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Closed Form Fuzzy Interpolation with Interval Type-2 Fuzzy Sets

Longzhi Yang, Chengyuan Chen, Nanlin Jin, Xin Fu and Qiang Shen

***Abstract*— Fuzzy rule interpolation enables fuzzy inference with sparse rule bases by interpolating inference results, and may help to reduce system complexity by removing similar (often redundant) neighbouring rules. In particular, the recently proposed closed form fuzzy interpolation offers a unique approach which guarantees convex interpolated results in a closed form. However, the difficulty in defining the required precise-valued membership functions still poses significant restrictions over the applicability of this approach. Such limitations can be alleviated by employing type-2 fuzzy sets as their membership functions are themselves fuzzy. This paper extends the closed form fuzzy rule interpolation using interval type-2 fuzzy sets. In this way, as illustrated in the experiments, closed form fuzzy interpolation is able to deal with uncertainty in data and knowledge with more flexibility.**

Keywords - Fuzzy rule interpolation, closed form interpolation, interval type-2 fuzzy sets.

I. Introduction

The compositional rule of inference [1] offers an effective mechanism to perform inference based on dense rule bases (by which an observation is always at least partially covered). However, for cases where a fuzzy rule base contains 'gaps' (i.e., the so-called sparse rule base as termed in [2]), if a given observation has no overlap with the antecedent values of any given rule, conventional fuzzy inference methods will fail. Fortunately, by using fuzzy rule interpolation [3], certain conclusions may still be obtained. Moreover, with the help of fuzzy rule interpolation techniques, the complexity of a rule base can be reduced by omitting those fuzzy rules which may be approximated from their neighboring ones.

Nevertheless, despite of these advantages, the application of traditional fuzzy rule interpolation methods may lead to invalid fuzzy conclusions. Also, the convexity of the derived fuzzy sets is not guaranteed [4], which is often a crucial requirement for fuzzy inference in order to attain better interpretability in the results. In order to overcome such drawbacks, a number of significant extensions to the original fuzzy rule interpolation methods have been proposed in the literature, including [5], [6], [7], [8], [9], [10], [11], although extensions in other dimensions have also been proposed, such as [12], [13], [14]. In particular, closed form fuzzy interpolation (CFFI) [5] not only guarantees convex interpolated results, but also does so in a closed form.

The majority of the existing fuzzy interpolation approaches, including CFFI, are developed based on type-1 fuzzy sets only. Membership functions play an important role in defining fuzzy sets, and it is sometimes difficult, if not impossible, to give precise values for such membership functions. Different types of uncertainty in fuzzy rule-based systems have been discussed in [15], which include: (1) that the variables that are used in the antecedents and consequents of rules may be indiscernible; (2) that the meanings of the words may be vague because words mean different things to different people; and (3) that an object can belong to a set to a given degree, but that degree may itself be uncertain. Most of such types of uncertainty translate into difficulties in determining the crisp membership functions of fuzzy sets.

The concept of type-2 fuzzy sets was proposed as an extension of the concept of type-1 fuzzy sets to address the above situations. A type-1 fuzzy set has a grade of membership that is crisp, whereas a type-2 fuzzy set has grades of membership that are fuzzy. Membership functions of type-1 fuzzy sets are two-dimensional, whereas membership functions of type-2 fuzzy sets are three-dimensional. It is the new third-dimension that provides additional degrees of freedom for handling uncertainty and that is useful under the circumstances where it is difficult to determine the precise membership functions [16]. Note that apart from the use of type-2 fuzzy sets, other extensions have also been developed to represent various types of uncertainty that are associated with defining type-1 fuzzy sets, including interval-valued fuzzy sets [17], R-fuzzy sets [18], and rough-fuzzy sets [19].

Of particular interest herein are the very recent developments that extend the existing fuzzy rule interpolation approaches using interval type-2 fuzzy sets [20], [21], [22]. However, common to their type-1 counterparts, all of these approaches do not perform interpolation in a closed form. This paper extends the existing CFFI approach with the use of interval type-2 fuzzy sets, which guarantees valid interpolated results in a closed form. It also briefly discusses the important properties of the proposed approach supported by examples.

The remainder of this paper is structured as follows. Section II reviews the background, including the existing closed form fuzzy interpolation for type-1 fuzzy sets, and type-2 fuzzy sets. Section III presents the proposed extension based on interval type-2 fuzzy sets. This is followed by an illustrative example given in Section IV, to show how the proposed approach may help in real-world applications. Section

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V concludes the paper with future research suggested.

II. Background

This section firstly summarises CFFI with type-1 fuzzy sets, starting with an introduction of the underpinning interval rule interpolation. Then, type-2 fuzzy sets, with a particular focus on interval type-2 fuzzy sets, are reviewed.

A. Closed Form Fuzzy Interpolation

A fuzzy set A is said to be convex if and only if $\mu_A(z) \geq \min(\mu_A(x),\ \mu_A(y))$, $\forall(x,y,z) \in \Re$ and $z \in [x,y]$. A is said to be normal if and only if $\exists x \in \Re$ such that $\mu_A(x) = 1$. According to the decomposition principle, a normal and convex fuzzy set C can be represented by a series of α-cut intervals, each denoted as C_α, $\alpha \in (0,1]$. The support of a fuzzy set A is denoted by $supp(A) = \{x|\mu_A(x) > 0\}$. The core of fuzzy set A is denoted as $core(A) = \{x|\mu_A(x) = 1\}$.

Definition 2.1: An interval C_i is said to be *less than* another interval C_j, denoted as $C_i \prec C_j$, if $\inf\{C_i\} < \inf\{C_j\}$ and $\sup\{C_i\} < \sup\{C_j\}$, where $\inf\{A_k\}$ and $\sup\{A_k\}$ denote the infimum and the supremum of A_k, respectively.

Definition 2.2: A normal and convex fuzzy set C_i is said to be *less than* another normal and convex fuzzy set C_j, denoted as $C_i \prec C_j$, if $C_{i\alpha} \prec C_{j\alpha}$, $\forall \alpha \in (0,1]$.

Definition 2.3: A pair of fuzzy (or closed interval-based) rules

$$\begin{array}{l} R_i\text{: If } x \text{ is } A_i\text{, then } y \text{ is } B_i \\ R_j\text{: If } x \text{ is } A_j\text{, then } y \text{ is } B_j \end{array} \tag{1}$$

are said to be *neighbouring rules* if and only if: i) $A_i \prec A_j$ or $A_j \prec A_i$; and ii) there is no individual rule "If x is A, then y is B" such that $A_i \prec A \prec A_j$ if $A_i \prec A_j$, or $A_j \prec A \prec A_i$ if $A_j \prec A_i$.

Note that fuzzy rules R_i and R_j are jointly numbered as "Equation 1" for convenient references hereafter.

Definition 2.4: A fuzzy set (or closed interval) C is *strictly less than* another fuzzy set (or closed interval) C', denoted as $C \prec_s C'$, if and only if $x < x', \forall x \in supp(C)$ and $\forall x' \in supp(C')$ (or $x < x', \forall x \in C$ and $\forall x' \in C'$, respectively).

Definition 2.5: Given a fuzzy (or closed interval) observation $x = A^*$, two neighbouring fuzzy (or closed interval) rules R_i and R_j, as expressed in Equation 1, are said to *strictly flank* the given observation vector if $A_i \prec_s A^* \prec_s A_j$ or $A_j \prec_s A^* \prec_s A_i$.

Definition 2.6: Given a fuzzy (or closed interval) observation $x = A^*$, two neighbouring fuzzy (or closed interval) rules in the form of Equation 1 are said to *partially flank* the given observation vector if ($A^* \not\prec_s A_i$ and $A_j \not\prec_s A^*$ when $A_i \prec A_j$) and ($A^* \not\prec_s A_j$ and $A_i \not\prec_s A^*$ when $A_j \prec A_i$).

Obviously, neighbouring rules that strictly flank an observation $x = A^*$ also partially flank $x = A^*$.

Fuzzy sets are an extension of classical crisp sets. If the membership of all the members of a real-valued fuzzy set degenerate to 1, the fuzzy set degenerates to a crisp interval. In order to facilitate the study of closed form fuzzy interpolation, interval-base rule interpolation (IRI) is introduced first.

1) Interval Rule Interpolation: The calculation of IRI with two single antecedent interval-based rules is outlined here, but the details can be found in [5].

Given an interval observation "x is A^*" and two neighbouring interval rules as of Equation 1, the relative placement factor Λ and the consequence B^* are both intervals, which can be calculated as follows:

$$\Lambda = [\frac{\min(A^*) - \max(A_i)}{\max(A_j) - \max(A_i)}, \frac{\max(A^*) - \min(A_i)}{\min(A_j) - \min(A_i)}] \cap [0,1]. \tag{2}$$

$$\begin{cases} \min(B^*) = \begin{cases} (1-\min(\Lambda))\min(B_i) + \min(\Lambda)\min(B_j), \\ \text{ if } min(B_i) \leq min(B_j) \\ (1-\max(\Lambda))\min(B_i) + \max(\Lambda)\min(B_j), \\ \text{ otherwise} \end{cases} \\ \max(B^*) = \begin{cases} (1-\max(\Lambda))\max(B_i) + \max(\Lambda)\max(B_j), \\ \text{ if } max(B_i) \leq max(B_j) \\ (1-\min(\Lambda))\max(B_i) + \min(\Lambda)\max(B_j), \\ \text{ otherwise.} \end{cases} \end{cases} \tag{3}$$

The interpolated consequence B^* by IRI satisfies:

$$B^* \subseteq [\min(B_i, B_j), \max(B_i, B_j)]. \tag{4}$$

Interpolation with multiple single antecedent interval-based rules is an extension of the interpolation with two single antecedent rules as introduced above. Suppose that there are n rules in the rule base with x and y being the antecedent and the consequent variable respectively, denoted by R_i (If x is A_i, then y is B_i), $i \in \{1,2,...,n\}$, with $A_j \prec A_{j+1}$, $1 \leq j \leq (n-1)$. Let $B^*_{j(j+1)}$ be the interpolated result from the neighbouring rules R_j and R_{j+1}. The interpolated result B^* by the interval-based rule interpolation with multiple rules is:

$$B^* = \bigcup_{j=1}^{n-1} B^*_{j(j+1)}. \tag{5}$$

The interpolated result B^* in Equation 5 is equivalent to the interpolated result using the set of rules which contain only those pairs of neighbouring rules that each partially flank the given observation. This set is referred to as the *set of rules used for interpolation*.

Suppose that rules R_i (If x is A_i, then y is B_i), $i \in \{1,2,...,n\}$ are the only ones with x and y being the antecedent and consequent variables within a given rule base, where $A_j \prec A_{j+1}$, $1 \leq j \leq n-1$. Given an observation A^*, suppose that the set of rules for interpolation is $\mathbb{S}^* = \{R_p, R_{p+1}, ..., R_q\}, 1 \leq p < q \leq n$. All the rules whose antecedents overlap with A^* are contained in this set. That is, $R_k \in \mathbb{S}$ if $A_k \cap A^* \neq \varnothing$. There are only two rules in this set whose antecedents may not overlap with the given observation at all, which are R_p and R_q.

Given an observation and a rule base, suppose that the *set of rules for interpolation* with regard to the given observation is: R_i (If x is A_i, then y is B_i), $i \in \{p, p+1, ..., q\}$, where R_j and $R_{j+1}, p \leq j \leq q-1$, are *neighbouring rules*. Let $B^*_{j(j+1)}$

be the interpolated result from the neighbouring rules R_j and R_{j+1}, then the interpolated result B^* by the interval-based rule interpolation with multiple single-antecedent rules is:

$$B^* = \bigcup_{j=p}^{q-1} B^*_{j(j+1)}. \tag{6}$$

The interpolated result following the interval-based rule interpolation approach is an interval.

2) Type-1 Fuzzy Rule Interpolation: CFFI can be preformed as an extension of IRI by employing the decomposition and resolution principles [23]. Let B^*_α ($\alpha \in (0, 1]$) be the interpolated result using the interval-based rule interpolation from observation "x is A^*_α" and neighbouring rules "If x is $A_{i\alpha}$, then y is $B_{i\alpha}$" and "If x is $A_{j\alpha}$, then y is $B_{j\alpha}$". The interpolated result B^* by CFFI with two single-antecedent rules is equivalent to the composition of the interpolated results by the interval-based rule interpolation from all α-level cut intervals of the corresponding fuzzy sets. That is:

$$B^* = \bigcup_{\alpha \in (0,1]} \alpha B^*_\alpha, \tag{7}$$

This can be further extended for the cases of interpolation with multiple fuzzy rules. Suppose that there are n fuzzy rules in a given rule base with x and y being the antecedent and the consequent variable respectively, which are denoted as R_i (If x is A_i, then y is B_i), $i \in \{1, 2, ..., n\}$, such that $A_j \prec A_{j+1}, 1 \leq j \leq (n-1)$. Let $B^*_{j(j+1)}$ be the interpolated result from the neighbouring rules R_j and R_{j+1}, $1 \leq j \leq n-1$, then the interpolated result B^* by CFFI with multiple rules is calculated by:

$$B^* = \bigcup_{j=1}^{n-1} B^*_{j(j+1)}. \tag{8}$$

Let $(B^*_{j(j+1)})_\alpha$ be the interpolated result using the interval-based rule interpolation from the interval observation "x is A^*_α" and the neighbouring interval rules "If x is $A_{j\alpha}$, then y is $B_{j\alpha}$" and "If x is $A_{(j+1)\alpha}$, then y is $B_{(j+1)\alpha}$". The interpolated result B^* in Equation 8 can then be calculated as:

$$B^* = \bigcup_{\alpha \in (0,1]} \alpha \left(\bigcup_{j=1}^{n-1} (B^*_{j(j+1)})_\alpha \right). \tag{9}$$

The interpolated result B^* by CFFI with multiple single-antecedent rules is convex. The interpolated result B^* by CFFI from a given observation "x is A^*" is normal if and only if there is at least one pair of neighbouring rules R_i (If x is A_i, then y is B_i) and R_j (If x is A_j, then y is B_j) which partially flank A^* such that $\exists x \in core(A^*)$, $\min(core(A_i)) \leq x \leq \max(core(A_j))$.

B. Type-2 Fuzzy Sets

Type-2 fuzzy sets and systems generalise type-1 fuzzy sets and systems so that more uncertainty in data and knowledge representation can be handled. Different from type-1 fuzzy sets, the membership function of a type-2 fuzzy set is two dimensional. In other words, the membership grade of each element of the variable value is a fuzzy number in $[0, 1]$.

Definition 2.7: [24]: A *type-2 fuzzy set*, denoted $\tilde{A}$, is characterised by a *type-2 membership function* $\mu_{\tilde{A}}(x, u)$, where $x \in X$ and $u \in J_x \subseteq [0, 1]$, i.e.,

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) | \forall x \in X, \forall u \in J_x \subseteq [0, 1]\}, \tag{10}$$

in which $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$. $\tilde{A}$ can also be expressed as:

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} \mu_{\tilde{A}}(x, u)/(x, u), \quad J_x \subseteq [0, 1], \tag{11}$$

where $\int\int$ denotes union over all admissible x and u.

Definition 2.8: [25]: When all $\mu_{\tilde{A}}(x, u) = 1$ then $\tilde{A}$ is an *interval type-2 fuzzy set*, which is expressed as:

$$\tilde{A} = \int_{x \in X} \int_{u \in J_x} 1/(x, u), \quad J_x \subseteq [0, 1] \tag{12}$$

Definition 2.9: [25]: Uncertainty in the primary memberships of an interval type-2 fuzzy set, $\tilde{A}$, consists of a bounded region that is called the *footprint of uncertainty* (FOU). It is the union of all primary memberships, i.e.,

$$\text{FOU}(\tilde{A}) = \bigcup_{x \in X} J_x \tag{13}$$

Definition 2.10: [25]: The *upper membership function* (UMF), denoted $\overline{\tilde{A}}$, and *lower membership function* (LMF), denoted $\underline{\tilde{A}}$, of $\tilde{A}$ are two type-1 membership functions that bound the FOU. The UMF is associated with the upper bound of FOU($\tilde{A}$) and is denoted $\overline{\mu}_{\tilde{A}}(x)$, $\forall x \in X$, and the LMF is associated with the lower bound of FOU($\tilde{A}$) and is denoted $\underline{\mu}_{\tilde{A}}(x)$, $\forall x \in X$, i.e.,

$$\begin{aligned} \underline{\mu}_{\tilde{A}}(x) &= \overline{\text{FOU}(\tilde{A})} \quad \forall x \in X, \\ \overline{\mu}_{\tilde{A}}(x) &= \underline{\text{FOU}(\tilde{A})} \quad \forall x \in X. \end{aligned} \tag{14}$$

As indicated above, an interval type-2 fuzzy set $\tilde{A}$ can be represented by the LMF $\underline{\tilde{A}}$ and the UMF $\overline{\tilde{A}}$, i.e., $\tilde{A} = < \underline{\tilde{A}}, \overline{\tilde{A}} >$. In particular, when triangular membership functions are used, such an interval type-2 fuzzy set can be illustrated as shown in Figure 1, where $\underline{\tilde{A}} = (\underline{\tilde{a}_0}, \underline{\tilde{a}_1}, \underline{\tilde{a}_2}; \underline{H_{\tilde{A}}})$, $\overline{\tilde{A}} = (\overline{\tilde{a}_0}, \overline{\tilde{a}_1}, \overline{\tilde{a}_2}; \overline{H_{\tilde{A}}})$, with $(\underline{\tilde{a}_0}, \underline{\tilde{a}_1}, \underline{\tilde{a}_2})$ and $(\overline{\tilde{a}_0}, \overline{\tilde{a}_1}, \overline{\tilde{a}_2})$ denoting the three key points of the LMF and those of the UMF, respectively, and $\underline{H_{\tilde{A}}}$ and $\overline{H_{\tilde{A}}}$ denoting the maximum membership values of $\underline{\tilde{A}}$ and $\overline{\tilde{A}}$, while $\overline{\tilde{a}_0} \leq \underline{\tilde{a}_0}$, $\underline{\tilde{a}_2} \leq \overline{\tilde{a}_2}$, and $0 < \underline{H_{\tilde{A}}} \leq \overline{H_{\tilde{A}}} = 1$. Clearly, the closer the shapes of $\underline{\tilde{A}}$ and $\overline{\tilde{A}}$ is, the lower the uncertain information contained within $\tilde{A}$ is. When $\underline{\tilde{A}}$ coincides with $\overline{\tilde{A}}$, i.e., FOU($\tilde{A}$) is empty, the interval type-2 fuzzy set degenerates to a type-1 fuzzy set.

III. The Extension

In this section, the extension of the existing CFFI using interval type-2 fuzzy sets is presented. Note that for simplicity this paper focuses on interpolation with interval type-2 fuzzy sets only, shorten as IT2 fuzzy sets.

A. Interpolation with Two Rules

For completeness, the definition of CFFI with type-1 fuzzy sets is first summarised as follows:

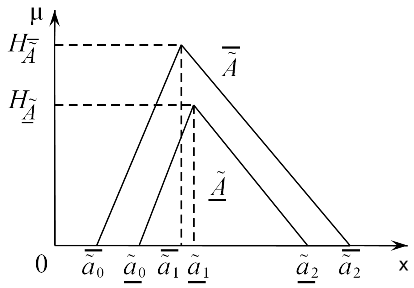


Fig. 1. Lower membership function $\tilde{\underline{A}}$ and upper membership function $\tilde{\overline{A}}$ of a triangular interval type-2 fuzzy set $\tilde{A}$

Theorem 3.1: [5] Given a fuzzy observation "x is A^*" and two neighbouring fuzzy rules, "If x is A_i, then y is B_i" and "If x is A_j, then y is B_j", the consequence B^* can be generated by fuzzy rule interpolation such that:

$$\mu_{B^*}(y) = \sup_{\substack{y=(1-\frac{x-x_i}{x_j-x_i})\cdot y_i+\frac{x-x_i}{x_j-x_i}\cdot y_j,\\ x_i<x_j,\ x_i\leq x,\ x\leq x_j,\\ \{x,x_i,x_j\}\in D_x,\ \{y_i,y_j\}\in D_y}} \min\{\mu_{A_i}(x_i), \mu_{A_j}(x_j), \mu_{B_i}(y_i), \mu_{B_j}(y_j), \mu_{A^*}(x)\}. \quad (15)$$

As the type-1 fuzzy sets involved in Equation 15 are extended to IT2 fuzzy sets, the binary operations sup (used as s-norm herein) and min (used as t-norm herein) need to be accordingly extended in order to handle type-1 fuzzy sets instead of crisp numbers. In particular, the extension of s-norm and t-norm operations of type-1 fuzzy sets are termed as $join$ (denoted as $\sqcup$) and $meet$ (denoted as $\sqcap$), respectively [26]. Let $\int_{x\in X}\mu_A(x)/x$ represent a type-1 fuzzy set A. The operations of the $join$ and $meet$ to type-1 fuzzy sets are defined as:

$$A \sqcup A' = \int_x \int_{x'} [\mu_A(x) \wedge \mu_A(x')]/(x \vee x'), \quad (16)$$

and

$$A \sqcap A' = \int_x \int_{x'} [\mu_A(x) \wedge \mu_A(x')]/(x \wedge x'). \quad (17)$$

In this work, only IT2 fuzzy sets are employed, that is, the membership of each given primary point is an interval. Also, only sup and min are utilised as s-norm an t-norm, respectively. Therefore, the calculation of $meet$ and $join$ operations can be simplified. The $join$ $\sqcup_{n=1}^{n} A_k$ of n intervals A_1, A_2, ..., A_n becomes:

$$\begin{aligned}\sqcup_{k=1}^{n} A_k &= [\min(A_1), \max(A_1)] \sqcup ... \sqcup [\min(A_n), \max(A_n)] \\ &= [\max(\min(A_1), ..., \min(A_n)), \max(\max(A_1), ..., \\ &\quad \max(A_n))].\end{aligned} \quad (18)$$

Similarly, the $meet$ $\sqcap_{n=1}^{n} A_k$ of n intervals A_1, A_2, ..., A_n is simplified as:

$$\begin{aligned}\sqcap_{k=1}^{n} A_k &= [\min(A_1), \max(A_1)] \sqcap ... \sqcap [\min(A_n), \max(A_n)] \\ &= [\min(\min(A_1), ..., \min(A_n)), \min(\max(A_1), ..., \\ &\quad \max(A_n))].\end{aligned} \quad (19)$$

Thanks to the closed form representation of CFFI and the well-developed type-2 arithmetic, CFFI with IT2 fuzzy sets can then be extended directly from CFFI with type-1 fuzzy sets.

Theorem 3.2: Given a fuzzy observation "x is $\tilde{A}^*$" and two neighbouring fuzzy rules, "If x is $\tilde{A}_i$, then y is $\tilde{B}_i$" and "If x is $\tilde{A}_j$, then y is $\tilde{B}_j$", the consequence $\tilde{B}^*$ can be generated by fuzzy rule interpolation as follows:

$$\mu_{\tilde{B}^*}(y) = \bigsqcup_{\substack{y=(1-\frac{x-x_i}{x_j-x_i})\cdot y_i+\frac{x-x_i}{x_j-x_i}\cdot y_j,\\ x_i<x_j,\ x_i\leq x,\ x\leq x_j,\\ \{x,x_i,x_j\}\in D_x,\ \{y_i,y_j\}\in D_y}} \sqcap \{\mu_{\tilde{A}_i}(x_i), \mu_{\tilde{A}_j}(x_j), \mu_{\tilde{B}_i}(y_i), \mu_{\tilde{B}_j}(y_j), \mu_{\tilde{A}^*}(x)\}. \quad (20)$$

As only IT2 fuzzy sets are considered in this work, the above equation implies the need for computation of only the upper and lower membership functions of $\tilde{A}_k$, i.e. $\overline{\mu}_{\tilde{A}_k}$ and $\underline{\mu}_{\tilde{A}_k}$, respectively. Therefore, $\tilde{B}$ is also an interval type-2 fuzzy set [27], where the FOU is determined by:

$$\overline{\mu}_{\tilde{B}}(y) = \sup_{\substack{y=(1-\frac{x-x_i}{x_j-x_i})\cdot y_i+\frac{x-x_i}{x_j-x_i}\cdot y_j,\\ x_i<x_j,\ x_i\leq x,\ x\leq x_j,\\ \{x,x_i,x_j\}\in D_x,\ \{y_i,y_j\}\in D_y}} \min\{\overline{\mu}_{\tilde{A}_i}(x_i), \overline{\mu}_{\tilde{A}_j}(x_j), \overline{\mu}_{\tilde{B}_i}(y_i), \overline{\mu}_{\tilde{B}_j}(y_j), \overline{\mu}_{\tilde{A}^*}(x)\}, \quad (21)$$

and

$$\underline{\mu}_{\tilde{B}}(y) = \sup_{\substack{y=(1-\frac{x-x_i}{x_j-x_i})\cdot y_i+\frac{x-x_i}{x_j-x_i}\cdot y_j,\\ x_i<x_j,\ x_i\leq x,\ x\leq x_j,\\ \{x,x_i,x_j\}\in D_x,\ \{y_i,y_j\}\in D_y}} \min\{\underline{\mu}_{\tilde{A}_i}(x_i), \underline{\mu}_{\tilde{A}_j}(x_j), \underline{\mu}_{\tilde{B}_i}(y_i), \underline{\mu}_{\tilde{B}_j}(y_j), \underline{\mu}_{\tilde{A}^*}(x)\}. \quad (22)$$

This means that the CFFI with IT2 fuzzy sets is also able to be represented in a closed form, which indicates one of the important features of CFFI compared with other existing interpolation approaches.

The embedded representation of IT2 fuzzy set has been widely used in the literature for theoretical derivation [28]. Without losing generality, suppose that the primary variable x is sampled at N values, x_1, x_2, ..., x_N, and at each of these values its primary memberships u_i are sampled at M_i values, u_{i_1}, u_{i_2}, ..., $u_{i_{M_i}}$. Let A_e^j denote the jth embedded set for IT2 fuzzy set A, then

$$\tilde{A} = \sum_{j=1}^{n_A} \tilde{A}_e^j, \quad (23)$$

where $j = 1, 2, ..., n_A$, and

$$\tilde{A}_e^j = \sum_{i=1}^{N} [1/u_i^j]/x_i \quad u_i^j \in \{u_{ik}, k = 1, 2, ..., M_i\}, \quad (24)$$

and

$$n_A = \sum_{i=1}^{N} M_i. \quad (25)$$

The meaning of embedded representation of IT2 fuzzy sets as given above is that any IT2 fuzzy set is able to be represented as a collection of type-1 fuzzy sets, and vice verse. To facilitate the description, an embedded type-1 configuration of a pair of neighbouring IT2 fuzzy rules is a pair of type-1 fuzzy rules where each involved IT2 fuzzy set is represented by one of its embedded type-1 fuzzy sets, respectively.

Remark 3.1: Each embedded fuzzy set of the conclusion using CFFI with IT2 fuzzy sets results from at least one embedded type-1 fuzzy set of the given observation, based on at least one type-1 configuration of the given fuzzy neighbouring rules by the CFFI using type-1 fuzzy sets. Similarly, given any embedded type-1 fuzzy set of the given observation and a type-1 configuration of the given fuzzy neighbouring rules by CFFI using type-1 fuzzy sets, the interpolated type-1 fuzzy set is an embedded fuzzy set of the conclusion.

Theorem 3.3: The proposed CFFI with IT2 fuzzy sets degenerates to CFFI with type-1 fuzzy sets when all the fuzzy sets in the given observation and the given rule base degenerate to type-1 fuzzy sets.

This can be readily proven but the proof is omitted due to lack of space.

Example 3.1: Extend Example 5.1 in [5] using IT2 triangular fuzzy sets to represent uncertain concept. The two fuzzy rules used for interpolation are "If x is $\tilde{A}_i$, then y is $\tilde{B}_i$" and "If x is $\tilde{A}_j$, then y is $\tilde{B}_j$", and the given observation is denoted as $\tilde{A}^*$. Suppose that the observation and the fuzzy sets involved in the two neighbouring rules are given as follows:

$$\begin{aligned}
\tilde{A}^* &=< (3.1, 3.4, 3.9; 0.6), (3, 3.5, 4; 1) >;\\
\tilde{A}_i &=< (1.2, 1.6, 1.9; 0.6), (1, 1.5, 2; 1) >;\\
\tilde{A}_j &=< (7.3, 7.9, 8.6; 0.6), (7, 8, 9; 1) >;\\
\tilde{B}_i &=< (1.1, 1.4, 1.8; 0.6), (1, 1.5, 2; 1) >;\\
\tilde{B}_j &=< (6.4, 7.2, 7.9; 0.6), (6, 7, 8; 1) > .
\end{aligned}$$

The interpolated result is illustrated in Figure 2.

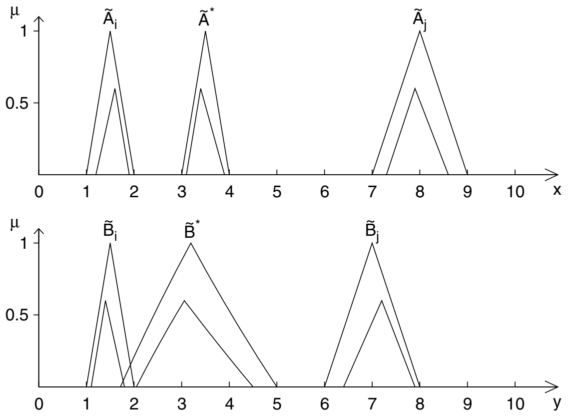


Fig. 2. CFFI with IT2 for two rules

B. Interpolation with Multiple Rules

CFFI with multiple rules using IT2 fuzzy sets is an extension of the CFFI with two rules using IT2 fuzzy sets in conjunction with CFFI involving multiple single antecedent rules using type-1 fuzzy sets. Suppose that there are n fuzzy rules in a given rule base with x and y being the antecedent and consequent variable respectively, which are denoted as R_i (If x is $\tilde{A}_i$, then y is $\tilde{B}_i$), $i \in \{1, 2, ..., n\}$, such that $\tilde{A}_j \prec \tilde{A}_{j+1}, 1 \le j \le (n-1)$. Let $\tilde{B}^*_{j(j+1)}$ be the interpolated result from the neighbouring rules R_j and R_{j+1}, $1 \le j \le (n-1)$. The interpolated result $\tilde{B}^*$ by CFFI with multiple rules is calculated by:

$$\tilde{B}^* = \bigcup_{j=1}^{n-1} \tilde{B}^*_{j(j+1)}. \tag{26}$$

To implement the above, the union operation of IT2 fuzzy sets is required. Fortunately, type-2 fuzzy set operations have been well studied in the literature [25]. In particular, binary operations on two IT2 fuzzy sets, $\tilde{A}$ and $\tilde{A}'$, are defined as:

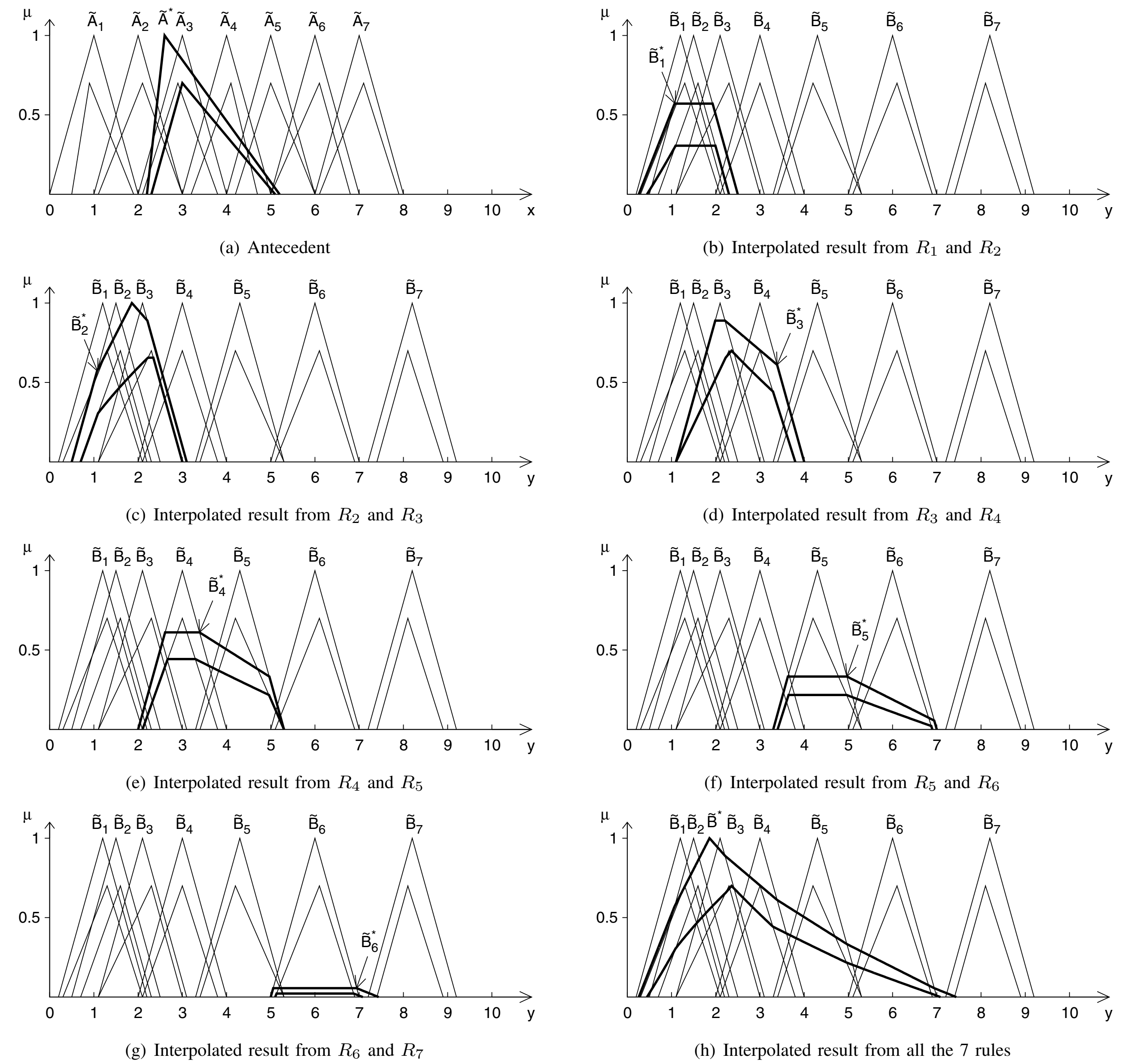
$$\begin{aligned}
\tilde{A} \cap \tilde{A}' = 1/[\underline{\mu}_{\tilde{A}}(x) \wedge \underline{\mu}_{\tilde{A}'}(x), \overline{\mu}_{\tilde{A}}(x) \wedge \overline{\mu}_{\tilde{A}'}(x)];\\
\tilde{A} \cup \tilde{A}' = 1/[\underline{\mu}_{\tilde{A}}(x) \vee \underline{\mu}_{\tilde{A}'}(x), \overline{\mu}_{\tilde{A}}(x) \vee \overline{\mu}_{\tilde{A}'}(x)].
\end{aligned} \tag{27}$$

Example 3.2: Reconsider Example 5.2 in [5] where IT2 fuzzy sets are utilised. The rule base is given as "If x is A_i, then y is B_i", where $i \in \{1, 2, ..., 7\}$. Suppose that the fuzzy sets involved in the given observation and the rules are defined as follows:

$$\begin{aligned}
\tilde{A}^* &=< (2.3, 3, 5.1; 0.7), (2.2, 2.6, 5.2; 1) >;\\
\tilde{A}_1 &=< (0.5, 0.9, 1.9; 0.7), (0, 1, 2; 1) >;\\
\tilde{A}_2 &=< (1.1, 2.1, 3.0; 0.7), (1, 2, 3; 1) >;\\
\tilde{A}_3 &=< (2.1, 2.9, 3.8; 0.7), (2, 3, 4; 1) >;\\
\tilde{A}_4 &=< (3.2, 4.1, 4.7; 0.7), (3, 4, 5; 1) >;\\
\tilde{A}_5 &=< (4.2, 5, 6; 0.7), (4, 5, 6; 1) >;\\
\tilde{A}_6 &=< (5, 6.1, 6.8; 0.7), (5, 6, 7; 1) >;\\
\tilde{A}_7 &=< (6.1, 7.1, 7.9; 0.7), (6, 7, 8; 1) >;\\
\tilde{B}_1 &=< (0.3, 1.3, 2.1; 0.7), (0.2, 1.2, 2.2; 1) >;\\
\tilde{B}_2 &=< (0.7, 1.6, 2.3; 0.7), (0.5, 1.5, 2.5; 1) >;\\
\tilde{B}_3 &=< (1.1, 2.3, 3; 0.7), (1.1, 2.1, 3.1; 1) >;\\
\tilde{B}_4 &=< (2.1, 3, 3.8; 0.7), (2, 3, 4; 1) >;\\
\tilde{B}_5 &=< (3.4, 4.2, 5.3; 0.7), (3.3, 4.3, 5.3; 1) >;\\
\tilde{B}_6 &=< (5.1, 6.1, 6.9; 0.7), (5, 6, 7; 1) >;\\
\tilde{B}_7 &=< (7.4, 8.1, 8.9; 0.7), (7.2, 8.2, 9.2; 1) > .
\end{aligned}$$

The interpolation process and the interpolated results are illustrated in Figure 3. It is clear from this figure that all the indeterminate (Figures 3(b)-3(f)) and the final interpolated (Figure 3(h)) are valid IT2 fuzzy sets. Note that the LMFs of all the fuzzy sets used in each individual example have a normalised maximum membership. This is deliberately set up in order to ensure that the LMFs of the consequences are convex. If each of the IT2 fuzzy sets involved in this example is degenerated to UMF, this problem becomes the one given in Example 5.2 [5]. Interestingly, the UMF of the interpolated result in this example is exactly the same as that generated in Example 5.2 [5], which forms a special case of Theorem 3.3.

The strengths of CFFI using IT2 fuzzy sets are mainly inherited from the original CFFI using type-1 fuzzy sets.



(a) Antecedent

(b) Interpolated result from R_1 and R_2

(c) Interpolated result from R_2 and R_3

(d) Interpolated result from R_3 and R_4

(e) Interpolated result from R_4 and R_5

(f) Interpolated result from R_5 and R_6

(g) Interpolated result from R_6 and R_7

(h) Interpolated result from all the 7 rules

Fig. 3. CFFI with IT2 fuzzy sets based on multiple rules

Compared to other existing fuzzy interpolation approaches using IT2 fuzzy sets, the proposed approach not only guarantees valid interpolated results, but is represented in a closed form. Also, in principle, the proposed approach is not restricted to any particular fuzzy set shapes, although only triangular fuzzy sets are utilised in this illustration. Trapezoidal fuzzy sets will be employed to represent the uncertainty in a real-world application, which is described in the next section.

IV. Experimentation

Research has shown that environmental change influences disease burden [29]. Great efforts have been made to identify logical relationships underlying such influences so that the consequences of a certain environmental change may be predicted. This is of significant importance in the assessment of potential impact of such changes upon the environment and society, before the starting of any large-scale infrastructure projects.

An investigation has recently been made on measuring how the construction of a new road or railway in a previously roadless area may affect the epidemiology of infectious diseases in northern coastal Ecuador [30]. A simplified version of this problem has been restudied in [5]. In this simplified case, the diarrhoeal disease rate of a remote village is directly affected by two factors. On the one hand, low social connectedness tends to failure in creating adequate water and sanitation infrastructure because the residents are

Variables	Fuzzy sets	Values
x_1	$\tilde{A}_1$	$<(0.39, 0.40, 0.42, 0.43; 0.50, 0.50), (0.38, 0.40, 0.42, 0.44; 1.00, 1.00)>$
x_1	$\tilde{A}_2$	$<(0.71, 0.72, 0.74, 0.75; 0.50, 0.50), (0.70, 0.72, 0.74, 0.76; 1.00, 1.00)>$
x_1	$\tilde{A}_3$	$<(0.91, 0.92, 0.94, 0.95; 0.50, 0.50), (0.90, 0.92, 0.94, 0.96; 1.00, 1.00)>$
x_2	$\tilde{B}_1$	$<(0.47, 0.48, 0.50, 0.51; 0.50, 0.50), (0.46, 0.48, 0.50, 0.52; 1.00, 1.00)>$
x_2	$\tilde{B}_2$	$<(0.66, 0.67, 0.69, 0.70; 0.50, 0.50), (0.65, 0.67, 0.69, 0.71; 1.00, 1.00)>$
x_2	$\tilde{B}_3$	$<(0.77, 0.78, 0.80, 0.81; 0.50, 0.50), (0.76, 0.78, 0.80, 0.82; 1.00, 1.00)>$
x_2	$\tilde{B}_6$	$<(0.31, 0.32, 0.34, 0.35; 0.50, 0.50), (0.30, 0.32, 0.34, 0.36; 1.00, 1.00)>$
x_2	$\tilde{B}_7$	$<(0.61, 0.62, 0.64, 0.65; 0.50, 0.50), (0.60, 0.62, 0.64, 0.66; 1.00, 1.00)>$
x_2	$\tilde{B}_8$	$<(0.93, 0.94, 0.96, 0.97; 0.50, 0.50), (0.92, 0.94, 0.96, 0.98; 1.00, 1.00)>$
x_3	$\tilde{C}_4$	$<(0.29, 0.30, 0.32, 0.33; 0.50, 0.50), (0.28, 0.30, 0.32, 0.34; 1.00, 1.00)>$
x_3	$\tilde{C}_5$	$<(0.56, 0.57, 0.59, 0.60; 0.50, 0.50), (0.55, 0.57, 0.59, 0.61; 1.00, 1.00)>$
x_4	$\tilde{D}_4$	$<(0.27, 0.28, 0.30, 0.31; 0.50, 0.50), (0.26, 0.28, 0.30, 0.32; 1.00, 1.00)>$
x_4	$\tilde{D}_5$	$<(0.62, 0.63, 0.65, 0.66; 0.50, 0.50), (0.61, 0.63, 0.65, 0.67; 1.00, 1.00)>$
x_4	$\tilde{D}_6$	$<(0.37, 0.38, 0.40, 0.41; 0.50, 0.50), (0.36, 0.38, 0.40, 0.42; 1.00, 1.00)>$
x_4	$\tilde{D}_7$	$<(0.59, 0.60, 0.62, 0.63; 0.50, 0.50), (0.58, 0.60, 0.62, 0.64; 1.00, 1.00)>$
x_4	$\tilde{D}_8$	$<(0.89, 0.90, 0.92, 0.93; 0.50, 0.50), (0.88, 0.90, 0.92, 0.94; 1.00, 1.00)>$
x_5	$\tilde{E}_6$	$<(0.19, 0.20, 0.22, 0.23; 0.50, 0.50), (0.18, 0.20, 0.22, 0.24; 1.00, 1.00)>$
x_5	$\tilde{E}_7$	$<(0.69, 0.70, 0.72, 0.73; 0.50, 0.50), (0.68, 0.70, 0.72, 0.74; 1.00, 1.00)>$
x_5	$\tilde{E}_8$	$<(0.77, 0.78, 0.80, 0.81; 0.50, 0.50), (0.76, 0.78, 0.80, 0.82; 1.00, 1.00)>$
x_1	$\tilde{A}^*$	$<(0.70, 0.74, 0.75, 0.76; 0.50, 0.50), (0.69, 0.75, 0.76, 0.77; 1.00, 1.00)>$
x_3	$\tilde{C}^*$	$<(0.51, 0.51, 0.53, 0.53; 0.50, 0.50), (0.50, 0.51, 0.53, 0.54; 1.00, 1.00)>$

TABLE I

THE IT2 FUZZY SETS UTILISED IN THE EXAMPLE

unlikely to know one another well and share social norms, thereby usually resulting in a high diarrhoeal disease rate. On the other hand, more frequent contact between the residents within a village and those outside tends to increasing the rate of introduction of pathogens, thereby also raising the diarrhoeal disease rate.

All factors considered in this example are represented as system variables and each relation between two directly connected factors is represented as a rule associating the relevant variables. In summary, there are five variables in the problem: contact outside of village, reintroduction of pathogenic strains, social connectedness, hygiene and sanitation infrastructure, and infections disease rate, denoted as x_1, x_2, ..., x_5, respectively.

In the work of [5], all the object values are represented as type-1 fuzzy sets, which may provide difficulties in defining these values as exact membership values are required. To ease this, IT2 fuzzy sets are utilised herein. Note that different variables are defined on different domains. To simply knowledge representation, variable domains are mapped onto the real line and normalised. Their values which are utilised in the rules and observations are listed in Table I and the part of the rule base employed is given as follows:

R_1: If x_1 is $\tilde{A}_1$, then x_2 is $\tilde{B}_1$;
R_2: If x_1 is $\tilde{A}_2$, then x_2 is $\tilde{B}_2$;
R_3: If x_1 is $\tilde{A}_3$, then x_2 is $\tilde{B}_3$;
R_4: If x_3 is $\tilde{C}_4$, then x_4 is $\tilde{D}_4$;
R_5: If x_3 is $\tilde{C}_5$, then x_4 is $\tilde{D}_5$;
R_6: If x_2 is $\tilde{B}_6$ and x_4 is $\tilde{D}_6$, then x_5 is $\tilde{E}_6$;
R_7: If x_2 is $\tilde{B}_7$ and x_4 is $\tilde{D}_7$, then x_5 is $\tilde{E}_7$;
R_8: If x_2 is $\tilde{B}_8$ and x_4 is $\tilde{D}_8$, then x_5 is $\tilde{E}_8$.

The interpolated results following the present work are illustrated in Figure 4. It is clear from this figure that the interpolated results generated from CFFI with IT2 are all valid IT2 fuzzy sets without any modification, which forms one of the main advantages of the present approach.

V. CONCLUSION

This paper reports on the work that extends the existing CFFI approach through the use of interval type-2 fuzzy sets, which offers a better way to deal with the uncertainty in fuzzy rule interpolation. Thanks to the closed form representation of the approach, the extension is relatively straightforward compared to the existing type-2 extensions for other fuzzy rule interpolation methods, given that type-2 fuzzy arithmetics and operations have been well studied in the literature. It is this simplicity which presents the primary advantage of the current research. This ensures that the approach may

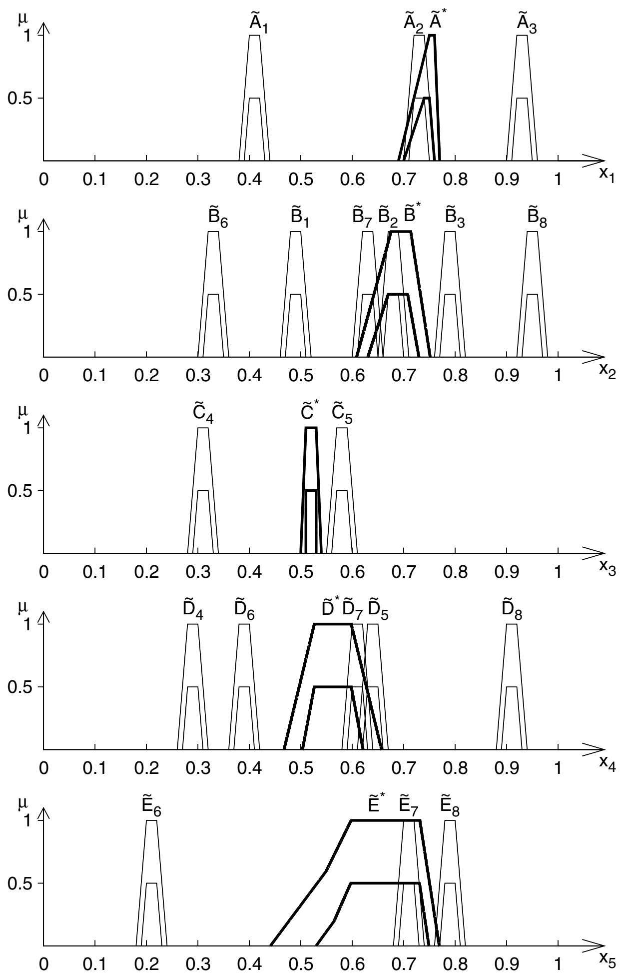


Fig. 4. Rule base, observations and interpolated results

be readily evolved along with the development of fuzzy set theory and fuzzy logic. Illustrative examples have been provided to demonstrate the approach, with a realistic real-world application also presented. The investigation indicates that the proposed approach is of natural appeal for interpolation while dealing with the uncertainty that the conventional CFFI may otherwise be difficult to handle.

Although the work is promising, much can be improved. In particular, only interval type-2 fuzzy sets are considered for interpolation in this work. It is beneficial to investigate how the approach may be further generalised to general type-2 fuzzy sets, although the application of such fuzzy sets are still at their early stage in general. It would also be interesting to compare the interpolated results of the proposed approach with those generated by existing interval type-2 fuzzy rule interpolation methods. In addition, scaled-up real-world applications are required to further evaluate the potential of this work.

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