^{Ll}-X_i(\alpha)^{Ll})*W_i(\alpha)^{Ll}$$
$$d_{Lr}(\alpha,k)=\sum_{i=1}^{k}(X_{k+1}(\alpha)^{Lr}-X_i(\alpha)^{Lr})*W_i(\alpha)^{Rl}$$
$$+\sum_{i=k+2}^{n}(X_{k+1}(\alpha)^{Lr}-X_i(\alpha)^{Lr})*W_i(\alpha)^{Rl}$$

$d_{Ll}(\alpha,k)$ is an increasing function with respect to k $(0\leq k\leq n-1)$, and there exists a value of k = k_{Ll}^* $(0\leq k_{Ll}^*\leq n-1)$ such that $d_{Ll}(\alpha,k_{Ll}^*-1)<0$ and $d_{Ll}(\alpha,k_{Ll}^*)\geq 0$. So k_{Ll}^* is the optimal switch point to $y_{Ll}(\alpha)$. By the same token, is the optimal switch point to $y_{Lr}(\alpha)$

(b) Let

$$d_{Rl}(\alpha,k)=-\sum_{i=1}^{k}(X_{k+1}(\alpha)^{Rl}-X_i(\alpha)^{Rl})*W_i(\alpha)^{Lr}$$
$$-\sum_{i=k+2}^{n}(X_{k+1}(\alpha)^{Rl}-X_i(\alpha)^{Rl})*W_i(\alpha)^{Lr}$$
$$d_{Rr}(\alpha,k)=-\sum_{i=1}^{k}(X_{k+1}(\alpha)^{Rr}-X_i(\alpha)^{Rr})*W_i(\alpha)^{Ll}$$
$$-\sum_{i=k+2}^{n}(X_{k+1}(\alpha)^{Rr}-X_i(\alpha)^{Rr})*W_i(\alpha)^{Rr}$$
$$\alpha\in[0,h_{\min}]$$

$d_{Rl}(\alpha,k)$ is an decreasing function with respect to k $(0\leq k\leq n-1)$, and there exists a value of $k=k_{Rl}^*$ $0<k_{Rl}^*<n-1$, such that $d_{Rl}(\alpha,k_{Rl}-1)\geq 0$ and $d_{Rl}(\alpha,k_{Rl}^*)<0$ .So k_{Rl}^* is the optimal switch point to $y_{Rl}(\alpha)$ .By the same token, k_{Rr}^* is the optimal switch point to $y_{Rr}(\alpha)$.

Proof: Similar to the process in [22]. Omitted

At first glance of Theorem I, it appears that for every α-cut level, we need to find the corresponding values of k_{Ll}^* , k_{Lr}^*, k_{Rl}^* and k_{Rr}^*, which are similar to the various current α-cut discrete algorithms; however, as we have analytical expressions for $d_{Ll}(\alpha,k_{Ll})$, $d_{Lr}(\alpha,k_{Lr})$, $d_{Rl}(\alpha,k_{Rl})$ and $d_{Rr}(\alpha,k_{Rr})$ with parameter α . We can, determine k_{Ll}^* , k_{Lr}^*, k_{Rl}^* and k_{Rr}^* for some domain of α instead of for a given specific value of α . With Theorems I, Table II gives detailed steps to obtain the analytical solution. According to Theorem I, we have changed the determination of optimal switch point k from an iterative algorithm for a specific value of α into inequalities. So, the analytical method is accurate and has no errors, which is very different from the approximate methods that connect the values for different α-cut levels of the FWA together, whose accuracy is largely dependent on the how many units one divides the α-cut domain [0, 1] into. This method also changes the optimal solution finding strategy from the point-based approximate solution to a patch-based exact solution which seamlessly covers the α-cut interval [0, 1]. Compared with other discrete α-cut based algorithms, this method is more accurate and efficient.

TABLE II THE ANALYTICAL METHOD TO COMPUTE LWA

STEP	LMF		UMF	
1	Sort $X_i(\alpha)^{Rl}$ $X_i(\alpha)^{Lr}$ (i = 1, 2, .. , n) in increasing order. $\alpha \in [0, h_{\min}]$		Sort $X_i(\alpha)^{Ll}$ $X_i(\alpha)^{Rr}$ (i = 1, 2, . , n) in increasing order. $\alpha \in [0,1]$	
2	Match the corresponding $W_i(\alpha)^{Lr}$, i = 1, 2 . . . , n, accordingly.	Match the corresponding $W_i(\alpha)^{Rl}$, i = 1, 2 . . . ,n, accordingly.	Match the corresponding $W_i(\alpha)^{Ll}$, i = 1, 2 . . . , n, accordingly.	Match the corresponding $W_i(\alpha)^{Rr}$, i = 1, 2 . . . , n, accordingly.
3	construct the left functions $d_{Lr}(\alpha, k_{Lr})$	construct the right functions $d_{Rl}(\alpha, k_{Rl})$	construct the left functions $d_{Ll}(\alpha, k_{Ll})$	construct the right functions $d_{Rr}(\alpha, k_{Rr})$
4	find optimal switch k_{Lr}^* with $d_{Lr}(\alpha, k_{Lr}^* - 1) < 0$ and $d_{Lr}(\alpha, k_{Lr}^*) \geq 0$	find optimal switch k_{Rl}^* with $d_{Rl}(\alpha, k_{Rl}^* - 1) \geq 0$ and $d_{Rl}(\alpha, k_{Rl}^*) < 0$	find optimal switch k_{Lr}^* with $d_{Lr}(\alpha, k_{Ll}^* - 1) < 0$ and $d_{Ll}(\alpha, k_{Ll}^*) \geq 0$	find optimal switch k_{Rr}^* with $d_{Rr}(\alpha, k_{Rr}^* - 1) \geq 0$ and $d_{Rr}(\alpha, k_{Rr}^*) < 0$
5	Compute $y_{Lr}(\alpha)$	Compute $y_{Rl}(\alpha)$	Compute $y_{Ll}(\alpha)$	Compute $y_{Rr}(\alpha)$
6	$\underline{Y}_{LWA}(\alpha) = [y_{Lr}(\alpha), y_{Rl}(\alpha)]$		$\overline{Y}_{LWA}(\alpha) = [y_{Ll}(\alpha), y_{Rr}(\alpha)]$	
7	$FOU(Y_{LWA}) = 1 / [\underline{Y}_{LWA}, \overline{Y}_{LWA}]$			

IV. NUMERICAL EXAMPLE

Here, we will present an LWA example to illustrate our new method. The example is taken from [15]. Let X_1=tiny, X_2=little, X_3=sizeable, W_1=small, W_2=medium, W_3=large, then the LWA can be expressed by

$$Y_{LWA} = \frac{tiny * smll + little * medium + sizable * large}{small + medium + large}$$

Here all the linguistic variables are represented by IT2 FS. FOU data for those words in the Table III. FOU data for those words. Each UMF and LMF is represented as a trapezoid. The fifth parameter for the LMF is its height.

Next, we will show the analytical solution process of obtaining UMF of the LWA problem.

The α-cuts of the above UMF numbers are:

$X_1(\alpha)^U = [X_1(\alpha)^{Ll}, X_1(\alpha)^{Rr}] = [0, 2.63 - 2\alpha]$

$X_2(\alpha)^U = [X_2(\alpha)^{Ll}, X_2(\alpha)^{Rr}] = [0.38 + 1.2\alpha, 5.62 - 2.12\alpha]$

$X_3(\alpha)^U = [X_3(\alpha)^{Ll}, X_3(\alpha)^{Rr}] = [4.38 + 2.12\alpha, 9.41 - 1.41\alpha]$

$W_1(\alpha)^U = [W_1(\alpha)^{Ll}, W_1(\alpha)^{Rr}] = [0.09 + 1.41\alpha, 4.62 - 1.22\alpha]$

$W_2(\alpha)^U = [W_2(\alpha)^{Ll}, W_2(\alpha)^{Rr}] = [3.59 + 1.16\alpha, 6.91 - 1.41\alpha]$

$W_3(\alpha)^U = [W_3(\alpha)^{Ll}, W_3(\alpha)^{Rr}] = [5.98 + 1.77\alpha, 9.52 - 0.92\alpha]$

Here we only give the computation process of the UMF of LWA with $\mu_{Y_{UMF}}(x)$.

The solution of $y_{Ll}(\alpha)$

TABLE III THE LINGUISTIC VARIABLE VALUES OF NUMERCAL EXAMPLE

	UMF	LMF
Tiny	[0, 0, 0.63, 2.63]	[0, 0, 0.09, 1.16, 1]
Small	[0.09, 1.5, 3.00, 4.62]	[1.79, 2.28, 2.28, 2.81, 0.4]
Little	[0.38, 1.58,3.5, 5.62]	[1.79, 2.2, 2.2, 2.4, 0.24]
Medium	[3.59, 4.75, 5.5, 6.91]	[4.86, 5.03, 5.03, 5.14, 0.27]
Sizeable	[4.38, 6.5, 8, 9.41]	[7.29, 7.56, 7.56, 8.21, 0.38]
Large	[5.98, 7.75, 8.6, 9.52]	[8.03, 8.36, 8.36, 9.17, 0.57]

Steps 1-2: It is obvious that for $\forall \alpha \in [0,1]$, $X_1(\alpha)^{Ll} < X_2(\alpha)^{Ll} < X_3(\alpha)^{Ll}$, so no re-ordering of the $X_i(\alpha)^{Ll}$ is need.

Step 3-4: Using the formulas for the α-cuts of UMF, construct the left difference functions $d_{Ll}(\alpha, k)$ for k = 0, 1, 2, as:

$$d_{LL}(\alpha,0)=(X_1(\alpha)^{Ll}-X_2(\alpha)^{Ll})*W_2(\alpha)^{Ll}$$
$$+(X_1(\alpha)^{Ll}-X_3(\alpha)^{Ll})*W_3(\alpha)^{Ll}$$
$$=\text{-}27.56\text{-}25.18*\alpha\text{-}5.14*\alpha^2$$
$$d_{LL}(\alpha,1)=(X_2(\alpha)^{Ll}-X_1(\alpha)^{Ll})*W_1(\alpha)^{Rr}$$
$$+(X_2(\alpha)^{Ll}-X_3(\alpha)^{Ll})*W_3(\alpha)^{Ll}$$
$$=\text{-}22.16\text{-}7.65*\alpha\text{-}3.57*\alpha^2$$
$$d_{LL}(\alpha,2)=(X_3(\alpha)^{Ll}-X_1(\alpha)^{Ll})*W_1(\alpha)^{Rr}$$
$$+(X_3(\alpha)^{Ll}-X_2(\alpha)^{Ll})*W_2(\alpha)^{Rr}$$
$$=47.88\text{-}4.73*\alpha\text{-}3.42*\alpha^2$$

Step 5: Observe that for $\forall\alpha\in[0,1], d_{Ll}(\alpha,1)<0$, and $d_{Ll}(\alpha,2)>0$, hence, $k_{\mathrm{Ll}}{}^*=2$

Step 6: Construct $y_{Ll}(\alpha)$ as :

$$y_{Ll}(\alpha)=\frac{X_1(\alpha)^{Ll}*W_1(\alpha)^{Rr}+X_2(\alpha)^{Ll}*W_2(\alpha)^{Rr}+X_3(\alpha)^{Ll}*W_3(\alpha)^{Ll}}{W_1(\alpha)^{Rr}+W_2(\alpha)^{Rr}+W_3(\alpha)^{Ll}}$$
$$=\frac{206.04\alpha^2+2818.64\alpha+2881.82}{1751\text{-}126\alpha}$$

2 The solution of $y_{Rr}(\alpha)$

Steps 1-2: It is obvious that for $\forall\alpha\in[0,1]$, $X_1(\alpha)^{Rr}\le X_2(\alpha)^{Rr}\le X_3(\alpha)^{Rr}$, so no re-ordering of the $X_i(\alpha)^{Rr}$ is need.

Step 3-4: Using the formulas for the α-cuts of UMF, construct the right differencial e functions $d_{Rr}(\alpha)$ for k = 0, 1, 2, as:

$$d_{Rr}(\alpha,0)=-(X_1(\alpha)^{Rr}-X_2(\alpha)^{Rr})*W_2(\alpha)^{Rr}$$
$$-(X_1(\alpha)^{Rr}-X_3(\alpha)^{Rr})*W_3(\alpha)^{Rr}$$
$$=85.21\text{-}5.67*\alpha\text{-}.374*\alpha^2$$
$$d_{Rr}(\alpha,1)=-(X_2(\alpha)^{Rr}-X_1(\alpha)^{Rr})*W_1(\alpha)^{Ll}$$
$$-(X_2(\alpha)^{Rr}-X_3(\alpha)^{Rr})*W_3(\alpha)^{Rr}$$
$$=35.81\text{-}0.93*\alpha\text{-}0.48*\alpha^2$$
$$d_{Rr}(\alpha,2)=-(X_3(\alpha)^{Rr}-X_1(\alpha)^{Rr})*W_1(\alpha)^{Ll}$$
$$-(X_3(\alpha)^{Rr}-X_2(\alpha)^{Rr})*W_3(\alpha)^{Ll}$$
$$=\text{-}14.22\text{-}16.56*\alpha\text{-}1.66*\alpha^2$$

Step 5: Observe that for $\forall\alpha\in[0,1]$, $d_{Rr}(\alpha,1)>0$ and $d_{Rr}(\alpha,2)<0$, hence, $k_{\mathrm{Rr}}{}^*=2$

Step 6: Construct $y_{Rr}(\alpha)$ as:

$$y_{Rr}(\alpha)=\frac{X_1(\alpha)^{Rr}*W_1(\alpha)^{Ll}}{W_1(\alpha)^{Ll}+W_2(\alpha)^{Ll}+W_3(\alpha)^{Rr}}$$
$$+\frac{X_2(\alpha)^{Rr}W_2(\alpha)^{Ll}}{W_1(\alpha)^{Ll}+W_2(\alpha)^{Ll}+W_3(\alpha)^{Rr}}$$
$$+\frac{X_3(\alpha)^{Rr}W_3(\alpha)^{Rr}}{W_1(\alpha)^{Ll}+W_2(\alpha)^{Ll}+W_3(\alpha)^{Rr}}$$
$$=\frac{110\text{-}19.64\alpha-3.98\alpha^2}{13.2+1.65\alpha}$$

Step 7: By computing inverse functions of $y_{Ll}(\alpha)$ and $y_{Rr}(\alpha)$, respectively, the closed form UMF solution of LMW, Y_{UMF} can be obtained as

$$\mu_{Y_{UMF}}(x)=\begin{cases}\text{-}6.84\text{-}0.31x \\ +0.97\sqrt{34.81+13.46x+0.1x^2}, \\ \qquad\qquad 1.65<x<0.63; \\ 1, \qquad 3.63<x<\ 5.82; \\ \text{-}2.47\text{-}0.21x \\ +0.13e\text{-}4\sqrt{2138\text{-}145x+2.72x^2}, \\ \qquad\qquad 5.82\ <x<8.33\end{cases}$$

In a similar way, the analytical solution of LMF of the LWA problem can also be obtained as

$$\mu_{Y_{LMF}}(x)=\begin{cases}31.64+2.02*x \\ \text{-}0.18*\sqrt{40071+2313.51*x+131.79*x^2}, \\ \qquad\qquad 4.2\ <\ x\ <\ 4.6; \\ 0.24, \qquad 4.6<x\ <5.2; \\ \text{-}0.64*x\text{-}9.98 \\ +0.9*\sqrt{0.51*x^2\text{-}1.72*x+221}, \\ \qquad\qquad 5.2<\ x\ <\ 5.6;\end{cases}$$

Thus, we have obtained the analytical solution of LWA with its FOU as $\mu_{Y_{LMF}}(x)$ and $\mu_{Y_{UMF}}(x)$ respectively.

V. Conclusions

For the linguistic weighted average (LWA) problems expressed with interval type-2 fuzzy sets, to overcome the computational inefficiency and in accurate of the current α -cut method, the paper proposes an analytical solution method for LWA problem. Some properties of the solutions are discussed, and new algorithms to obtain the analytical solution are designed. A numerical example is used to illustrate our new proposed methods and algorithms.

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