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On the α-universal multiple I restriction method for general fuzzy reasoning

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***Abstract*— Based on the differently implicational idea, the α-universal multiple I restriction method is put forward for general fuzzy reasoning, which contains the α-multiple I restriction method as its specific case. First of all, we give the α-universal multiple I restriction principle, which improves the previous restriction principle, and then provide the existing condition of the solutions of the new method. Furthermore, we obtain the optimal solution of the new method for the fuzzy implications with residual pair, the R-implications, as well as some particular fuzzy implications. Finally, it is found that the new method is more reasonable by contrast with the α-multiple I restriction method.**

I. INTRODUCTION

THE fundamental form of fuzzy reasoning [1]-[3] is the following fuzzy modus ponens (FMP):

$$
\begin{array}{lll}
\text{From} & & \\
\quad\text{Rule:} & A & \text{implies } B \\
\quad\text{and input} & A^* & \\
\hline
\text{Compute} & & B^*
\end{array}
\tag{1}
$$

where $A, A^* \in F(U)$, $B, B^* \in F(V)$, in which $F(U), F(V)$ respectively represent the set of all fuzzy subsets of U, V. For providing the solution of FMP, the most famous strategy is the compositional rule of inference (CRI) method [4]-[7]. To improve the CRI method, the triple I (the abbreviation of triple implications) method was presented (in [8], [9]) and further investigated (see e.g. [10]-[14]). The basic principle of the triple I method is to find the smallest fuzzy set $B^* \in F(V)$ making ($\alpha \in [0,1]$)

$$(A(u) \to B(v)) \to (A^*(u) \to B^*(v)) \geq \alpha \tag{2}$$

holds for any $u \in U, v \in V$, in which $\to$ is defined by a fuzzy implication. It is verified that the triple I method has many wonderful advantages including strict logic basis, reversibility and so forth (see [15]-[18]).

As an extension of (1), the general form of fuzzy reasoning is as follows:

$$
\begin{array}{llll}
\text{From} & & & \\
\quad\text{Rule 1:} & A_{11}, A_{12}, \cdots, A_{1n} & \text{imply} & B_1 \\
\quad\text{Rule 2:} & A_{21}, A_{22}, \cdots, A_{2n} & \text{imply} & B_2 \\
\quad\cdots & \cdots\cdots\cdots\cdots\cdots & \cdots & \cdots \\
\quad\text{Rule m:} & A_{m1}, A_{m2}, \cdots, A_{mn} & \text{imply} & B_m \\
\quad\text{and input} & A_1^*,\ A_2^*,\ \cdots,\ A_n^* & & \\
\hline
\text{Compute} & & & B^*
\end{array}
\tag{3}
$$

where $A_{ij}, A_j^* \in F(U_j)$, $B_i, B^* \in F(V)$ ($i = 1, \cdots, m; j = 1, \cdots, n$). For this general fuzzy reasoning, Wang established a fully implicational multiple I method (see [19]). Later, it was generalized to the α-multiple I method, which focused on ($u_j \in U_j, v \in V, j = 1, \cdots, n$):

$$
\begin{aligned}
&R_{\to,1} \to (R_{\to,2} \to (\cdots (R_{\to,m} \to (A_1^*(u_1) \to \\
&(A_2^*(u_2) \to (\cdots (A_n^*(u_n) \to B^*(v)) \cdots) \cdots) \geq \alpha,
\end{aligned}
\tag{4}
$$

in which the i-th rule $R_{\to,i}$ was as follows ($i = 1, \cdots, m$):

$$
\begin{aligned}
&R_{\to,i} = R_{\to,i}(u_1, \cdots, u_n, v) \\
&= A_{i1}(u_1) \to (A_{i2}(u_2) \to (\cdots (A_{in}(u_n) \to B_i(v)) \cdots)).
\end{aligned}
$$

Following that, the restriction idea (see [20]) was introduced to analyze the condition opposite to (4), and then the α-multiple I restriction method was investigated with discussing its unified forms [21]. In detail, the basic idea of the α-multiple I restriction method is to seek out the largest $B^* \in F(V)$ such that ($\alpha \in (0,1]$)

$$
\begin{aligned}
&R_{\to,1} \to (R_{\to,2} \to (\cdots (R_{\to,m} \to (A_1^*(u_1) \to \\
&(A_2^*(u_2) \to (\cdots (A_n^*(u_n) \to B^*(v)) \cdots) \cdots) < \alpha
\end{aligned}
\tag{5}
$$

holds for any $u_j \in U_j$, $v \in V$ ($j = 1, \cdots, n$).

From the point of view of some kind of fuzzy system, it is unfortunate that the effect of the triple I method is imperfect because of its inferior response ability and practicability (see [22]-[24] for more details). Aiming at such problem, we generalized in [25] the triple I method to the differently implicational universal triple I method of (1, 2, 2) type (the universal triple I method for short). The ideal solution of the universal triple I method takes the smallest $B^* \in F(V)$ such that ($\alpha \in [0,1]$)

$$(A(u) \to_1 B(v)) \to_2 (A^*(u) \to_2 B^*(v)) \geq \alpha \tag{6}$$

holds for any $u \in U, v \in V$, where $\to_1$ and $\to_2$ can employ different fuzzy implications. Furthermore, it was obtained that the universal triple I method can generate more

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practicable and better fuzzy systems by contrast with the CRI method and the triple I method (see [25]). Then, we analyzed the reversibility of universal triple I method for the problem of fuzzy modus tollens (in which $\to_2$ employed I_L), and drew the conclusion that its reversibility seemed excellent (see [26]). In [27], we put forward the reverse universal triple I method, and investigated related principles, solutions, as well as its reversibility. In [28], we investigated the universal triple I method for the FMP problem (in which $\to_2$ took I_{FD}), which focused on its reversibility properties and also the corresponding more general fuzzy systems, and then applied the universal triple I method to textual emotion polarity recognition. In [29], the universal triple I method was researched for the FMP problem from the viewpoints of both fuzzy reasoning and fuzzy controller, which included its solution, reversibility, and response ability, where involved the (0,1)-implications, R-implications, together with the expansion, reduction and other type operators.

Based on these works, we should introduce the differently implicational idea [25] to the α-multiple I restriction method, that is, we should investigate the general fuzzy reasoning derived from ($\alpha \in (0,1]$)

$$R_{\to_1,1} \to_2 (R_{\to_1,2} \to_2 (\cdots (R_{\to_1,m} \to_2 (A_1^*(u_1) \to_2 (A_2^*(u_2) \to_2 (\cdots (A_n^*(u_n) \to_2 B^*(v)) \cdots) \cdots) < \alpha, \tag{7}$$

where the new i-th rule $R_{\to_1,i}$ is as follows ($i = 1, \cdots, m$):

$$R_{\to_1,i} = R_{\to_1,i}(u_1, \cdots, u_n, v) = A_{i1}(u_1) \to_1 (A_{i2}(u_2) \to_1 (\cdots (A_{in}(u_n) \to_1 B_i(v)) \cdots)),$$

which is said to be the α-universal multiple I restriction method. It evidently contains the α-multiple I restriction method as its specific case. The aim of this paper is to establish unified forms of the α-universal multiple I restriction method, which allow different fuzzy implications to be employed in the same manner.

The rest of this paper is organized as follows: Section II is the preliminaries. In Sections III, the fundamental characteristics of the α-universal multiple I restriction method are researched, which include its basic principle, the existing condition of solutions. In Section IV, the optimal solutions of the α-universal multiple I restriction method are obtained for some kinds of fuzzy implications. Section V concludes this paper.

II. Preliminaries

There are several definitions of fuzzy implications, and here we choose the acknowledged definition employed in [1],[30], which is equivalent to the one in [31].

Definition *2.1:* A fuzzy implication on $[0,1]$ is a function $I : [0,1]^2 \to [0,1]$ satisfying the following three properties:

(P1) I is decreasing in the first variable.

(P2) I is increasing in the second variable.

(P3) $I(0,0) = 1$, $I(1,1) = 1$, $I(1,0) = 0$.

$I(a,b)$ is also written as $a \to b$ for any $a, b \in [0,1]$.

For any fuzzy implication I, obviously I satisfies

(P4) $I(0,a) = I(a,1) = 1$ ($a \in [0,1]$)

and obviously $I(0,1) = 1$.

Definition *2.2:* ([6]) A function $T : [0,1]^2 \to [0,1]$ is called a triangular norm (t-norm, for short), if T is associative, increasing, commutative, and satisfies $T(1,a) = a$ ($a \in [0,1]$).

Definition *2.3:* ([32]) Let T, I be two $[0,1]^2 \to [0,1]$ functions, (T,I) is called a residual pair or, T and I are residual to each other, if the following residual condition satisfies:

$$T(a,b) \le c \text{ if and only if } b \le I(a,c) \quad (a,b,c \in [0,1]). \tag{8}$$

In Definitions 2.3, the function T which is residual to I is unique, and vice versa.

Proposition *2.1:* If I is a fuzzy implication satisfying

(P5) I is right-continuous w.r.t. the second variable,

then the function $T : [0,1]^2 \to [0,1]$ defined by

$$T(a,b) = \inf\{x \in [0,1] \mid b \le I(a,x)\}, \quad a,b \in [0,1] \tag{9}$$

is residual to I, and the following formula holds:

$$I(a,b) = \sup\{x \in [0,1] \mid T(a,x) \le b\}, \quad a,b \in [0,1]. \tag{10}$$

Proof: Denote $\Gamma = \{x \in [0,1] \mid b \le I(a,x)\}$ ($a, b \in [0,1]$).

(i) Assume that $b \le I(a,c)$ ($c \in [0,1]$). Then $c \in \Gamma$. Since $T(a,b) = \inf \Gamma$, it follows that $T(a,b) \le c$ holds.

(ii) Assume that $T(a,b) \le c$ ($c \in [0,1]$). Taking into account that the fuzzy implication I satisfies (P4), we get $b \le 1 = I(a,1)$, and thus $\Gamma \ne \varnothing$. We have two cases to be considered. (a) For the case $T(a,b) < c$, it follows from the definition of infimum that there is $x_0 \in \Gamma$ making $x_0 < T(a,b) + \varepsilon < c$ for any $\varepsilon \in (0, c - T(a,b))$. Noting that $x_0 \in \Gamma$ and that the fuzzy implication I satisfies (P2), we get $b \le I(a,x_0) \le I(a,c)$. (b) For the case $T(a,b) = c$, assume on the contrary that $c \notin \Gamma$. Then in Γ there exist $c_0 > c_1 > \cdots > c_n > \cdots$ such that $\lim_{i\to\infty} c_i = c$ and $c_i > c$, so c is the right-limit of $\{c_i\}$ ($i = 0, 1, \cdots$). Because I satisfies (P5) and $b \le I(a, c_i)$ ($i = 0, 1, \cdots$), it follows that $b \le \lim_{i\to\infty} I(a, c_i) = I(a,c)$ (i.e., $c \in \Gamma$), which is a contradiction. So we get $c \in \Gamma$ and then $b \le I(a,c)$.

To sum up, we obtain that the residual condition (8) holds.

Moreover, because (T,I) is a residual pair, it follows that

$$\sup\{x \in [0,1] \mid T(a,x) \le b\} = \sup\{x \in [0,1] \mid x \le I(a,b)\} = I(a,b).$$

That is, (10) holds. ■

Definition *2.4:* ([1]) A function $I : [0,1]^2 \to [0,1]$ is said to be an R-implication if there exists a left-continuous t-norm T such that (10) holds.

The following lemma is from [31],[33].

Lemma *2.1:* If I is an R-implication derived from a left-continuous t-norm T, then (T,I) is a residual pair, and I satisfies (P5) together with the following properties:

(P6) $a \le I(b,c) \Longleftrightarrow b \le I(a,c)$,

(P7) $I(a, I(b,c)) = I(b, I(a,c))$,

(P8) $I(T(a,b),c) = I(a, I(b,c))$,

(P9) $I(1,a)=a$,

(P10) $a\le b \Longleftrightarrow I(a,b)=1$,

(P11) $I(\sup_{x\in X} x, a)=\inf_{x\in X} I(x,a)$,

(P12) $I(a,\inf_{x\in X} x)=\inf_{x\in X} I(a,x)$,

(P13) I is left-continuous w.r.t. the first variable,

in which $a,b,c,x\in[0,1]$ and $X\subset[0,1],\ X\neq\varnothing$.

III. The Fundamental Characteristics of the α-Universal Multiple I Restriction Method

***Definition** 3.1:* Let Z be any nonempty set and $F(Z)$ the set of all fuzzy subsets on Z, define partial order relation $\le_F$ on $F(Z)$ (on the basis of pointwise order) as follows ($A,B\in F(Z)$):

$$A\le_F B \text{ if and only if } A(z_0)\le B(z_0) \text{ for any } z_0\in Z.$$

***Lemma** 3.1:* ([34]) $<F(Z),\le_F>$ is a complete lattice.

For the general fuzzy reasoning (3), from the viewpoint of the α-universal multiple I restriction method, we can obtain the following basic principle:

α-universal multiple I restriction principle: The conclusion B^* of general fuzzy reasoning (3) is the largest fuzzy set satisfying (7) in $<F(V),\le_F>$.

It is obvious that such α-universal multiple I restriction principle for general fuzzy reasoning improves the previous α-multiple I restriction principle w.r.t. (5) in [21].

***Definition** 3.2:* Let $A_{ij},A_j^*\in F(U_j)$, $B_i\in F(V)$, if B^* (in $<F(V),\le_F>$) makes (7) hold for any $u_j\in U_j$ and $v\in V$ ($i=1,\cdots,m; j=1,\cdots,n$), then B^* is called an α-universal multiple I restriction solution.

***Theorem** 3.1:* Assume that $A_{ij},A_j^*\in F(U_j)$, $B_i\in F(V)$ ($i=1,\cdots,m; j=1,\cdots,n$), $\alpha\in(0,1]$. Then there exists a $B^*\in F(V)$ as an α-universal multiple I restriction solution if and only if the following inequality holds for any $u_j\in U_j, v\in V$ ($j=1,\cdots,n$):

$$\begin{aligned}&R_{\to_1,1}\to_2(R_{\to_1,2}\to_2(\cdots(R_{\to_1,m}\to_2(A_1^*(u_1)\to_2\\&(A_2^*(u_2)\to_2(\cdots(A_n^*(u_n)\to_2 0)\cdots)\cdots)<\alpha.\end{aligned}\tag{11}$$

Proof: (i) Suppose that (11) holds. We take $B^*(v)\equiv 0$, so B^* satisfies (7), and thus B^* is an α-universal multiple I restriction solution.

(ii) Suppose that there exists a $B^*\in F(V)$ which is an α-universal multiple I restriction solution. Then B^* satisfies (7). Since the fuzzy implication $\to_2$ satisfies (P2), we have $A_n^*(u_n)\to_2 B^*(v)\ge A_n^*(u_n)\to_2 0$, and then

$$\begin{aligned}\alpha>\ &R_{\to_1,1}\to_2(R_{\to_1,2}\to_2(\cdots(R_{\to_1,m}\to_2(A_1^*(u_1)\to_2\\&\quad(A_2^*(u_2)\to_2(\cdots(A_n^*(u_n)\to_2 B^*(v))\cdots)\cdots)\\&\ge R_{\to_1,1}\to_2(R_{\to_1,2}\to_2(\cdots(R_{\to_1,m}\to_2(A_1^*(u_1)\to_2\\&\quad(A_2^*(u_2)\to_2(\cdots(A_n^*(u_n)\to_2 0)\cdots)\cdots),\end{aligned}$$

i.e., (11) holds. ■

Similar to Theorem 3.1, we can prove Proposition 3.1.

***Proposition** 3.1:* Suppose that D_1 is an α-universal multiple I restriction solution, and that $D_2\le_F D_1$ (in which $D_1,D_2\in<F(V),\le_F>$). Then D_2 is an α-universal multiple I restriction solution.

***Remark** 3.1:* Assume that (11) holds. For an α-universal multiple I restriction solution D_1^* for general fuzzy reasoning, every fuzzy set D_2^* which is less than D_1^*, will be an α-universal multiple I restriction solution (by virtue of Proposition 3.1, where $D_1^*,D_2^*\in<F(V),\le_F>$). This implies that there are many α-universal multiple I restriction solutions, including $D_3^*(v)\equiv 0$ ($v\in V$). The last D_3^* is a special solution, for which (7) always holds no matter what $A_{ij},A_j^*\in F(U_j)$, $B_i\in F(V)$ are adopted. Therefore, when the optimal α-universal multiple I restriction solution exists, it should be the largest one; in other words, it should be the supremum of all solutions.

***Proposition** 3.2:* Let I be a fuzzy implication satisfying (P5), and T the function residual to I. If T is associative, then I satisfies

(P14) $I(a,I(b,c))=I(T(b,a),c)$ $(a,b,c\in[0,1])$.

Proof: Let any $x\in[0,1]$. Since T is associative, then it follows from residual condition (8) that we can get the following formulas are equivalent to each other:

$$\begin{aligned}&x\le I(a,I(b,c)),\ T(a,x)\le I(b,c),\\&T(b,T(a,x))\le c,\ T(T(b,a),x)\le c,\\&x\le I(T(b,a),c).\end{aligned}$$

Thus, because x is arbitrary, we get that $I(a,I(b,c))=I(T(b,a),c)$ $(a,b,c\in[0,1])$, i.e., I satisfies (P14). ■

For a binary operation T, denote

$$T(x_1,x_2,\cdots,x_n)\triangleq T(T(x_1,\cdots,x_{n-1}),x_n)$$

and

$$T(x_n,x_{n-1},\cdots,x_2,x_1)\triangleq T_{i=1}^n x_i,$$

where $x_i\in[0,1]$, $i=1,\cdots,n$.

***Proposition** 3.3:* If $\to$ is a fuzzy implication satisfying (P5), and its residual function T is associative, then

$$x_1\to(x_2\to(\cdots(x_n\to y)\cdots))=(T_{i=1}^n x_i)\to y\tag{12}$$

holds for any $x_i,y\in[0,1]$ ($i=1,\cdots,n;\ n\ge 1$).

Proof: Based on the conditions that $T,\to$ possess, it follows from Proposition 3.2 that $\to$ satisfies (P14). If $n=1,2$, then it is obvious that the conclusion holds. If $n\ge 3$, then we have

$$\begin{aligned}&x_1\to(x_2\to(\cdots(x_n\to y)\cdots))\\&=\ x_1\to(x_2\to(\cdots(x_{n-1}\to(x_n\to y))\cdots))\\&=\ x_1\to(x_2\to(\cdots(T(x_n,x_{n-1})\to y)\cdots))\\&=\ x_1\to(x_2\to(\cdots(T(T(x_n,x_{n-1}),x_{n-2})\to y)\cdots))\\&=\ x_1\to(x_2\to(\cdots(T(x_n,x_{n-1},x_{n-2})\to y)\cdots))\\&=\ \cdots=\ T(x_n,x_{n-1},\cdots,x_2,x_1)\to y\\&=\ (T_{i=1}^n x_i)\to y.\end{aligned}$$

■

By Proposition 3.3, we can get Theorem 3.2.

***Theorem** 3.2:* If $\to_2$ is a fuzzy implication satisfying (P5), and its residual function T is associative, then (7) and (11) are equivalent to the following formulas, respectively:

$$\begin{aligned}&(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))\to_2\\&((T_{j=1}^n A_j^*(u_j))\to_2 B^*(v))<\alpha,\end{aligned}\tag{13}$$

$$\begin{aligned}&(T_{i=1}^{m}R_{\to_1,i}(u_1,\cdots,u_n,v))\to_2\\&((T_{j=1}^{n}A_j^*(u_j))\to_2 0)<\alpha,\end{aligned} \tag{14}$$

IV. Optimal Solution of the α-universal multiple I restriction method

Theorem 4.1: If $\to_2$ is a fuzzy implication satisfying (P5), and its residual function T is associative, and (14) holds, then the supremum of α-universal multiple I restriction solutions can be computed as follows ($v \in V, i = 1,\cdots,m; j = 1,\cdots,n$):

$$\begin{aligned}B^*(v)=\inf_{u_j\in U_j}\{&T(T_{j=1}^{n}A_j^*(u_j),\\&T(T_{i=1}^{m}R_{\to_1,i}(u_1,\cdots,u_n,v),\ \alpha))\}.\end{aligned} \tag{15}$$

Proof: Because $\to_2$ is a fuzzy implication satisfying (P5), it follows from Proposition 2.1 that the residual condition (8) holds for $(T,\to_2)$.

Let

$$G_1=\{v\in V\mid B^*(v)=0\},\ G_2=\{v\in V\mid B^*(v)>0\}.$$

Assume that $C\in F(V)$, and that $C(v)=0$ for $v\in G_1$, and that $C(v)<B^*(v)$ for $v\in G_2$. We shall verify that C is an α-universal multiple I restriction solution, that is, the following inequality holds for any $u_j\in U_j, v\in V$ ($j=1,\cdots,n$):

$$\begin{aligned}&(T_{i=1}^{m}R_{\to_1,i}(u_1,\cdots,u_n,v))\to_2\\&((T_{j=1}^{n}A_j^*(u_j))\to_2 C(v))<\alpha.\end{aligned} \tag{16}$$

If $v\in G_1$, then it follows from (14) that $C(v)=0$ satisfies (16) for any $u_j\in U_j$ ($j=1,\cdots,n$).

If $v\in G_2$, then it follows from (15) and $C(v)<B^*(v)$ that

$$C(v)<T(T_{j=1}^{n}A_j^*(u_j),T(T_{i=1}^{m}R_{\to_1,i}(u_1,\cdots,u_n,v),\ \alpha)) \tag{17}$$

holds for any $u_j\in U_j$ ($j=1,\cdots,n$). Assume on the contrary that (16) does not hold. Then there exist $u_j^\circ\in U_j$ ($j=1,\cdots,n$) such that

$$\begin{aligned}&(T_{i=1}^{m}R_{\to_1,i}(u_1^\circ,\cdots,u_n^\circ,v))\to_2\\&((T_{j=1}^{n}A_j^*(u_j^\circ))\to_2 C(v))\geq\alpha.\end{aligned}$$

Noting that $(T,\to_2)$ satisfies the residual condition (8), we get that

$$T(T_{i=1}^{m}R_{\to_1,i}(u_1^\circ,\cdots,u_n^\circ,v),\ \alpha)\leq(T_{j=1}^{n}A_j^*(u_j^\circ))\to_2 C(v),$$

and

$$T(T_{j=1}^{n}A_j^*(u_j^\circ),\ T(T_{i=1}^{m}R_{\to_1,i}(u_1^\circ,\cdots,u_n^\circ,v),\alpha))\leq C(v),$$

which contradicts (17). Together we have that (16) holds for any $u_j\in U_j, v\in V$ ($j=1,\cdots,n$). Consequently, C is an α-universal multiple I restriction solution.

Furthermore, we shall prove that B^* expressed by (15) is the supremum of α-universal multiple I restriction solutions. Assume that $D\in F(V)$, and that there exists $v^\circ\in V$ such that $D(v^\circ)>B^*(v^\circ)$. We shall show that D is not an α-universal multiple I restriction solution. In fact, we get from (15) that there exist $u_j^\circ\in U_j$ ($j=1,\cdots,n$) such that

$$D(v^\circ)>T(T_{j=1}^{n}A_j^*(u_j^\circ),\ T(T_{i=1}^{m}R_{\to_1,i}(u_1^\circ,\cdots,u_n^\circ,v^\circ),\alpha)).$$

Since $(T,\to_2)$ satisfies the residual condition (8), it follows that

$$\begin{aligned}&T(T_{i=1}^{m}R_{\to_1,i}(u_1^\circ,\cdots,u_n^\circ,v^\circ),\ \alpha)\\&\leq(T_{j=1}^{n}A_j^*(u_j^\circ))\to_2 D(v^\circ),\end{aligned}$$

and thus

$$\begin{aligned}\alpha\leq{}&(T_{i=1}^{m}R_{\to_1,i}(u_1^\circ,\cdots,u_n^\circ,v^\circ))\to_2\\&((T_{j=1}^{n}A_j^*(u_j^\circ))\to_2 D(v^\circ)).\end{aligned}$$

Therefore, D is not an α-universal multiple I restriction solution.

To sum up, B^* expressed by (15) is the supremum of α-universal multiple I restriction solutions. ∎

For an R-implication $\to$ derived from left-continuous t-norm T, then $\to$ satisfies (P5), and T is associative (by the definition of t-norm). Thus it follows from Theorem 4.1 that we get Corollary 4.1.

Corollary 4.1: If $\to_2$ is an R-implication derived from left-continuous t-norm T, and (14) holds, then the supremum of α-universal multiple I restriction solutions can be expressed as follows ($v\in V, i=1,\cdots,m; j=1,\cdots,n$):

$$\begin{aligned}B^*(v)=\inf_{u_j\in U_j}\{&T(T_{j=1}^{n}A_j^*(u_j),\\&T(T_{i=1}^{m}R_{\to_1,i}(u_1,\cdots,u_n,v),\ \alpha))\}.\end{aligned}$$

Example 4.1: The following functions are fuzzy implications satisfying (P5), in which its residual function T is associative ($a,b\in[0,1]$, and $x\,'$ denotes $1-x$). Here $I_{y-0.5}, I_{ep}$ are from [25,35].

$I_L(a,b)=\begin{cases}1, & a\leq b\\ a\,'+b, & a>b\end{cases}$
(Łukasiewicz implication),

$I_G(a,b)=\begin{cases}1, & a\leq b\\ b, & a>b\end{cases}$ (Gödel implication),

$I_{Go}(a,b)=\begin{cases}1, & a=0\\ (b/a)\wedge 1, & a\neq 0\end{cases}$ (Goguen implication),

$I_{FD}(a,b)=\begin{cases}1, & a\leq b\\ a\,'\vee b, & a>b\end{cases}$ (Fodor implication),

$I_{GR}(a,b)=\begin{cases}1, & a\leq b\\ 0, & a>b\end{cases}$ (Gaines-Rescher implication),

$I_{y-0.5}(a,b)=\begin{cases}1, & a\leq b\\ 1-(\sqrt{1-b}-\sqrt{1-a})^2, & a>b\end{cases}$,

$I_{ep}(a,b)=\begin{cases}1, & a\leq b\\ (2b-ab)/(a+b-ab), & a>b\end{cases}$.

Moreover, the functions respectively residual to $I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_{y-0.5}, I_{ep}$ are as follows.

$T_L(a,b)=\begin{cases}a+b-1, & a+b>1\\ 0, & a+b\leq 1\end{cases}$,

$T_G(a,b)=a\wedge b$,

$T_{Go}(a,b)=a\times b$,

$T_{FD}(a,b)=\begin{cases}a\wedge b, & a+b>1\\ 0, & a+b\leq 1\end{cases}$,

$T_{GR}(a,b) = \begin{cases} a, & b > 0 \\ 0, & b = 0 \end{cases}$,

$T_{y-0.5}(a,b) = \begin{cases} 1-(g(a,b))^2, & g(a,b) \le 1 \\ 0, & g(a,b) > 1 \end{cases}$ where $g(a,b) = \sqrt{1-a} + \sqrt{1-b}$,

$T_{ep}(a,b) = ab/[2-(a+b-ab)]$.

Here T_{ep} is the Einstein product; and $T_{y-0.5}$ is the t-norm of Yager (where ω takes 0.5), which is defined as

$$T_{y-\omega}(a,b) = 1 - \min[1, ((1-a)^\omega + (1-b)^\omega)^{1/\omega}],$$

in which $\omega \in (0,\infty)$. It is easy to know that $T_L, T_G, T_{Go}, T_{FD}, T_{y-0.5}, T_{ep}$ are left-continuous t-norm, thus $I_L, I_G, I_{Go}, I_{FD}, I_{y-0.5}, I_{ep}$ are R-implications.

***Example** 4.2:* From Theorem 3.1 and Theorem 3.2, we get that if $\to_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_{y-0.5}, I_{ep}\}$, then there exists a $B^* \in F(V)$ which is an α-universal multiple I restriction solution if and only if (11) (or (14)) holds.

***Proposition** 4.1:* Suppose that $\to_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_{y-0.5}, I_{ep}\}$, and that T is the function residual to $\to_2$, and finally that (14) holds, then the supremum of α-universal multiple I restriction solutions is as follows, respectively ($v \in V, i = 1, \cdots, m; j = 1, \cdots, n$):

(i) If $\to_2$ takes I_L, then

$$B^*(v) = \inf_{u_j \in U_j} \{(T_{j=1}^n A_j^*(u_j)) + (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v)) + \alpha - 2\}.$$

(ii) If $\to_2$ takes I_G, then

$$B^*(v) = \inf_{u_j \in U_j} \{(T_{j=1}^n A_j^*(u_j)) \wedge (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v))\} \wedge \alpha.$$

(iii) If $\to_2$ takes I_{Go}, then

$$B^*(v) = \inf_{u_j \in U_j} \{(T_{j=1}^n A_j^*(u_j)) \times (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v)) \times \alpha\}.$$

(iv) If $\to_2$ takes I_{FD}, then

$$B^*(v) = \inf_{u_j \in U_j} \{(T_{j=1}^n A_j^*(u_j)) \wedge (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v))\} \wedge \alpha.$$

(v) If $\to_2$ takes I_{GR}, then

$$B^*(v) = \inf_{u_j \in U_j} \{T_{j=1}^n A_j^*(u_j)\}.$$

(vi) If $\to_2$ takes $I_{y-0.5}$, then

$$B^*(v) = \inf_{u_j \in U_j} \left\{ 1 - \left(\sqrt{1 - (T_{j=1}^n A_j^*(u_j))} + \sqrt{1 - (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v))} + \sqrt{1-\alpha} \right)^2 \right\}.$$

(vii) If $\to_2$ takes I_{ep}, then

$$B^*(v) = \inf_{u_j \in U_j} \left\{ \frac{(T_{j=1}^n A_j^*(u_j))\Psi_{ep}}{2 - (T_{j=1}^n A_j^*(u_j)) - \Psi_{ep} + (T_{j=1}^n A_j^*(u_j))\Psi_{ep}} \right\}$$

where

$$\Psi_{ep} = \{\alpha(T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v))\} \Big/ \{2 - (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v)) - \alpha + \alpha(T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v))\}.$$

Proof: If $\to_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_{y-0.5}, I_{ep}\}$, then it follows from Theorem 4.1 that the supremum of α-universal multiple I restriction solutions is ($v \in V$):

$$B^*(v) = \inf_{u_j \in U_j} \{T(T_{j=1}^n A_j^*(u_j), T(T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v), \alpha))\}.$$

where T is the function residual to $\to_2$. Then, we need to get the specific expression of B^*. We only prove the case of I_L as an example, the remainders can be proved similarly.

Suppose that $\to_2$ takes I_L. It follows from Example 4.1 that T_L is the function residual to I_L. By (14), we have that $T_{j=1}^n A_j^*(u_j) > 0$, and

$$T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v) > 1 - (T_{j=1}^n A_j^*(u_j)),$$

$$1 - (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v)) + 1 - (T_{j=1}^n A_j^*(u_j)) < \alpha$$

hold for any $u_j \in U_j, v \in V$ $(j = 1, \cdots, n)$. We further get

$$(T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v)) + \alpha > 1,$$

$$(T_{j=1}^n A_j^*(u_j)) + (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v)) + \alpha - 1 > 1$$

hold for any $u_j \in U_j, v \in V$ $(j = 1, \cdots, n)$. Thus it follows from Theorem 4.1 that ($v \in V$)

$$\begin{aligned} B^*(v) &= \inf_{u_j \in U_j} T(T_{j=1}^n A_j^*(u_j), T(T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v), \alpha)) \\ &= \inf_{u_j \in U_j} T(T_{j=1}^n A_j^*(u_j), ((T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v)) + \alpha - 1)) \\ &= \inf_{u_j \in U_j} \{(T_{j=1}^n A_j^*(u_j)) + (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v)) + \alpha - 2\}. \end{aligned}$$

■

***Proposition** 4.2:* If $\to_2 \in \{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_{y-0.5}, I_{ep}\}$, and T is the function residual to $\to_2$, and (14) holds, then the condition (which the supremum B^* of α-universal multiple I restriction solutions is the maximum) is as follows, respectively (for any $u_j \in U_j,\ v \in V,\ i = 1, \cdots, m;\ j = 1, \cdots, n$):

(i) Let $\to_2$ take I_L, then the condition is as follows:

$$(T_{j=1}^n A_j^*(u_j)) + (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v)) > \inf_{u_j \in U_j} \{(T_{j=1}^n A_j^*(u_j)) + (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v))\}.$$

(ii) Let $\to_2$ take I_G, then the condition is as follows:

$$(T_{j=1}^n A_j^*(u_j)) \wedge (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v)) \wedge \alpha > \inf_{u_j \in U_j} \{(T_{j=1}^n A_j^*(u_j)) \wedge (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v))\}.$$

(iii) Let $\to_2$ take I_{Go}, then the condition is as follows:

$$(T_{j=1}^n A_j^*(u_j)) \times (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v)) > \inf_{u_j \in U_j} \{(T_{j=1}^n A_j^*(u_j)) \times (T_{i=1}^m R_{\to_1,i}(u_1, \cdots, u_n, v))\}.$$

(iv) Let $\to_2$ take I_{FD}, then the condition is as follows:

$$(T_{j=1}^n A_j^*(u_j)) \wedge (T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v)) \wedge \alpha > \inf_{u_j\in U_j}\{(T_{j=1}^n A_j^*(u_j)) \wedge (T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))\}.$$

(v) Let $\to_2$ take I_{GR}, then the condition is as follows:

$$T_{j=1}^n A_j^*(u_j) > \inf_{u_j\in U_j}\{T_{j=1}^n A_j^*(u_j)\}.$$

(vi) Let $\to_2$ take $I_{y-0.5}$, then the condition is as follows:

$$1-\Big(\sqrt{1-(T_{j=1}^n A_j^*(u_j))}+ \sqrt{1-(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))}+\sqrt{1-\alpha}\ \Big)^2 > \inf_{u_j\in U_j}\Big\{1-\Big(\sqrt{1-(T_{j=1}^n A_j^*(u_j))}+ \sqrt{1-(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))}+\sqrt{1-\alpha}\ \Big)^2\Big\}.$$

(vii) Let $\to_2$ take I_{ep}, then the condition is as follows:

$$\frac{(T_{j=1}^n A_j^*(u_j))\Psi_{ep}}{2-(T_{j=1}^n A_j^*(u_j))-\Psi_{ep}+(T_{j=1}^n A_j^*(u_j))\Psi_{ep}} > \inf_{u_j\in U_j}\left\{\frac{(T_{j=1}^n A_j^*(u_j))\Psi_{ep}}{2-(T_{j=1}^n A_j^*(u_j))-\Psi_{ep}+(T_{j=1}^n A_j^*(u_j))\Psi_{ep}}\right\}.$$

Proof: If the supremum B^* (of α-universal multiple I restriction solutions) is an α-universal multiple I restriction solution, then B^* is the maximum one. Therefore, it is enough to prove that B^* is an α-universal multiple I restriction solution, i.e., B^* should make (13) hold for any $u_j \in U_j, v \in V$ $(j = 1,\cdots,n)$. We prove the cases of $I_L, I_{y-0.5}$ as examples.

(i) Let $\to_2$ take I_L. From Proposition 4.1, the supremum of α-universal multiple I restriction solutions is

$$B^*(v) = \inf_{u_j\in U_j}\{(T_{j=1}^n A_j^*(u_j))+ (T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))+\alpha-2\}.$$

By the condition given in (i), we obtain

$$B^*(v) < \alpha-2+(T_{j=1}^n A_j^*(u_j))+ (T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v)).$$

Note that $0 \geq \alpha-2+(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))$, then

$$\begin{aligned}&T_{j=1}^n A_j^*(u_j)\\ &\geq (T_{j=1}^n A_j^*(u_j))+\alpha-2+(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))\\ &> B^*(v)\end{aligned}$$

holds. Thus

$$\begin{aligned}&T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v)\\ &\geq \alpha-1+(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))\\ &= 1-(T_{j=1}^n A_j^*(u_j))+[\alpha-2+(T_{j=1}^n A_j^*(u_j))+ (T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))]\\ &> 1-(T_{j=1}^n A_j^*(u_j))+B^*(v).\end{aligned}$$

To sum up, we get $(u_j \in U_j, v \in V, j = 1,\cdots,n)$

$$\begin{aligned}&(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v)) \to_2 ((T_{j=1}^n A_j^*(u_j)) \to_2 B^*(v))\\ &= (T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v)) \to_2 [1-(T_{j=1}^n A_j^*(u_j))+ B^*(v)]\\ &= 1-(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))+1-(T_{j=1}^n A_j^*(u_j))+ B^*(v)\\ &< 1-(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))+1-(T_{j=1}^n A_j^*(u_j))+ [\alpha-2+(T_{j=1}^n A_j^*(u_j))+(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))]\\ &= \alpha.\end{aligned}$$

Thus B^* makes (13) hold for any $u_j \in U_j, v \in V$ $(j = 1,\cdots,n)$, and hence it is an α-universal multiple I restriction solution.

(vi) Let $\to_2$ employ $I_{y-0.5}$. It follows from Proposition 4.1 that the supremum of α-universal multiple I restriction solutions is

$$B^*(v) = \inf_{u_j\in U_j}\Big\{1-\Big(\sqrt{1-(T_{j=1}^n A_j^*(u_j))}+ \sqrt{1-(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))}+\sqrt{1-\alpha}\ \Big)^2\Big\}.$$

It follows from the condition given in (vi) that we have

$$B^*(v) < 1-\Big(\sqrt{1-(T_{j=1}^n A_j^*(u_j))}+ \sqrt{1-(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))}+\sqrt{1-\alpha}\ \Big)^2,$$

which implies

$$\sqrt{1-B^*(v)} > \sqrt{1-(T_{j=1}^n A_j^*(u_j))}+\sqrt{1-\alpha}+ \sqrt{1-(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))}.$$

Then we can respectively obtain $T_{j=1}^n A_j^*(u_j) > B^*(v)$, and

$$T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v) > 1-\Big(\sqrt{1-B^*(v)}- \sqrt{1-(T_{j=1}^n A_j^*(u_j))}\ \Big)^2,$$

$$\alpha > 1-\left(\sqrt{1-B^*(v)}-\sqrt{1-(T_{j=1}^n A_j^*(u_j))}- \sqrt{1-(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))}\right)^2.$$

To sum up, we have $(u_j \in U_j, v \in V, j = 1,\cdots,n)$

$$\begin{aligned}&(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v)) \to_2 ((T_{j=1}^n A_j^*(u_j)) \to_2 B^*(v))\\ &= (T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v)) \to_2 \Big[1-\Big(\sqrt{1-B^*(v)}- \sqrt{1-(T_{j=1}^n A_j^*(u_j))}\ \Big)^2\Big]\\ &= 1-\left(\sqrt{1-B^*(v)}-\sqrt{1-(T_{j=1}^n A_j^*(u_j))}- \sqrt{1-(T_{i=1}^m R_{\to_1,i}(u_1,\cdots,u_n,v))}\right)^2\\ &< \alpha.\end{aligned}$$

Consequently, B^* lets (13) hold for any $u_j \in U_j, v \in V$ $(j = 1, \cdots, n)$, and then it is an α-universal multiple I restriction solution. ■

Table 1 shows some results of the α-universal multiple I restriction method from Proposition 4.1 and Proposition 4.2, where denote $T_{i=1}^{m} R_{\rightarrow_1,i} = T_{i=1}^{m} R_{\rightarrow_1,i}(u_1, \cdots, u_n, v)$ for short.

***Remark** 4.1:* It is noted that we can always find better optimal solution of the α-universal multiple I restriction method (than the α-multiple I restriction method). For example, we take $\rightarrow = I_{Go}$ in (5) for the α-multiple I restriction method, then we can choose $\rightarrow_2 = I_{Go}$, $\rightarrow_1 = I_L$ in (7) for the α-universal multiple I restriction method. Then by computing, we can found that, for the same $A_{ij}, A_j^*, B_i, \alpha$ $(i = 1, \cdots, m; j = 1, \cdots, n)$, the optimal solution of the α-universal multiple I restriction method (i.e., the supremum of α-universal multiple I restriction solutions) is larger than the optimal one from the α-multiple I restriction method (noting that they are equal only for some extreme cases). From the viewpoint of the α-universal multiple I restriction principle (which seeks the largest B^* satisfying (7)), the α-universal multiple I restriction method can let the inference closer, then it is more reasonable than the α-multiple I restriction method.

***Remark** 4.2:* Aiming at the set of fuzzy implications $\{I_L, I_G, I_{Go}, I_{FD}, I_{GR}, I_{y-0.5}, I_{ep}\}$, it follows from the α-universal multiple I restriction method that there are $7 * 7 = 49$ kinds of specific fuzzy reasoning methods for general fuzzy reasoning, in which $6 * 7 = 42$ kinds of actual specific methods for general fuzzy reasoning exist (noting that the expressions are the same for the case $\rightarrow_2 = I_G$ and $\rightarrow_2 = I_{FD}$, accroding to Table 1). However, from the α-multiple I restriction method, there are only 7 kinds of specific methods for general fuzzy reasoning (including 6 kinds of actual specific methods for general fuzzy reasoning), see Table 2. These further demonstrates that the α-universal multiple I restriction method is superior to the α-multiple I restriction method.

As a very hot research field, affective computing (see [36], [37]) is the research of systems and devices which are able to recognize, interpret, process, and simulate human affects, which especially focuses on emotional recognition, emotional expression as well as emotional interaction. In this field, the emotional response [36] is a vital research topic for emotional interaction (for example, one man may be glad when his good friend is happy).

The emotions of human are uncertain, dynamic and complex, then we can analyze the relationship of main emotions via fuzzy sets, and obtain related fuzzy rules of normal emotion responses from the interaction process, and establish the corresponding rule base. However there exist different situations in some special cases. For example, when two people are opposed, the emotion response is not normal (one may be sad when the other is happy). For such case, we can carry through the general fuzzy reasoning to get emotion response with the normal emotion response rule base, which can be computed by the α-universal multiple I restriction method. Due to the limit of the length of this paper, we shall show the details in the further work.

V. Conclusions

The α-universal multiple I restriction method for general fuzzy reasoning is put forward. First of all, we provide the α-universal multiple I restriction principle, which improves the previous restriction principle, and then prove the existing condition of α-universal multiple I restriction solutions. Moreover, we obtain the optimal solution (i.e., the supremum of solutions) of the α-universal multiple I restriction method for the fuzzy implications with residual pair, and the R-implications, and also some specific fuzzy implications. And then we achieve the condition which the corresponding supremum is the maximum. Finally, it is verified that the α-universal multiple I restriction method is superior to the α-multiple I restriction method.

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TABLE I

SOME RESULTS OF THE α-UNIVERSAL MULTIPLE I RESTRICTION METHOD

$\to_2$	The supremum B^*	The condition which B^* is the maximum
I_L	$\inf_{u_j \in U_j}\{(T_{j=1}^n A_j^*(u_j)) + (T_{i=1}^m R_{\to_1,i}) + \alpha - 2\}$	$(T_{j=1}^n A_j^*(u_j)) + (T_{i=1}^m R_{\to_1,i}) > \inf_{u_j \in U_j}\{(T_{j=1}^n A_j^*(u_j)) + (T_{i=1}^m R_{\to_1,i})\}$
I_G	$\inf_{u_j \in U_j}\{(T_{j=1}^n A_j^*(u_j)) \wedge (T_{i=1}^m R_{\to_1,i})\} \wedge \alpha$	$((T_{j=1}^n A_j^*(u_j)) \wedge (T_{i=1}^m R_{\to_1,i})) \wedge \alpha > \inf_{u_j \in U_j}\{(T_{j=1}^n A_j^*(u_j)) \wedge (T_{i=1}^m R_{\to_1,i})\}$
I_{Go}	$\inf_{u_j \in U_j}\{(T_{j=1}^n A_j^*(u_j)) \times (T_{i=1}^m R_{\to_1,i}) \times \alpha\}$	$(T_{j=1}^n A_j^*(u_j)) \times (T_{i=1}^m R_{\to_1,i}) > \inf_{u_j \in U_j}\{(T_{j=1}^n A_j^*(u_j)) \times (T_{i=1}^m R_{\to_1,i})\}$
I_{FD}	$\inf_{u_j \in U_j}\{(T_{j=1}^n A_j^*(u_j)) \wedge (T_{i=1}^m R_{\to_1,i})\} \wedge \alpha$	$((T_{j=1}^n A_j^*(u_j)) \wedge (T_{i=1}^m R_{\to_1,i})) \wedge \alpha > \inf_{u_j \in U_j}\{(T_{j=1}^n A_j^*(u_j)) \wedge (T_{i=1}^m R_{\to_1,i})\}$
I_{GR}	$\inf_{u_j \in U_j}\{T_{j=1}^n A_j^*(u_j)\}$	$T_{j=1}^n A_j^*(u_j) > \inf_{u_j \in U_j}\{T_{j=1}^n A_j^*(u_j)\}$
$I_{y-0.5}$	$\inf_{u_j \in U_j}\left\{1 - \left(\sqrt{1-(T_{j=1}^n A_j^*(u_j))} + \sqrt{1-(T_{i=1}^m R_{\to_1,i})} + \sqrt{1-\alpha}\right)^2\right\}$	$1 - \left(\sqrt{1-(T_{j=1}^n A_j^*(u_j))} + \sqrt{1-(T_{i=1}^m R_{\to_1,i})} + \sqrt{1-\alpha}\right)^2 > \inf_{u_j \in U_j}\left\{1 - \left(\sqrt{1-(T_{j=1}^n A_j^*(u_j))} + \sqrt{1-(T_{i=1}^m R_{\to_1,i})} + \sqrt{1-\alpha}\right)^2\right\}$
I_{ep}	$\inf_{u_j \in U_j}\{(T_{j=1}^n A_j^*(u_j))\Psi_{ep}/[2 - (T_{j=1}^n A_j^*(u_j)) - \Psi_{ep} + (T_{j=1}^n A_j^*(u_j))\Psi_{ep}]\}$	$(T_{j=1}^n A_j^*(u_j))\Psi_{ep}/[2 - (T_{j=1}^n A_j^*(u_j)) - \Psi_{ep} + (T_{j=1}^n A_j^*(u_j))\Psi_{ep}] > \inf_{u_j \in U_j}\{(T_{j=1}^n A_j^*(u_j))\Psi_{ep}/[2 - (T_{j=1}^n A_j^*(u_j)) - \Psi_{ep} + (T_{j=1}^n A_j^*(u_j))\Psi_{ep}]\}$

TABLE II

SPECIFIC FUZZY REASONING METHODS FOR GENERAL FUZZY REASONING

	α-universal multiple I restriction method	α-multiple I restriction method
Specific methods	49 kinds	7 kinds
Actual specific methods	42 kinds	6 kinds

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