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Robust Stability Analysis of PD Type Single Input Interval Type-2 Fuzzy Control Systems

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***Abstract*—In this paper, the robust stability of a PD type Single input Interval Type-2 Fuzzy Logic Controller (SIT2-FLC) structure will be examined via the well-known Popov criterion and Lyapunov's direct method approach. Since a closed form formulation of the SIT2-FLC output is possible, the type-2 fuzzy functional mapping is analyzed in a two dimensional domain. Thus, mathematical derivations are presented to show that type-2 fuzzy functional mapping is a symmetrical function and always sector bounded. Consequently, the type-2 fuzzy system can be transformed into a perturbed Lur'e system to examine its robust stability. It has been proven that the stability of the PD type SIT2-FLC system is guaranteed with the aids of the Popov-Lyapunov method. A robustness measure of the type-2 fuzzy control system is also presented to give the bound of allowable uncertainties/ nonlinearities of the control system. Moreover, if this bound is known, the exact region of stability of the type-2 fuzzy system can be found since SIT2-FLC output can be presented in a closed form. An illustrate example is presented to demonstrate the robust stability analysis of the PD type SIT2-FLC system.**

Keywords—Interval type-2 fuzzy logic controllers; Lur'e system; Robust stability

I. Introduction

Recently, the main focus of the fuzzy control researches is on Interval Type-2 Fuzzy Logic Controllers (IT2-FLCs) since they have been demonstrated significant performance improvements. The structure of the IT2-FLC is similar to the type-1 counterpart. Though, the IT2-FLCs employ and use Interval Type-2 Fuzzy Sets (IT2-FSs), rather than Type-1 Fuzzy Sets (T1-FSs), so there is a need of an extra type-reduction process [1]. Generally, IT2-FLCs achieve better control performance because of the additional degree of freedom provided by the Footprint of Uncertainty (FOU) in their Membership Functions (MFs) which also provides robustness against uncertainties [2-7]. The design problem of the IT2-FLCs is usually solved by blurring/extending the T1-FSs of its type-1 counterpart (since IT2-FS is a generalization of T1-FS). Thus, several studies have been performed in order to design PID type IT2-FLCs [4-8]. Recently, a new design strategy for PID type Single input IT2-FLCs (SIT2-FLCs) has been presented [9], [10].

It has been reported that the IT2-FLCs are generally more robust than type-1 counterparts [8]. However, since their internal structure is typically more relatively more complex than their type-1 counterpart, it is difficult to analyze the robustness of the IT2-FLCs [8-12]. Several studies have been presented to investigate the robustness and stability of the IT2-FLCs [8], [13-15]. However, the existing PD/PID type IT2-FLCs do not provide any methodology to show the stability of the control system [5-9].

In this paper, the robust stability of a PD type SIT2-FLC system is examined with the aids of the Popov-Lyapunov approach for the first time in literature. The most important feature of the examined SIT2-FLC is the closed form presentation of its output. This gives the opportunity to investigate the type-2 fuzzy mapping in a two dimensional domain. Thus, mathematical derivations and analysis will be presented on the SIT2-FLC to show that it is a symmetrical function and is always sector bounded. This will give the opportunity to transform the type-2 fuzzy system into a perturbed Lur'e system to examine its robustness. In this context, the well-known Popov-Lyapunov method will be employed to guarantee the robust stability of the system. Moreover, a robustness measure will be presented to give a bound on allowable uncertainties/nonlinearities of the system. If this bound is known, then the exact region of stability (safe operating region) of the type-2 fuzzy system can be found since SIT2-FLC output has a closed form representation. The presented stability analysis will be illustrated a mass-spring-damper system.

Section II will briefly present the structure of the PD type SIT2-FLC, Section III will present robust stability analysis of the SIT2-FLC system, Section IV will present an example to illustrate the stability analysis procedure, and Section IV will present conclusions and future work.

II. The General Structure of PD Type SIT2-FLC

In this section, the general structure of the PD type SIT2-FLC is presented. The SIT2-FLC is constructed by choosing the input to be the error signal (e) and the output as the control signal (u) as shown in Fig.1a [9]. Here, the input Scaling Factor (SF) K_e can be defined such that the input is normalized to the universe of discourse where the antecedent MFs of the SIT2-FLC are defined. Thus, K_e is defined as:

$$K_e = 1/\left(r(t_f) - y(t_f)\right) \tag{1}$$

where $r(t_f)$ and $y(t_f)$ are the values of the reference and system output at the time of the reference variation (t=t_f) [6]. Thus e is converted after normalization into σ which is the input of the SIT2-FLC while the its output (φ_o) is converted into the control signal (u) as follows:

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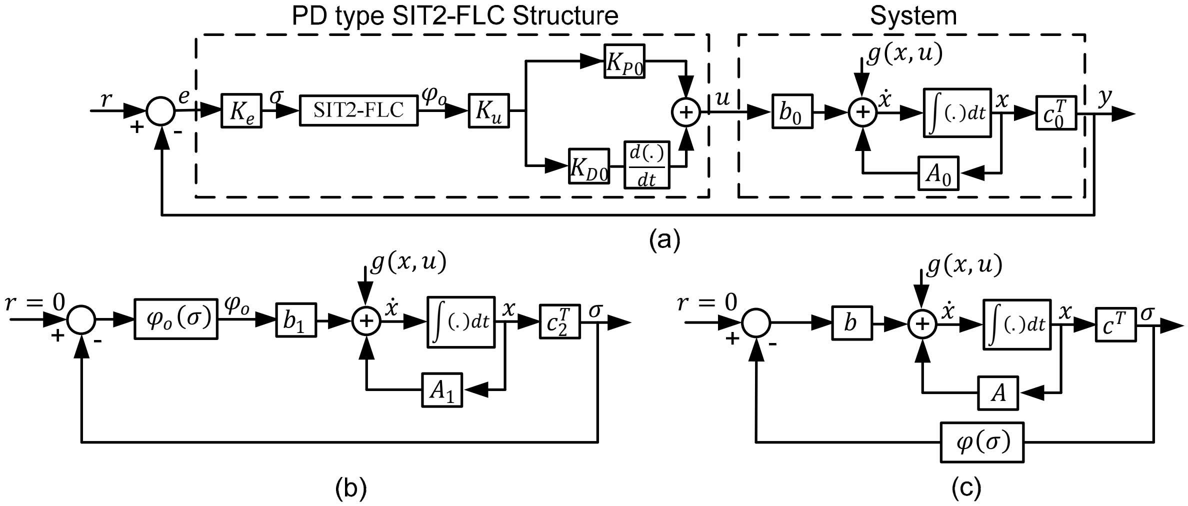

Fig. 1. Illustration of the (a) PD type SIT2-FLC system (b) equivalent type-2 fuzzy control system (c) Perturbed Lur'e system

$$u = K_P \varphi_o + K_D \frac{d\varphi_o}{dt} \quad (2)$$

where

$$K_P = K_{P0} K_u \quad K_D = K_{D0} K_u \quad (3)$$

Here, K_{P0} and K_{D0} are the baseline PD controller gains, and K_U is the output SF defined as $K_u = {K_e}^{-1}$. It can be seen that, the output of PD type SIT2-FLC is analogous to a conventional PD structure [9], [10]. Note that, the SFs and the baseline PD gains of the SIT2-FLC can directly affect the performance and robustness of the type-2 fuzzy control system [4-10].

A. The Internal Structure of the SIT2-FLC

In this study, a SIT2-FLC composed of three rules ($I = 3$) is employed and preferred for simplicity. The rule structure of the SIT2-FLC is as follows:

$$R^i\text{: IF } \sigma \text{ is } \tilde{A}_i \text{ THEN } \varphi_o \text{ is } B_i, \qquad i = 1,2,3 \quad (4)$$

where $\tilde{A}_i$ are triangular IT2-FSs (boomerang IT2-FSs) defining the antecedent MFs and B_i are the consequent MFs of the SIT2-FLC and defined as $B_1 = -1, B_2 = 0$ and $B_3 = +1$. Here, the antecedent IT2-FSs can be described in terms of upper MFs ($\overline{\mu}_{\tilde{A}_i}$) and lower MFs ($\underline{\mu}_{\tilde{A}_i}$) which creates the FOU (which provides extra degree of freedom) in IT2-FSs [2-7]. As shown in Fig. 2, m_i's represent the height of the lower antecedent MFs and are the main design parameters of the presented SIT2-FLC. In order to have symmetrical MFs, the value of m_1 must be equal to the value of m_3. Thus, the membership grades of the antecedent MFs for a crisp input σ' are defined as:

$$\overline{\mu}_{\tilde{A}_1}(\sigma') = \overline{\mu}_{\tilde{A}_3}(\sigma') = |\sigma|, \underline{\mu}_{\tilde{A}_1}(\sigma') = \underline{\mu}_{\tilde{A}_3}(\sigma') = |\sigma'| m_1 \quad (5)$$

$$\overline{\mu}_{\tilde{A}_2}(\sigma') = |1 - \sigma'|, \underline{\mu}_{\tilde{A}_2}(\sigma') = |1 - \sigma'| m_2 \quad (6)$$

In [1] and [9], it has been demonstrated that the defuzzified output of the SIT2-FLC is as follows:

$$\varphi_o = (\varphi_o^r + \varphi_o^l)/2 \quad (7)$$

where φ_o^r and φ_o^l are the end points of the type reduced set and are defined for each region as follows:

$$\varphi_o^l = \frac{\sum_{i=1}^{L} \overline{\mu}_{\tilde{A}_i}(\sigma') \cdot B_i + \sum_{L+1}^{N} \underline{\mu}_{\tilde{A}_i}(\sigma') \cdot B_i}{\sum_{i=1}^{L} \overline{\mu}_{\tilde{A}_i}(\sigma') + \sum_{L+1}^{N} \underline{\mu}_{\tilde{A}_i}(\sigma')} \quad (8)$$

$$\varphi_o^r = \frac{\sum_{i=1}^{R} \underline{\mu}_{\tilde{A}_i}(\sigma') \cdot B_i + \sum_{R+1}^{N} \overline{\mu}_{\tilde{A}_i}(\sigma') \cdot B_i}{\sum_{i=1}^{R} \overline{\mu}_{\tilde{A}_i}(\sigma') + \sum_{R+1}^{N} \overline{\mu}_{\tilde{A}_i}(\sigma')} \quad (9)$$

Here, (*R, L*) is the solution set such that which minimize/maximize Equations (8) and (9), respectively [1]. Since it is always guaranteed that a crisp value of σ always belongs to two successive IT2-FSs, the switching points (*R, L*) are always equal to "1" (for any crisp input only two rules (*N=2*) are always activated) [9], [10].

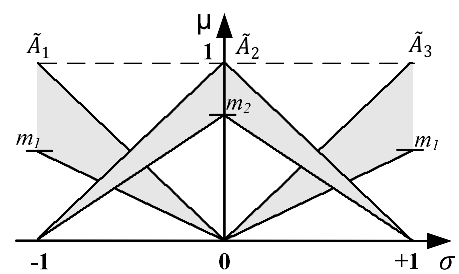

Fig. 2. Illustration of the antecedent MFs of the IT2-FLC

Theorem-1: Let $\varphi_o(\sigma)$ denote the functional mapping achieved by the SIT2-FLC structure. Then, it holds that $\varphi_o(\sigma) = -\varphi_o(-\sigma)$ for $\forall \sigma \neq 0$ and $\varphi_o(0) = 0$, i.e. $\varphi_o(\sigma)$ is symmetrical function with respect the input σ.

Proof: First, the output of the SIT2-FLC given in Equation (7) is derived for the input interval $\sigma \in [0, +1]$ as follows:

$$\varphi_o(\sigma) = \frac{1}{2}\left(\frac{\underline{\mu}_{\tilde{A}_3}(\sigma)}{\underline{\mu}_{\tilde{A}_3}(\sigma) + \overline{\mu}_{\tilde{A}_2}(\sigma)} + \frac{\overline{\mu}_{\tilde{A}_3}(\sigma)}{\overline{\mu}_{\tilde{A}_3}(\sigma) + \underline{\mu}_{\tilde{A}_2}(\sigma)}\right) \quad (10)$$

Replacing the membership grades given in Equations (5) and (6) into (10):

$$\varphi_o(\sigma) = \sigma \cdot k(\sigma) \quad (11)$$

where $k(\sigma)$ is the nonlinear gain generated from the type-2 fuzzy inference and is defined as:

$$k(\sigma) = \frac{1}{2}\left(\frac{1}{\sigma + (1-\sigma)m_2} + \frac{m_1}{\sigma m_1 + (1-\sigma)}\right) \quad (12)$$

Similarly, for the input interval $\sigma \in [-1,0]$, the output of the SIT2-FLC can be derived as:

$$\varphi_o(\sigma) = \frac{1}{2}\left(\frac{-\overline{\mu}_{\tilde{A}_1}(\sigma)}{\overline{\mu}_{\tilde{A}_1}(\sigma) + \underline{\mu}_{\tilde{A}_2}(\sigma)} + \frac{-\underline{\mu}_{\tilde{A}_1}(\sigma)}{\underline{\mu}_{\tilde{A}_1}(\sigma) + \overline{\mu}_{\tilde{A}_2}(\sigma)}\right) \quad (13)$$

Similarly, replacing Equations (5) and (6) into Equation (13):

$$\varphi_o(\sigma) = -\sigma \cdot k(\sigma) \quad (14)$$

It can be concluded from Equations (11) and (14) that, $\varphi_o(\sigma)$ is symmetrical function with respect the σ, i.e. $\varphi_o(\sigma) = -\varphi_o(-\sigma)$ and if $\sigma = 0$, then $\varphi_o(0) = 0$.■

B. Gain analysis of the SIT2-FLC

In this section, the gain analysis of the SIT2-FLC is presented. As it has been derived in the Equation (11) and (14), the SIT2-FLC output can be explicitly derived in the input domain. This simplifies the SIT2-FLC design method to a nonlinear control curve generation, instead of a control curve design [9]. Thus, we will examine the effect of the design parameters $(m_2, m_1 = m_3)$ on the SIT2-FLC output. For an easier analysis, we will employ $m_2 = \alpha$ and $m_1 = 1 - \alpha$ in the rest of the paper. The SIT2-FLC output (φ_o) and the corresponding nonlinear gain $(k(\sigma))$ variations are illustrated in Fig.3a and Fig.3b, respectively. As it can be seen, for $\alpha_1 = 0.2$ an aggressive control curve while for $\alpha_2 = 0.8$ a smooth control curve is generated. The presented control curves are commonly used and preferred in nonlinear control theory. Moreover, it can be seen that the SIT2-FLC output is bounded in a sector as shown Fig.3b.

Theorem-2: If $m_2 = \alpha$ and $m_1 = 1 - \alpha$, then the output of the SIT2-FLC always belongs to a sector $[K_{min}, K_{max}]$:

$$K_{min}\sigma^2 \leq \varphi_o\sigma \leq K_{max}\sigma^2, \text{for } \forall\sigma \neq 0 \quad (15)$$

where

$$K_{min} \equiv \inf_{\substack{\sigma\in[-1,+1] \\ \alpha\in[0,+1]}} k(\sigma), \quad K_{max} \equiv \sup_{\substack{\sigma\in[-1,+1] \\ \alpha\in[0,+1]}} k(\sigma) \quad (16)$$

Here, $k(\sigma)$ will map to different values of K_{min} and K_{max} with respect to α, σ values as tabulated in Table I where:

$$\alpha_{c1} = \frac{1}{2}(3 - \sqrt{5}), \quad \alpha_{c2} = \frac{1}{2}(-1 + \sqrt{5})$$
$$\sigma_{c1} = 0, \sigma_{c2} = 1 \quad (17)$$
$$\sigma_{c3} = \frac{-1 - \alpha + \alpha^2}{-1 - 3\alpha + \alpha^2} - \sqrt{\frac{(1 - \alpha + \alpha^2)^2}{\alpha(1 - 3\alpha + \alpha^2)^2}}$$

TABLE I. THE EXTREMA VALUES OF THE TYPE-2 FUZZY GAIN

	$0 \leq \alpha < \alpha_{c1}$	$\alpha_{c1} \leq \alpha < \alpha_{c2}$	$\alpha_{c2} \leq \alpha \leq 1$
K_{min}	$k(\sigma_{c2})$	$k(\sigma_{c3})$	$k(\sigma_{c3})$
K_{max}	$k(\sigma_{c1})$	$k(\sigma_{c1})$	$k(\sigma_{c2})$

Proof: If $m_0 = \alpha$ and $m_1 = 1 - \alpha$, then the functional mapping of the SIT2-FLC for the interval $\sigma \in [0, +1]$ can be formulated as:

$$\varphi_o(\sigma) = \sigma\, k(\sigma) \quad (18)$$

where

$$k(\sigma) = \frac{1}{2}\left(\frac{1}{\alpha + \sigma - \alpha\sigma} + \frac{-1 + \alpha}{-1 + \alpha\sigma}\right) \quad (19)$$

The candidate extrema values of $k(\sigma)$ are the boundary points $\sigma_{c1} = 0$, $\sigma_{c2} = 1$. Moreover, the critical point σ_{c3} $(dK(\sigma_{c3})/d\sigma = 0)$ is a candidate point and is found as:

$$\sigma_{c3} = \frac{-1 - \alpha + \alpha^2}{-1 - 3\alpha + \alpha^2} - \sqrt{\frac{(1 - \alpha + \alpha^2)^2}{\alpha(1 - 3\alpha + \alpha^2)^2}} \quad (20)$$

By simply examining the first order derivative test $dK(\sigma)/d\sigma > 0$ under the constraints $0 < \alpha < 1$ and $0 < \sigma \leq 1$; the following two cases can be derived that,

Case-1: If $0 < \alpha \leq \alpha_{c1}$, then $k(\sigma)$ is always an increasing function with respect to σ for the interval $0 < \sigma \leq 1$ where α_{c1} is found as:

$$\alpha_{c1} = \frac{1}{2}(3 - \sqrt{5}) \quad (21)$$

Case-2: If $\alpha_{c1} < \alpha < 1$, then $k(\sigma)$ is an increasing function with respect to σ for the interval $0 < \sigma \leq \sigma_{c3}$ while a decreasing function with respect to σ for the interval $\sigma_{c3} \leq e \leq 1$.

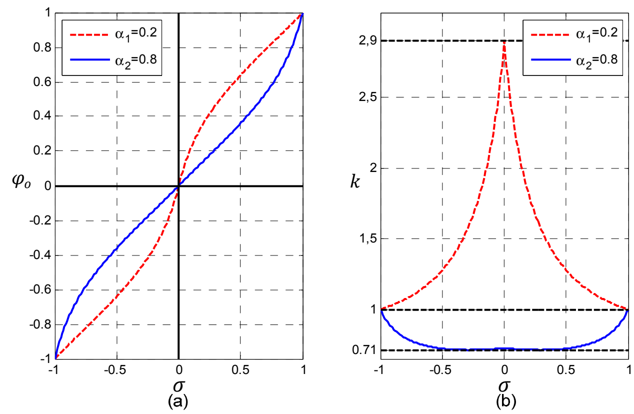

Fig. 3. Illustration of the (a) nonlinear control curves (b) $k(\sigma)$ variations

In **Case-1**, since $k(\sigma)$ is always an increasing function, the maximum and minimum values of $k(\sigma)$ will be the found in the boundary values σ_{c1} and σ_{c2}. Thus,

$$K_{min} = \min\big(k(\sigma_{c1}), k(\sigma_{c2})\big) \quad (22)$$
$$K_{max} = \max\big(k(\sigma_{c1}), k(\sigma_{c2})\big) \quad (23)$$

where

$$k(\sigma_{c1}) = \frac{1}{2}\left(\frac{1 + \alpha - \alpha^2}{\alpha}\right), \qquad k(\sigma_{c2}) = 1 \quad (24)$$

It can be simply derived from Equation (24) that, $k(\sigma_{c1}) \geq k(\sigma_{c2})$ if and only if $\alpha \leq \alpha_{c2}$ where α_{c2} is found as:

$$\alpha_{c2} = \frac{1}{2}(-1 + \sqrt{5}) \quad (25)$$

Thus, this inequality is always satisfied since α_{c2} is always greater then α_{c1} $(\alpha_{c2} > \alpha_{c1})$ in the interval $0 < \alpha \leq \alpha_{c1}$. Consequently, the sector bounds for Case-1 are:

$$K_{min} = k(\sigma_{c2}) \quad (26)$$
$$K_{max} = k(\sigma_{c1}) \quad (27)$$

In **Case-2**, it is clear that σ_{c3} is the minimizing value of $k(\sigma)$, thus the minimum value of $k(\sigma)$ will be always

$$K_{min} = k(\sigma_{c3}) \quad (28)$$

for the interval $\alpha_{c1} < \alpha < 1$. Whereas, the maximum value of $k(\sigma)$ will be found in the boundary values σ_{c1} and σ_{c2}. Thus, in the light of $k(\sigma_{c1}) \geq k(\sigma_{c2})$ if and only if $\alpha < \alpha_{c2}$, the upper bound for the interval $\alpha_{c1} \leq \alpha < \alpha_{c2}$ is:

$$K_{max} = k(\sigma_{c1}) \tag{29}$$

while for the interval $\alpha_{c2} \leq \alpha < 1$

$$K_{max} = k(\sigma_{c2}) \tag{30}$$

Thus, since the functional mapping of the SIT2-FLC is a symmetrical function (proved in Theorem-1), the nonlinearity always belongs to the sector $[K_{min}, K_{max}]$ for $\forall\sigma \neq 0$. ■

III. Stability Analysis of the SIT2-FLC System

In this section, the stability analysis of the type-2 fuzzy control system is presented. At first, the control system will be transformed to the perturbed Lur'e systems and then the stability analysis are presented with the aids of the Lyapunov's direct method and the Popov criterion.

A. The PD type SIT2-FLC System

In this subsection, the type-2 fuzzy control system (shown in Fig.1a) will be transformed to the Lur'e systems. Thus, let us first define a SISO nonlinear system as follows:

$$\begin{aligned} \dot{X} &= f(X, U) \\ Y &= c_0^T X \end{aligned} \tag{30}$$

where $X \in R^n$ is the state vector, n is the number of the states, U is the scalar system input and $f \in R^n$ represents the nonlinear mapping. Let $X = x + x_0$, and $U = u + u_0$ where x_0 and u_0 denote the nominal operating point of the system given Equation (30). Then, by simply expanding the nonlinear system into a Taylor series around (x_0, u_0), the following equation are obtained:

$$\dot{x} = x\frac{\partial f}{\partial X}\bigg|_{(x_0,u_0)} + u\frac{\partial f}{\partial U}\bigg|_{(x_0,u_0)} + g(x, u) \tag{31}$$

where $g(x, u)$ inherits higher-order terms in x and u or the uncertainties of the system. Letting A_0 denote the Jacobian matrix of $\partial f/\partial X$ and b_0 denote the Jacobian matrix of $\partial f/\partial U$ at the nominal operating point, we can obtain:

$$\begin{aligned} \dot{x} &= A_0 x + b_0 u + g(x, u) \\ y &= c_0^T x \end{aligned} \tag{32}$$

Here, if the nominal open-loop transfer function of the system has a relative degree 2 or more, then $c_0^T b_0 = 0$. Besides, it will be assumed that the system satisfies $c_0^T g(x, u) = 0$.

In order to examine the stability of the fuzzy system, let the reference to be zero $(r = 0)$. Thus, the output of the system is

$$e = -y = -c_1^T x \tag{33}$$

$$\sigma = K_e e = -K_e c_1^T x = -c_2^T x \tag{34}$$

Moreover, the input of the system is:

$$u = K_P \varphi_o + K_D \frac{d\varphi_o}{dt} \tag{35}$$

It can be observed that the input signal inherits a derivative action. Thus, let us define a new state space model where the input signal is φ_o and the output is σ as follows:

$$\begin{aligned} \dot{x} &= A_1 x + b_1 \varphi_o(\sigma) + g(x, u) \\ \sigma &= -c_2^T x \end{aligned} \tag{36}$$

where A_1, b_1 and c_1^T can be obtained by defining new state variables or applying the integration method presented in [16]. The block diagram of Equation (36) is given in Fig. 1b.

It has been proven in Theorem 2 that the output of the IT2-FLC (φ_o) always belongs to the sector $[K_{min}, K_{max}]$. Thus, by defining a new functional mapping as:

$$\varphi(\sigma) = \varphi_o(\sigma) - K_{min}\sigma \tag{37}$$

Then, the inequality given in Equation (15) can be defined as

$$0 \leq \varphi\sigma \leq K\sigma^2, \text{for } \forall\sigma \neq 0 \tag{38}$$

where $K = K_{max} - K_{min}$. Thus, the nonlinearity φ will be in the sector $[0, K]$ and Equation (36) can be rewritten as:

$$\begin{aligned} \dot{x} &= Ax - b\varphi(\sigma) + g(x, u) \\ \sigma &= c^T x \end{aligned} \tag{39}$$

where

$$A = A_1 - K_{min} b_1 c_2^T \quad b = -b_1 \quad c = -c_2^T \tag{40}$$

The new block diagram of the closed system is shown in Fig. 1c. It can be seen that Equation (39) is a perturbed Lur'e system [17], [18]. Note that, this transformation is possible since the functional mapping of the IT2-FLC is symmetrical and sector bounded (Theorem-1 and Theorem-2).

B. Stability Analysis of the Type-2 Fuzzy Control System

In this section, the following theorem will be used to guarantee the stability of the type-2 fuzzy system [17], [18].

Theorem-3: If the system described by Equation (39) satisfies the following conditions, then the point $x = 0$ is uniformly asymptotically stable.

C1) the nonlinearity φ belongs to the sector $[0, K]$ where K is a known and positive number**.**

C2) the system matrix A is Hurwitz (has all its eigenvalues strictly in the left-plane), i.e., $G(s) = c^T(sI - A)^{-1}b$ is stable and there exists a scalar $r > 0$ such that

$$\frac{1}{K} + Re[(1 + jwr)G(jw)] > 0 \ \forall w \in R \tag{41}$$

C3) Let

$$v = \frac{1}{2}(rA^T c + c) \quad \gamma = rc^T b + \frac{1}{K} \tag{42}$$

where r is chosen such that $\gamma \geq 0$. Given a symmetric positive define matrix W; there exists a scalar $\varepsilon > 0$, a vector q symmetric positive define matrix P and W_0, and a scalar $\delta > 0$ satisfiying:

$$A^T P + PA = -qq^T - \varepsilon W \tag{43}$$

$$Pb - v = \sqrt{\gamma} q \tag{44}$$

$$\varepsilon W = \varepsilon W_0 + \delta I \tag{45}$$

C4) the nonlinearity $g(x, u)$ is bounded and satisfies:

$$\|g(x, u)\|_2 \leq \beta\|x\|_2 \leq \frac{\delta}{2\|P\|_{i2} + rK\|c\|_2^2}\|x\|_2 \tag{46}$$

where β is a robustness measure, $\|P\|_{i2}$ denotes the spectral norm of the matrix P, $\|.\|_2$ represents 2-norm, and M is the domain where $g(x, u)$ is bounded.

Proof: The proof of **C1** was presented in Theorem 2 while the proof of **C2** can be derived with the aids of the Popov criterion [17], [18]. The proofs for **C3** and **C4** can be derived by using following Lyapunov function

$$V(x) = x^T P x + r \int_0^{\sigma} \varphi(y) dy \quad (47)$$

Moreover, a stability domain (Ω) where $x = 0$ is stable is defined as:

$$\Omega = \{x \in R^n | V(x) \leq \theta\} \quad (48)$$

The stability domain can be explicitly presented since the type-2 fuzzy mapping can be presented in a closed form [9]. The proofs of C3 and C4 can be found in [17], [18]. ■

The stability analysis of the SIT2-FLC system will be accomplished with respect to Theorem 3. Thus, first an appropriate r value from the Popov plot of Gjw) must be found such that Equation (41) is satisfied. In other words, a slope r ($r > 0$) of a line that intercepts the point $-1/K + j0$ must be found such that the Popov plot is always to the right to that line [18]. After evaluating v and γ from Equation (42), a symmetric positive-definite matrix W and a positive real number ε must be selected to obtain the matrix P via the following Riccati equation:

$$A_r^T P + P A_r - P R_r P + Q_r = 0 \quad (49)$$

where

$$A_r = A - \frac{1}{\gamma} b v^T \quad Q_r = \varepsilon W + \frac{v v^T}{\gamma} \quad R_r = -\frac{b b^T}{\gamma} \quad (50)$$

Once a positive-definite matrix P is obtained, a positive-definite matrix W_0 is selected and an appropriate δ value can be determined to obtain the robustness measure β given in Equation (46). Note that, if the value of β is relatively small, then the system might become unstable in the presence of uncertainties/nonlinearities [17].

IV. Illustrative Example

In this section, the stability analysis of the SIT2-FLC system is presented on a mass-damper-spring system. As it has been derived in Section 3, the boundaries (K_{min}, K_{max}) of the functional mapping varies with respect the design parameter α. Thus, the robustness of the PD type SIT2-FLCs for the values $\alpha_1 = 0.2$ (an aggressive control curve) and $\alpha_2 = 0.8$ (a smooth control curve) and their corresponding region of stability (Ω) will be examined and compared. In this example, the mass and damper constants of the system (shown in Fig. 4) are chosen to be equal and set as $m = c = 1$ while the spring characteristic is defined as $t(x_1) = 3 - x_1^3$ [17]. Thus, the dynamic equations of the mass-damper-spring system are as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix}}_{A_0} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b_0} u + \underbrace{\begin{bmatrix} 0 \\ x_1^3 \end{bmatrix}}_{g(x)}$$
$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{c_0^T} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (51)$$

where x_1 and x_2 are the position and the velocity of the mass-damper-spring system, respectively.

As it has been asserted in Section 2, the PD type SIT2-FLC is constructed where the error $e = r - x_1$ is the input and $u = K_P \varphi_o + K_D\, d\varphi_o/dt$ is the output. Thus, to be able to employ the stability analysis presented in Section 3, there is a need of defining new state space matrices since the output of the PD type SIT2-FLC inherits a derivative action. In this context, new state variables are defined as $x_1 = y$, $x_2 = \dot{y} - K_D \varphi_o$ and the following state space model is obtained.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -3 & -1 \end{bmatrix}}_{A_1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} K_D \\ K_P - K_D \end{bmatrix}}_{b_1} \varphi_o + \underbrace{\begin{bmatrix} 0 \\ x_1^3 \end{bmatrix}}_{g(x)}$$
$$\sigma = \underbrace{\begin{bmatrix} -1 & 0 \end{bmatrix}}_{c_2^T} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (52)$$

Consequently, the configuration shown in Fig. 1b is obtained. Here, the SFs of the SIT2-FLC structure are set as $K_e = 1$, $K_U = 1$ and the baseline PD parameters are chosen as $K_{P0} = 1$, $K_{D0} = 2$, accordingly $K_P = 1$, $K_D = 2$. The bounding sector for $\alpha_1 = 0.2$ is found as $K_{\min\_\alpha_1} = 1.0$, $K_{\max\_\alpha_1} = 2.9$ from Table I and $K_{\alpha_1} = 1.9$ from Equation (38) while for $\alpha_2 = 0.8$ the corresponding values are $K_{\min\_\alpha_2} = 0.71$, $K_{\max\_\alpha_2} = 1.0$ and $K_{\alpha_2} = 0.29$.

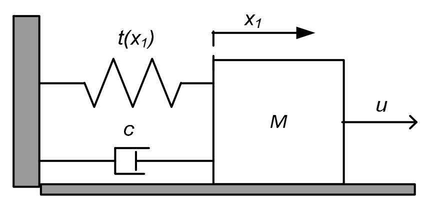

Fig. 4. Schematic diagram of the mass-damper-spring system

The state space representation of the SIT2-FLC system for $\alpha_1 = 0.2$ and $\alpha_2 = 0.8$ can be obtained from the Equation (39), and the corresponding transfer functions are:

$$G_{\alpha_1}(s) = \frac{2s + 1}{s^2 + 3s + 4} \text{ for } \alpha_1 = 0.2 \quad (53)$$

$$G_{\alpha_2}(s) = \frac{2s + 1}{s^2 + 2.43s + 3.71} \text{ for } \alpha_2 = 0.8 \quad (54)$$

The Popov plots of $G_{\alpha_1}(jw)$and $G_{\alpha_2}(jw)$ are given Fig.5a and Fig.5b, respectively. From the Popov plots, it can be found that Equation (41) is satisfied for $G_{\alpha_1}(jw)$ if $r \geq 0.17$ while for $G_{\alpha_2}(jw)$ if $r \geq 0.06$. However, since the aim is to compare the robust stability for the cases α_1 and α_2, r will be chosen as $r = 0.17$ which satisfies both conditions. Thus, v and γ values can be found as follows:

$$\left. v_{\alpha_1} = \begin{bmatrix} -0.3 \\ -0.1 \end{bmatrix};\ \gamma_{\alpha_1} = 0.93 \right\} \text{ for } \alpha_1 = 0.2 \quad (55)$$

$$\left. v_{\alpha_2} = \begin{bmatrix} -0.46 \\ -0.03 \end{bmatrix};\ \gamma_{\alpha_2} = 3.6 \right\} \text{ for } \alpha_2 = 0.8 \quad (56)$$

Then, if we let εW for both cases as

$$\varepsilon W = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} \quad (57)$$

and solve the Riccati Equation given in Equation (49), the following P matrices are obtained.

$$P_{\alpha_1} = \begin{bmatrix} +0.046 & -0.010 \\ -0.010 & +0.057 \end{bmatrix} \text{ for } \alpha_1 = 0.2 \quad (58)$$

$$P_{\alpha_2} = \begin{bmatrix} +0.062 & -0.010 \\ -0.010 & +0.041 \end{bmatrix} \text{ for } \alpha_2 = 0.8 \quad (59)$$

Here, if we set $\delta = 0.1$, Equation (46) becomes as follows:

$$M_{\alpha_1} = \{[x_1 \quad x_2]^T \in R^2 | (x_1^3)^2 \leq 0.1978(x_1^2 + x_2^2)\} \quad (60)$$

$$M_{\alpha_2} = \{[x_1 \quad x_2]^T \in R^2 | (x_1^3)^2 \leq 0.6764(x_1^2 + x_2^2)\} \quad (61)$$

where the robustness measures of the SIT2-FLC for $\alpha_1 = 0.2$ and $\alpha_2 = 0.8$ are obtained as $\beta_{\alpha_1} = 0.197$ and $\beta_{\alpha_2} = 0.676$, respectively. Moreover, the corresponding regions of attraction (Ω) can be determined via the Lyapunov function $V(x)$ (given in Equation (47)) since the SIT2-FLC output has a closed form analytical structure. Thus, the θ values are obtained with respect to Equations (60) and (61) of the SIT2-FLC systems for α_1 and α_2 as $\theta_{\alpha_1} = 0.103$ and $\theta_{\alpha_2} = 0.063$, respectively. The corresponding regions of robust stability of the SIT2-FLC systems are given in Fig.6.

It can be concluded that, the stability of both SIT2-FLC systems is guaranteed but in different robustness measures and stability regions. The SIT2-FLC for α_2 is potentially more robust against nonlinearities/uncertainties since it has a bigger β_{α_2} value in comparison to the one for the case α_1 $(\beta_{\alpha_2} > \beta_{\alpha_1})$. Also, as shown in Fig.6, the SIT2-FLC system for α_2 has a wider region of s robust stability in comparison to the one for α_1 (it has wider safe operating region). The results coincide with the results presented in [9] since the SIT2-FLC for α_2 generates a smooth control curve which provides robustness against uncertainties and nonlinearities.

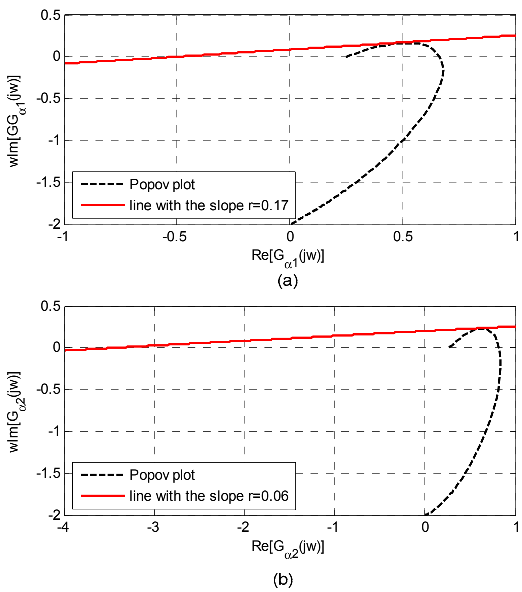

Fig. 5. Illustration of the Popov plots for (a) $G_{\alpha_1}(jw)$ (b) $G_{\alpha_2}(jw)$

V. Conclusions

In this paper, the robust stability of a PD type SIT2-FLC system is examined. Since the SIT2-FLC output has a closed form formulation, the type-2 fuzzy mapping is analyzed in a two dimensional domain. Thus, it has been proven that it's a symmetrical function (Theorem-1) and its output is sector bounded (Theorem-2). Then, the SIT2-FLC system is transformed into a perturbed Lur'e system to examine the stability of the fuzzy system. The stability of the fuzzy system is guaranteed with aids of the Popov-Lyapunov method (Theorem-3). Moreover, a robustness measure is presented to define the bound on allowable uncertainties/ nonlinearities of the system. If this bound is known, then the exact region of stability can be presented since the output of the SIT2-FLC has a closed form representation. It has been shown that the SIT2-FLC can robustly stabilize certain class of systems.

Future work will focus on extending the robust stability analysis to generalized type-2 fuzzy logic control systems.

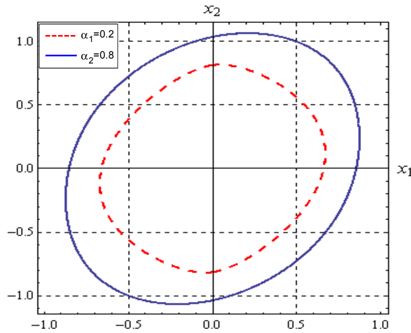

Fig. 6. The region of attractions for the IT2-FLC PD for α_1 and α_2

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