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Building a Type-2 Fuzzy Regression Model Based on Credibility Theory and Its Application on Arbitrage Pricing Theory

Yicheng Wei and Junzo Watada
Graduate School of Information, Production and Systems, Waseda University
2-7 Hibikino, Wakamatsu-ku, Kitakyushu, Fukuoka 808-0135, Japan
E-mail:weiyicheng0000@gmail.com, junzow@osb.att.ne.jp

***Abstract*—Real life circumstances used to provide us with linguistically vague expression of data in nature. Thus, type-1 fuzzy set (T1F set) was introduced to model this uncertainty. Additionally, same words will mean variously to different people, which means ambiguous uncertainty also exists when associated with the membership function of a T1F set. Type-2 fuzzy set(T2F set) is then invented to express the hybrid uncertainty of both primary fuzziness and secondary one of membership functions. On the one hand, T2F variable models the vagueness of information better. On the other hand, those variables are hard to deal with its three-dimensional feature given. To address problems in presence of such variables with hybrid fuzziness, a new class of T2F regression model is built based on credibility theory,called the T2F expected value regression model. The new model will be developed in this paper. This paper is a further work based on our former research of T2F qualitative regression model.**

***Index Terms*—Type-2 fuzzy set, regression model, Credibility theory, expected value,**

I. Introduction

In daily life people face linguistically vague information. Using characteristic function to define whether an element belongs to a certain set (event),traditional set theory is rigorous without such uncertainty. To deal with the problem, fuzzy set (T1F set) was first introduced in 1965 by Lofti A Zadeh [29]. After that, Watada and Tanaka developed a fuzzy quantification method in 1987 [23]. From then on, it is able to describe an artificial membership function with its output called primary membership grades, to which extend one element belongs to a certain set (event).

On the background that the membership function of a T1F set may also have uncertainty associated with it, Lotfi A. Zadeh invented Type-2 fuzzy sets(T2F set) in 1975 [33]. A T2F set lets us incorporate fuzziness about the membership function into fuzzy set theory and is a way to address the above concern of T1F sets head-on. However, T2F set didn't become popular immediately given its complexity of calculation. T2F sets are difficult to understand and use because: (1) the three-dimensional nature of T2F sets makes them difficult to handel. (2) using T2F sets is computationally more complicated than using T1F sets. Thus, the conception was only investigated by a few researchers; for example, Mizumoto and Tanaka [12] discussed what kinds of algebraic structures the grades of T1F sets form under join, meet and negation; Dubois and Prade [3] investigated the operations in a fuzzy-valued logic. It is not until recent days that T2F sets have been applied successfully to T2F logic systems to handle linguistic and numerical uncertainties [4], [5], [7], [8], [28].

On the other hand, various fuzzy regression models were introduced to cope with qualitative data coming from uncertain environments where human (expert) subjective estimates are concerned. The first fuzzy linear regression model was proposed by Tanaka et al. [17]. Tanakat et al.[20], Tanaka and Watada[18], Watada and Tanaka[22] presented possibilistic regression based on the concept of possibility measure. Chang[2] discussed a fuzzy least-squares regression, by using weighted fuzzy-arithmetic and the least-squares fitting criterion. Watada[24] developed models of fuzzy time-series by exploiting the concept of intersection of fuzzy numbers.

Most of the existing studies on modeling fuzzy regression analysis have focused on data consisting of numeric values or T1F variables without T2 hybrid uncertainty into consideration. In practical situations, there exists a growing need to cope with data in presence of more complicated uncertainty. However, with regard to the complexity of T2F variables, there are only a few mathematical algorithms learning T2F inputs and predicting T2F outputs [1] [6] [11] [15] [16] [34] [21]. Recently, Wei and Watada developed a T2F qualitative regression model [25][26][27] using possibilistic structure built by Tanaka and Watada.

However, the model is not completed, for it only applies T2F variables as coefficients of system but inputs and outputs. Liu [9] and Liu and Liu [10] created a notion of credibility measure for fuzzy sets, which is a convex combination of possibility measure and necessity measure. We are able to make the calculation of fuzzy sets much easier than before by using the credibility theory. We obtained the idea to build an advanced T2F regression algorithm.

Motivated by the above reasoning, the objective of this paper is to introduce a class of T2F regression model based on credibility theory to deal with T2F inputs and outputs. We use credibility theory introduced by Liu [9] to define the expected value of a T2F variable. After that, we transfer the T2F variable into T2F expected value and build an credibility-based T2F expected value regression model with the expected value. This paper will be a further work based on our former research of T2F qualitative regression model.

The remainder of this paper is organized as follows. In Section 2, we cover some preliminaries of credibility theory and T2F sets. Then we define the expected value T2F set and T2F variable. Notice that these two conceptions of expected values are different. Section 3 discusses a general T2F regression and introduce a new approach to T2F regression in section 4. In section 5, an algorithm is offered to solve the model. Section 6 provides an application for our model. Finally, concluding remarks are presented in Section 6.

II. Preliminaries

A. Credibility Theory

Recently, Liu has succeeded in establishing an axiomatic foundation for uncertainty. He created a notion of credibility measure, which is a convex combination of possibility measure and necessity measure. Given some universe Γ, let Pos be a possibility measure defined on the power set $\mathcal{P}(\Gamma)$ of Γ. Let $\Re$ be the set of real numbers. A function $A : \Gamma \to \Re$ is said to be a fuzzy variable defined on Γ . The possibility distribution μ_A of A is defined by $\mathrm{Pos}\{A = t\} \equiv \mu_A(t)$, $t \in \Re$, which is the possibility of event $\{A = t\}$. For fuzzy variable A with possibility distribution μ_A, the possibility, necessity and credibility of event $\{A \le r\}$ are given as follows

$$\begin{aligned} &\mathrm{Pos}\{A \le r\} = \sup_{t \le r} \mu_A(t), \\ &\mathrm{Nec}\{A \le r\} = 1 - \sup_{t > r} \mu_A(t), \\ &\mathrm{Cr}\{A \le r\} = \frac{1}{2}\left(1 + \sup_{t \le r} \mu_A(t) - \sup_{t > r} \mu_A(t)\right). \end{aligned} \tag{1}$$

Note that equqtion (1) defines the credibility measure with an average of the possibility and the necessity measure, i.e., $\mathrm{Cr}\{\cdot\} = (\mathrm{Pos}\{\cdot\} + \mathrm{Nec}\{\cdot\})/2$, and it is a self-dual set function, i.e., $\mathrm{Cr}\{A\} = 1 - \mathrm{Cr}\{A^c\}$ for any A in $\mathcal{P}(\Gamma)$. The credibility theory is first introduced by Liu [9]. The motivation behind the introduction of the credibility measure is to develop a certain measure which is a sound aggregate of the two extreme cases such as the possibility (expressing a level of overlap and being highly optimistic in this sense) and necessity (articulating a degree of inclusion and being pessimistic in its nature). Moreover, we are able to calculate the expected value of a fuzzy set from then on. For fuzzy variables, there are many ways to define an expected value operator. See, for example, Dubois and Prade [3]]. The most general definition of expected value operator of fuzzy variable was given by Liu and Liu [9]. Based on credibility measure, the expected value of a fuzzy variable is presented as follows.

Let A be a fuzzy variable. The expected value of A is defined as

$$E[A] = \int_0^{\infty} \mathrm{Cr}\{A \ge r\}\mathrm{d}r - \int_{-\infty}^{0} \mathrm{Cr}\{A \le r\}\mathrm{d}r \tag{2}$$

provided that the two integrals are finite.

Assume that $A = (a, c^l, c^r)_T$ is a triangular fuzzy variable whose possibility distribution is

$$\mu_A(x) = \begin{cases} \dfrac{x - c^l}{a - c^l}, & c^l \le x \le a \\ \dfrac{c^r - x}{c^r - a}, & a \le x \le c^r \\ 0, & \text{otherwise.} \end{cases}$$

Making use of (2), we calculate the expected value of A to be

$$E[A] = \frac{c^l + 2a + c^r}{4}.$$

B. Type-2 Fuzzy Set

Developed from T1F sets, T2F sets express the non-numeric membership with imprecision and uncertainty. A T2F set denoted by $\widetilde{A}$, is characterized by a T2 membership function $\mu_{\widetilde{A}}(x, \mu_A(x))$, where $x \in X$ and $\mu_A(x) \in J_x \subseteq [0, 1]$. $\mu_A(x)$ are called primary memberships of x in A and the memberships of the primary memberships, $\mu_{\widetilde{A}}(x)$, are called secondary memberships of x in $\widetilde{A}$. i.e.,

$$\begin{aligned} A &= \{x, \mu_A(x) | x \in X\} && (3) \\ \widetilde{A} &= \{(x, \mu_A(x)), \mu_{\widetilde{A}}(x, \mu_A(x)) | \\ & \qquad x \in X \;\; \mu_A(x) \in J_x \subseteq [0, 1]\} && (4) \end{aligned}$$

in which $\mu_A(x) \in J_x \subseteq [0, 1]$ and $\mu_{\widetilde{A}} \subseteq [0, 1]$. T2F sets $\widetilde{A}$ can also be expressed as

$$\int_{(x \in X)} \int_{(\mu_A(x) \in J_x \subseteq [0,1])} \mu_{\widetilde{A}}(x, \mu_A(x))/(x, \mu_A) \tag{5}$$

Regarding T2F set, another important concept is the footprint of uncertainty. Uncertainty in the primary memberships of a T2F set, consists of a bounded region that we call the footprint of uncertainty (FOU). It is the union of all primary memberships, i.e.,

$$FOU(\widetilde{A}) = \bigcup_{x \in X} J_x \tag{6}$$

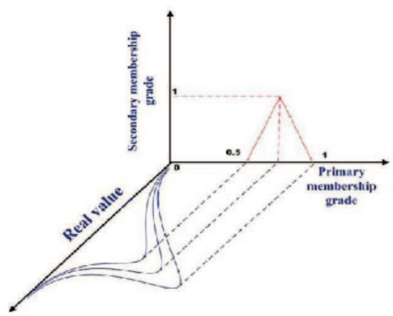

Fig. 1. Pictorial Representation of a T2F set

The term footprint of uncertaintyis useful, because it not only focuses our attention on the certainties inherent in a specific T2 membership function, whose shape is a direct consequence of the nature of these uncertainties, but it also provides a convenient verbal description of the entire domain of support for all the secondary grades of a T2 membership function. It also enables us to depict a T2F set graphically in two-dimensions instead of three dimensions, which let us overcome the first difficulty about T2F sets-their three-dimensional nature which makes them very difficult to draw. The shaded FOUs imply that there is a distribution that sits on top of it the new third dimension of T2F sets. What that distribution looks like depends on the specific choice made for the secondary grades.

C. Expected Value of T2F set and T2F variable based on Credibility Theory

After introducing credibility and T2F sets, we define the expected value of T2F set using credibility measure here. We may see later that using the expected value and variance to model T2F variables will reduce huge complexity in the process of calculation and enable us to deal with T2F variables.

Suppose that $(\Gamma_1, \mathcal{P}(\Gamma_1), \mathrm{Pos}_1)$ is a possibility space. Let Γ_1 be the universe of discourse, and $\mathcal{P}(\Gamma_1)$ on Γ_1 is a class of subsets of Γ_1 that is closed under arbitrary unions, intersections, and complement in Γ_1.

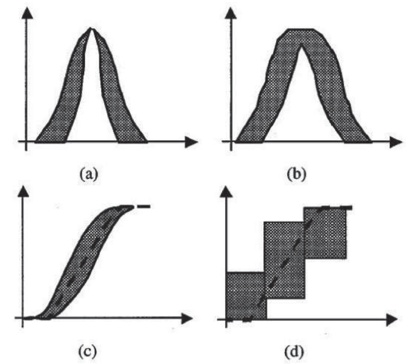

Fig. 2. FOU of a T2F set

Let $\Re$ be the set of real numbers. Then a map $A : \Gamma_1 \to \Re$ is said to be a T1F set defined on Γ_1. We also define another possibility space for T2F sets, which is $(\Gamma_2, \mathcal{P}(\Gamma_2), \mathrm{Pos}_2)$. A function $\mathrm{Pos}_2 : \mathcal{P}(\Gamma_2) \to [0, 1]$ and a T2F set is a mapping $\widetilde{A} : \Gamma_2 \to \Gamma_1$. The most normal T2F set is constructed in a triangular style which is illustrated as follows:

Let A be a fuzzy set defined on possibility space $(\Gamma_1, \mathcal{P}(\Gamma_1), \mathrm{Pos})$. Define that for every $A \in \Gamma_1$, $\widetilde{A}(A) = (A + a, A + c^l, A + c^r)\Gamma_2$, which is a triangular T2F set defined on some possibility space $(\Gamma_2, \mathcal{P}(\Gamma_2), \mathrm{Pos})$.

Moreover, a T2F variable is defined as a mapping $\widetilde{A}_v : \Gamma_2 \to \Re$. The normal form of a T2F variable is in the form of interval and a more complicated one is in the form of triangle defined as follows:

Let A be a fuzzy variable defined on possibility space $(\Gamma_1, \mathcal{P}(\Gamma_1), \mathrm{Pos})$. For every $x \in \Re$, it follows a triangular primary membership function as $A(x) = (x + a, x + c^l, x + c^r)\Gamma_1$ and for every $A \in \Gamma_1$, it also follows a triangular secondary membership function $\widetilde{A}(\mu_A) = (\mu_A(x_0), \mu_A(x_0) + e^l, \mu_A(x_0) + e^r)\Gamma_2$ which is a T2F set defined on some possibility space $(\Gamma_2, \mathcal{P}(\Gamma_2), \mathrm{Pos})$. Here, T1F set A and its membership function $\mu_A(x)$ roughly show the same meaning while they have a difference indeed.

For any T2F set $\widetilde{A}$ on Γ_2, for each A in Γ_1, the expected value of the T2F set $\widetilde{A}$ is denoted by $E[\widetilde{A}]$ given μ_A, which has been proved to be a measurable function of A i.e., it is a T1F variable. The mathmatical definition is as follows:

Let $\widetilde{A}_v$ be a T2F variable and $\widetilde{A}$ be a T2F set defined on a possibility space $(\Omega, P(\Omega), \mathrm{Pos})$, which describes the secondary membership grades for$\widetilde{A}_v$. The expected value of T2F set $\widetilde{A}_v$ is defined as follows.

$$\boldsymbol{E}[\widetilde{A}] = \int_{0}^{\infty} \mathrm{Cr}_{\widetilde{A}}\{\widetilde{A} \geq \mu_A(x)\}\mathrm{d}\mu_A(x) - \int_{-\infty}^{0} \mathrm{Cr}_{\widetilde{A}}\{\widetilde{A} \leq \mu_A(x)\}(\mu_A(x)) \quad (7)$$

Assume that original outputs for the primary membership function of $\widetilde{A_v}$ is the following possibility distribution of $\mu_A(x_0)$, where $x_0 \in \Re$. To get the expected value of T2F variable, we may take the place of the original primary grades by using the result of equation(7) Hence, μ_A will be transformed into a new function defined as expected primary membership function denoted as $\mu_{E[\widetilde{A}]}$.Its credibility is denoted as $\mathrm{Cr}_{E[\widetilde{A}]}$ instead of Cr_A. We called this process as "**Reduction**". The expected value of T2F variable is then defined as follows:

$$E[\widetilde{A}_v] = \int_{0}^{\infty} \mathrm{Cr}_{E[\widetilde{A}]}\{A \geq \boldsymbol{x}\}\mathrm{d}\boldsymbol{x} - \int_{-\infty}^{0} \mathrm{Cr}_{E[\widetilde{A}]}\{A \leq \boldsymbol{x}\}\mathrm{d}\boldsymbol{x} \quad (8)$$

Furthermore, we will give a simple example here to help understand the definition. Assume that there is a triangular T2F variable $\widetilde{A}_0$. Its primary membership function for real values is $A = (a, c^l, c^r)\Gamma_1$ whose possibility distribution is

$$\mu_A(x) = \begin{cases} \dfrac{x - c^l}{a - c^l}, & c^l \leq x \leq a \\ \dfrac{c^r - x}{c^r - a}, & a \leq x \leq c^r \\ 0, & \text{otherwise.} \end{cases}$$

Meanwhile, for any $x_0 \in \Re$ included in $A = (a, c^l, c^r)\Gamma_1$ with its primary grades expressed as $\mu_A(x_0)$, the secondary one for $\mu_A(x_0)$ is assumed to be $\widetilde{A}(\mu_A(x_0)) = (\mu_A(x_0), \mu_A(x_0) + e^l, \mu_A(x_0) + e^r)\Gamma_2$, whose possibility distribution is

$$\mu_{\widetilde{A}}(\mu_A(x)) = \begin{cases} \dfrac{\mu_A(x) - e^l}{\mu_A(x_0) - e^l}, & e^l \leq \mu_A(x) \leq \mu_A(x_0) \\ \dfrac{e^r - \mu_A(x)}{e^r - \mu_A(x_0)}, & \mu_A(x_0) \leq \mu_A(x) \leq e^r \\ 0, & \text{otherwise.} \end{cases}$$

where $A \in \Gamma_1$, $\widetilde{A} \in \Gamma_2$. Notice that $\mu_A(x_0)$ will be the center of the T2F membership grades.

We may calculate those expected values for T2F sets first.

For boundaries of primary grades of $\widetilde{A}_0$ where values of them are 0, the secondary nonmembership function is $\widetilde{A}(0) = (0, 0 + e^l, 0 + e^r)\Gamma_2$, and the expected value of them according to equation (7) will be

$$E[A] = \frac{e^l + e^r}{4}.$$

For center of primary grades where its value is 1, the secondary membership function is $\widetilde{A}(b) = (1, 1 + e^l, 1 + e^r)\Gamma_2$, and the expected value will be

$$E[A] = \frac{e^l + 2 + e^r}{4}.$$

It should be noticed that even there will be surplus more than 1 in the calculation process, it will not affect the final result.

Then we will perform the "**Reduction**" by substituting expected primary grades for original ones and form an expected primary membership function. The original one satisfies $\mu_A(a) = 1$, $\mu_A(c^l) = 0$ and $\mu_A(c^r) = 0$ as shown in Fig. 3. After transformation, the expected one follows distribution as the following equation and satisfies $\mu_A(a) = \dfrac{e^l + 2b + e^r}{4}$, $\mu_A(c^l) = \dfrac{e^l + e^r}{4}$ and $\mu_A(c^r) = \dfrac{e^l + e^r}{4}$ as shown in (1) in Fig. 4.

$$\mu_{E[\widetilde{A}]}(x) = \begin{cases} \dfrac{x - c^l}{2(a - c^l)} + \dfrac{e^l + e^r}{4}, & c^l \leq x \leq a \\ \dfrac{c^r - x}{2(c^r - a)} + \dfrac{e^l + e^r}{4}, & a \leq x \leq c^r \\ \dfrac{e^l + e^r}{4}, & \text{otherwise.} \end{cases}$$

Noticed that after transformation into the expected primary function, the function biases a value at $\frac{e^l + e^r}{4}$ higher for the universe, which is not reasonable. Thus we need to make some adjustment through shaping the function into triangular one again We call it the process of **"Shaving"**, which looses the original boundaries a little bit. Thus, the new expected T1F set of $\widetilde{A}_0$ will be $A = (a, c^l - \dfrac{(e^r + e^l)(a - c^l)}{2}, c^r + \dfrac{(e^r + e^l)(c^r - a)}{2})\Gamma_1$, whose possibility distribution follows:

$$\mu_{E[\widetilde{A}]}(x) = \begin{cases} \dfrac{x - c^l}{2(a - c^l)}, & c^l - \frac{(e^r + e^l)(a - c^l)}{2} \leq x \leq a \\ \dfrac{c^r - x}{2(c^r - a)}, & a \leq x \leq c^r + \dfrac{(e^r + e^l)(c^r - a)}{2} \\ 0, & \text{otherwise.} \end{cases}$$

According to the definition:

$$\mathrm{Cr}\{A \leq r\} = \frac{1}{2}\left(1 + \sup_{t \leq r} \mu_A(t) - \sup_{t > r} \mu_A(t)\right).$$

The credibility of the expected primary function will be written as

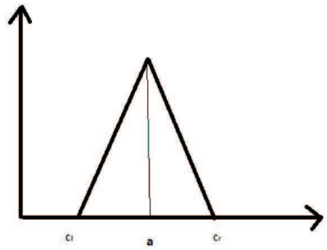

Fig. 3. Original primary function

$$\Phi(x) = \mathrm{Cr}_{E[\widetilde{A}]}(\epsilon \le x) = \begin{cases} 0, & x \le \gamma \\ \dfrac{x - \dfrac{(e^r+e^l)(a-c^l)}{2}}{4(a - c^l + \dfrac{(e^r+e^l)(a-c^l)}{2})}, & \gamma \le x \le a \\ \dfrac{c^r + \dfrac{(e^r+e^l)(c^r-a)}{2} + x - 2a}{4(c^r - a + \dfrac{(e^r+e^l)(c^r-a)}{2})}, & a \le x \le \zeta \\ 1, & \zeta \ge x \end{cases} \tag{9}$$

where $\gamma{=}c^l - \frac{(e^r+e^l)(a-c^l)}{2}$ and $\zeta{=}c^r + \frac{(e^r+e^l)(c^r-a)}{2}$

Now we are able to calculate the expected value according to equations (8) and (9) as follows

$$\begin{aligned} E[\widetilde{A}_0] &= a + \frac{\frac{(e^r+e^l)(c^r-a)}{2} + c^r - a}{8} \\ &+ \frac{c^l - \frac{(e^r+e^l)(a-c^l)}{2} - a}{8} \\ &= \frac{12a + 2c^r + 2c^l + (e^r+e^l)(c^r+c^l-2a)}{16} \end{aligned}$$

III. Formulation of a class of T2F regression model

The arithmetic operations of T1F set have been studied by making use of the extension principle [13], [14], [33], [33], [33]. These studies have involved the definition of possibility. Tanaka and Watada figured out that fuzzy equations discussed by Sanchez can be regarded as a possibilistic structure [20] [18]. In the sequel, a possibilistic system has been applied to the linear regression analysis, A possibility structure has much advantage to deal with inputs and outputs with uncertainty. We will use the structure here to formulate the T2F regression.

For a T2F regression model, input data $\widetilde{X}_{ij}$ and output data $\widetilde{Y}_i$, for all $i = 1.\cdots,N$ and $k = 1,\cdots,M$ are T2F variables, which are defined as

$$\begin{aligned} &\widetilde{Y}_i = (y_i, \mu_{Y_i}(y_i), \mu_{\widetilde{Y_i}}(y_i)) \\ &\widetilde{X}_{ij} = (x_{ij}, \mu_{X_{ij}}(x_{ij}), \mu_{\widetilde{X_{ij}}}(x_{ij})) \end{aligned} \tag{10}$$

respectively.

y_i and x_{ij} are the crisp value; $\mu_{Y_i}(y_i)$ and $\mu_{X_{ij}}(x_{ij})$ are primary membership grades for y_i and x_{ij}; $\mu_{\widetilde{Y_i}}(y_i)$ and $\mu_{\widetilde{X_{ij}}}(x_{ij})$ are secondary membership grades for $\mu_{Y_i}(y_i)$ and $\mu_{X_{ij}}(x_{ij})$. These three factors construct the basis for a T2F variable. i denotes sample i for $i = 1,\cdots,N$; j denotes for the jth attributes for $j = 1,2,\cdots,M$.

As discussed before, we will use a possibilistic structure here. Let us denote fuzzy linear regression model with T1F coefficients $A_1,\cdots,A_M$. Then the T2F regression is in the form as follows:

$$\widetilde{Y}_i = A_1\widetilde{X}_{i1} + A_2\widetilde{X}_{i2} + \cdots + A_M\widetilde{X}_{iM}, \tag{11}$$

where $\widetilde{Y}_i$ denotes an estimate of the T2F output and $A_j = \left(\frac{A_j^l+A_j^r}{2}, A_j^l, A_j^r\right)_T$ are symmetric triangular fuzzy coefficients when triangular T2F data $\widetilde{X_{ij}}$ are given for $i = 1,\cdots,N$ and $j = 1,\cdots,M$.

When outputs and inputs are defined as crisp value or T1F variables, it is easy to determine the linear regression model's parameters by satisfying the estimated model includes all given outputs. We will mimic this process to formulate the T2F regression model as follows, while all the inputs and outputs are T2 fuzzy variables.

$$\begin{aligned} \widetilde{Y}_i &= A_1\widetilde{X}_{i1} + A_2\widetilde{X}_{i2} + \cdots + A_M\widetilde{X}_{iM} \underset{FR}{\supset} \widetilde{Y}_i, \\ & i = 1,\dots N, \end{aligned} \tag{12}$$

where $\underset{FR}{\supset}$ is a T2F inclusion relation whose precise meaning will be explained later on.

IV. Building an expected value T2F regression model with credibility theory

We may reform an expected primary membership function range for $\widetilde{X}_v(ij)$ through reduction process that we have defined above to reduce the dimensions of data. According to the process, the $\widetilde{X}_v(ij)$ will transformed into a range of $[E[\widetilde{X}_v(ij)^C], E[\widetilde{X}_v(ij)^B]]$. We will assume the inputs will all be symmetric triangular T2F variables whose primary membership functions and secondary ones are all in triangular form with a center C and two equal distance boundaries B. Thus, $E[\widetilde{X}_v(ij)^C]$ represents the center. $E[\widetilde{X}_v(ij)^B]$ represents the bound. To simplify, we will use $[e_{\widetilde{X}_v(ij)}, \delta_{\widetilde{X}_v(ij)}]$ instead of the original expression and $[e_{\widetilde{Y}_v(i)}, \delta_{\widetilde{Y}_v(i)}]$ instead of $Y_{E[\widetilde{Y}_v(i)]}$.Here is $e_{\widetilde{X}_v(ij)}$

represents the center of expected values calculated by credibility theory $e_{\widetilde{X}_v(ij)} - \delta_{\widetilde{X}_v(ij)}$ and $e_{\widetilde{X}_v(ij)} + \delta_{\widetilde{X}_v(ij)}$ are the bonds of set of expected values calculated by credibility theory. The same is to $[e_{\widetilde{Y}_v(i)}, \delta_{\widetilde{Y}_v(i)}]$.

Thus the T2F regression model will be reformulated as follows

[T2F expected value regression model]

$$\left.\begin{array}{ll} \min\limits_{A} & J(A) = \sum\limits_{j=1}^{M} \left(A_j^r - A_j^l \right) \\ \text{subject to} & \\ & A_j^r \geq A_j^l, \\ & Y_i = \sum\limits_{j=1}^{M} A_j \cdot [e_{\widetilde{X}_v(ij)}, \delta_{\widetilde{X}_v(ij)}] \\ & \quad \underset{h}{\underset{\sim}{\supseteq}} [e_{\widetilde{Y}_v(i)}, \delta_{\widetilde{Y}_v(i)}], \\ & \text{for} \quad i = 1, \ldots N, j = 1, \cdots, M. \end{array}\right\} \quad (13)$$

TABLE I
T2F INPUT -T2F OUTPUT DATA TRANSFORMED IN THE RANGE OF EXPECTED PRIMARY GRADES

Sample	Output	Inputs		
i	$[e_Y, \delta_Y]$	$[e_{X_1}, \delta_{X_1}]$	$\cdots$	$[e_{X_K}, \delta_{X_K}]$
1	$[e_{Y_1}, \delta_{Y_1}]$	$e_{X_{11}}, \delta_{X_{11}}]$	$\cdots$	$[e_{X_{1K}}, \delta_{X_{1K}}]$
2	$[e_{Y_2}, \delta_{Y_2}]$	$[e_{X_{21}}, \delta_{X_{21}}]$	$\cdots$	$[e_{X_{2K}}, \delta_{X_{2K}}]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$[e_{Y_i}, \delta_{Y_i}]$	$[e_{X_{i1}}, \delta_{X_{i1}}]$	$\cdots$	$[e_{X_{iK}}, \delta_{X_{iK}}]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
N	$[e_{Y_N}, \delta_{Y_N}]$	$[e_{X_{N1}}, \delta_{X_{N1}}]$	$\cdots$	$[e_{X_{NK}}, \delta_{X_{NK}}]$

V. THE SOLUTION OF THE MODEL

The solution of the model can be rewritten as a problem of N samples with one output and M input interval values. This problem is hard to solve, since it consists of $N \times M$ products between the fuzzy coefficients and primary grades intervals. In order to solve the proposed model, we can employ a vertex method as given below. That is, these multidimensional vertices are taken as new sample points with fuzzy output numbers. In the sequel, we can solve this problem using the conventional method.

T2F regression model can be developed to include the mean interval values of all samples in the model. Therefore, it is sufficient and necessary to consider only both two vertices of the end points of the interval of each dimension of a sample. For example, one sample with one input interval feature can be expressed with two vertices of the end points of the interval with a fuzzy output value. As a consequence, in T2F regression form, if we denote I_{ij}^L and I_{ij}^U left and right end points of the expected primary grade intervals of the input X_{ik}, respectively, that is

$I_{ij}^L = e_{\widetilde{X}_v(ij)} - \delta_{\widetilde{X}_v(ij)}, \quad I_{ij}^U = e_{\widetilde{X}_v(ij)} + \delta_{\widetilde{X}_v(ij)},$

for $i = 1, 2, \cdots, N, j = 1, 2, \cdots, M$, the original model can be converted into the following conventional fuzzy regression model by making use of the vertice method:

$$\left.\begin{array}{lll} & \min\limits_{A} & J(A) = \sum\limits_{j=1}^{M} \left(A_j^r - A_j^l \right) \\ \text{subject to} & & \\ & & A_j^r \geq A_j^l, \\ (1) \rightarrow & & Y_i = A_1 \cdot I_{i1}^L + A.I_i^L + \cdots \\ & & \quad + A_M \cdot I_{iM}^L \underset{h}{\underset{\sim}{\supseteq}} [e_{\widetilde{Y}_v(i)}, \delta_{\widetilde{Y}_v(i)}], \\ (2) \rightarrow & & Y_i = A_1 \cdot I_{i1}^U + A_2 \cdot I_{i2}^L + \cdots \\ & & \quad + A_M \cdot I_{iM}^L \underset{h}{\underset{\sim}{\supseteq}} [e_{\widetilde{Y}_v(i)}, \delta_{\widetilde{Y}_v(i)}], \\ (3) \rightarrow & & Y_i = A_1 \cdot I_{i1}^L + A_2 \cdot I_{i2}^U + \cdots \\ & & \quad + A_M \cdot I_{iM}^L \underset{h}{\underset{\sim}{\supseteq}} [e_{\widetilde{Y}_v(i)}, \delta_{\widetilde{Y}_v(i)}], \\ \vdots & \vdots & \vdots \\ (2^M) \rightarrow & & Y_i = A_1 \cdot I_{i1}^U + A_2 \cdot I_{i2}^U + \cdots \\ & & \quad + A_K \cdot I_{iM}^U \underset{h}{\underset{\sim}{\supseteq}} [e_{\widetilde{Y}_v(i)}, \delta_{\widetilde{Y}_v(i)}], \\ & & \text{for} \quad i = 1, \ldots N, j = 1, \cdots, M \end{array}\right\} \quad (14)$$

where $\underset{h}{\underset{\sim}{\supseteq}}$ denotes the fuzzy inclusion relation realized at level h.

We may also define for the output:

$I_{Y_i}^L = [e_{\widetilde{Y}_v(i)} - \delta_{\widetilde{Y}_v(i)}], \quad I_{Y_i}^U = [e_{\widetilde{Y}_v(i)} + \delta_{\widetilde{Y}_v(i)}]$

The regression model can be easily solved by transferring $\underset{h}{\underset{\sim}{\supseteq}}$ into inequalities. And this conventional solution is explained in detail in our former work [**?**].

VI. APPLICATION ON ARBITRAGE PRICING THEORY

In this section, we will expand a famous pricing theory in finance, which is the arbitrage pricing theory(APT). We will use T2F regression model based on credibility theory to redefine the formula of APT and to show the benefit of the new model in practice.

A. Mathmetical model of Arbitrage Pricing Theory

In finance, arbitrage pricing theory (APT) is a general theory of asset pricing that holds that the expected return of a financial asset can be modeled as a linear function of various macro-economic factors or theoretical market indices, where sensitivity to changes in each factor is represented by a factor-specific beta coefficient.

The form of arbitrage pricing theory is like:

$$\boldsymbol{E}(r_j) = r_f + b_{j1} * RP_1 + \cdots, b_{jn} * RP_n,$$

where $E(r_j)$ is the expected return of the jth asset; RP_k is the risk premium of the kth factor; r_f is the risk-free rate. That is, the expected return of an asset j is a linear function of the asset's sensitivities to the n factors.

B. Arbitrage Pricing Theory with T2F regression based on Credibility Theory

Chen, Roll and Ross (1986) identified the following macro-economic factors as significant in explaining security returns: 1)surprises in inflation;2)surprises in GNP as indicated by an industrial production index; 3)surprises in investor confidence due to changes in default premium in corporate bonds; 4)surprise shifts in the yield curve. However, these factors are with great fuzziness due to their macro feature. So it comes to us to express these factors in the form of T2F sets. The form of T2F sets are capable of containing more information and will cater the feature of macroeconomical data. Moreover, we try to use our new model to deal with these T2F data.

Thus, we expanded the traditional APT into the T2F APT as follows:

$$\widetilde{E(r_j)} = r_f + A_1\widetilde{RP_1} + A_2\widetilde{RP_2} + \cdots + A_M\widetilde{RP_M} \quad (15)$$

where coefficients A in the formula are T1F sets. The expected return of the jth asset $E(r_j)$ is in the form of T2F sets. The risk premium of the kth factor RP_k is T2F sets as well ;

Because we don't have macro data on hand, we have choosed the yearly return of the stock of one lisiting company as the output for our model. Meanwhile, we have choosed the yearly performance of the whole stock market as one variable and the growth of the company's sales as the other variable. All the data are showing in the form of percentage.

First, we need to reduce the T2F inputs and outputs into T1F set. Then it will meet the form of T2FR. According to our former deduction, the new expected T1F set of $\widetilde{A}$ will be $A = (a, c^l - \frac{(e^r+e^l)(a-c^l)}{2}, c^r + \frac{(e^r+e^l)(c^r-a)}{2})\Gamma_1$. Thus the T2F regression model for the given data reads as follows:

$$\bar{Y}_i = \bar{A}_1 I[e_{X_{i1}}, \delta_{X_{i1}}] + \bar{A}_2 I[e_{X_{i2}}, \delta_{X_{i2}}],$$

where $I[e_{X_{ik}}, \delta_{X_{ik}}]$ for $k = 1, 2$ are the new expected T1F set of $\bar{A}$. Since $N = 4, K = 2$, taken $(\bar{A}_k)_{h^0} = [\bar{A}_k^l, \bar{A}_k^r], k = 1, 2$, the T2FR model can be built as

$$\left.\begin{array}{rl} \min\limits_{\bar{A}} & J(\bar{A}) = \bar{A}_1^r - \bar{A}_1^l + \bar{A}_2^r - \bar{A}_2^l \\ \text{subject to} & \\ \bar{A}_1^r \geq & \bar{A}_1^l, \bar{A}_2^r \geq \bar{A}_2^l \\ \bar{Y}_1 = & (\bar{A}_1)_{h^0} I[e_{X_{11}}, \delta_{X_{11}}] \\ & +(\bar{A}_2)_{h^0} I[e_{X_{12}}, \delta_{X_{12}}] \supsetsim I[e_{Y_1}, \delta_{Y_1}], \\ \bar{Y}_2 = & (\bar{A}_1)_{h^0} I[e_{X_{21}}, \delta_{X_{21}}] \\ & +(\bar{A}_2)_{h^0} I[e_{X_{22}}, \delta_{X_{22}}] \supsetsim I[e_{Y_2}, \delta_{Y_2}], \\ \bar{Y}_3 = & (\bar{A}_1)_{h^0} I[e_{X_{31}}, \delta_{X_{31}}] \\ & +(\bar{A}_2)_{h^0} I[e_{X_{32}}, \delta_{X_{32}}] \supsetsim I[e_{Y_3}, \delta_{Y_3}], \\ \bar{Y}_4 = & (\bar{A}_1)_{h^0} I[e_{X_{41}}, \delta_{X_{41}}] \\ & +(\bar{A}_2)_{h^0} I[e_{X_{42}}, \delta_{X_{42}}] \supsetsim I[e_{Y_4}, \delta_{Y_4}]. \end{array}\right\} \quad (16)$$

First of all, we need to calculate all the $I[e_{X_{ik}}, \delta_{X_{ik}}]$ and $I[e_{Y_k}, \delta_{Y_k}]$ for $i = 1, 2, 3, 4, k = 1, 2$. By using the calculation $A = (a, c^l - \dfrac{(e^r+e^l)(a-c^l)}{2}, c^r + \dfrac{(e^r+e^l)(c^r-a)}{2})\Gamma_1$, we obtain all the pairs $\left(e_{X_{ik}}, \delta_{X_{ik}}\right)$ and $\left(e_{Y_k}, \delta_{Y_k}\right)$.

TABLE II
EXPECTED T1F INPUT DATA

i	$I[e_{X_{i1}}, \delta_{X_{i1}}]$	$I[e_{X_{i2}}, \delta_{X_{i2}}]$
1	$[2.94, 4.06]$	$[3.05, 4.75]$
2	$[4.75, 9.25]$	$[2.94, 4.06]$
3	$[11.63, 15.37]$	$[9.50, 17.90]$
4	$[12.75, 17.25]$	$[18.40, 22.40]$

TABLE III
EXPECTED T1F OUTPUT DATA

i	$I[e_{Y_i}, \delta_{Y_i}]$
1	$[8.52, 23.88]$
2	$[15.19, 20.01]$
3	$[20.12, 29.48]$
4	$[26.16, 42.64]$

Thus, the T2FR regression model is given in the form:

$$\begin{aligned} \bar{Y}_i &= \bar{A}_1 I[e_{X_{i1}}, \delta_{X_{i1}}] + \bar{A}_2 I[e_{X_{i2}}, \delta_{X_{i2}}] \\ &= \bar{A}_1 I[e_{X_{i1}}, \delta_{X_{i1}}] \\ &\quad + \left(\frac{\bar{A}_2^l + \bar{A}_2^r}{2}, \bar{A}_2^l, \bar{A}_2^r\right)_T I[e_{X_{i2}}, \delta_{X_{i2}}] \\ &= 1.31 I[e_{X_{i1}}, \delta_{X_{i1}}] \\ &\quad +(3.29, 0.0, 6.57)_T I[e_{X_{i2}}, \delta_{X_{i2}}]. \end{aligned}$$

According to the example, we have successfully expanded the ATP. When compared with traditional models, the new model introduces a new algorithm to deal with the dimensions of T2F inputs and outputs and have more accurate prediction rate. We have compared the prediction accurate rate by using the method of percent of overlap of the range. The T2F APT model get over 67 %t accurate rate while the APT model only get 43 %.

VII. Concluding Remarks and future works

In this paper, we built a T2F expected value regression model based on credibility theory. The innovation of this paper stands on several stakes as follows: 1) we defined the expected value for T2F variable based on credibility theory. 2) we formulated a general form of T2F regression model with a possiblistic structure. 3) Moreover, we transform the general form into a T2F expected value form. 4) We have expanded the traditional arbitrage pricing theory and build a new T2F APT. This paper generalized our previous work [16] [1]. Our further work will establish on the definition of variance for T2F variable.

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