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SOS-based Fuzzy Stability Analysis via Homogeneous Lyapunov Functions

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Abstract

The class of polynomial fuzzy-model-based (PFMB) control systems has gained considerable attention in fuzzy control. The PFMB control system under consideration often assumes that the Lyapunov functions are quadratic, allowing use of semidefinite programming and the sum of squares (SOS) decomposition. This paper introduces a homogeneously polynomial Lyapunov function for a stabilization problem in which the state feedback synthesis based on SOS decomposition is proposed. To verify the analytical theories regarding PFMB stabilization with the proposed method, two examples are demonstrated to show the effectiveness of the proposed approach.

Keywords: Polynomial TS Fuzzy Models, Homogeneous Lyapunov function, Sum Of Squares (SOS).

I. Introduction

To complement nonlinear control methodologies that require rather involved knowledge, TS fuzzy model-based control scheme provides a simple and effective design that attracts a great deal of attention over the last decade. In particular, the research activity in synthesis and analysis of fuzzy control systems based on linear matrix inequalities has gained its popularity [1].

Recently, the concept of TS fuzzy models has been extended to the polynomial systems [2], [3] whose polynomial subsystems are allowed in the consequent of the fuzzy rules. Based on the Taylor-series expansion [4], a systematic approach extended from the sector-nonlinearity technique [1] to construct the polynomial fuzzy model was proposed. The sum-of-squares (SOS) approach [5], [6] was then applied to investigate the stability of PFMB control systems under the PDC design approach. Thanks to the Lyapunov stability theory, SOS-based stability conditions [2] were derived to guarantee the system stability and facilitate the controller synthesis. However, it is said in [6], [2] that to avoid introducing non-convex condition, the proposed scheme assumes that $P(\tilde{x})$ only depends on state $\tilde{x}$ whose dynamics is not directly affected by the control input, namely states whose corresponding rows in $B(x)$ are zero. With that said, the solution to the stability conditions can be found numerically with the third-party MATLAB toolbox SOSTOOLS [7] or YALMIP [8]. With extension to the PFMB control systems, the drawback of the LMI-based analysis that the membership-function information not being considered is inherent. The work in [4] produces less-conservative stability analysis results by considering some constraints of local operating domain and membership-function-shape information. Others [9], [10], [11], [12] incorporate membership shape information that may relax conservativeness of fuzzy control of nonlinear systems [13].

With the help of the sum of squares decomposition, many problems in polynomial control systems analysis and design have been attacked successfully, due to the fact that the sum of squares decomposition using semidefinite programming can be solved reliably and efficiently on a computer.

In the present paper, it is shown that semidefinite programming and the sum of squares decomposition can be also used for the case of homogeneous Lyapunov functions. This is established using Euler's homogeneity relation for positive homogeneous functions.

The organization of this paper is as follows. Following the introduction, Section II will rehearse some key notions of classical TS models, working lemmas and theorems, characterized by homogeneous Lyapunov method. Section III introduces polynomial fuzzy-model-based system and the relevant Lyapunov results, establishing the theory applicable to PFMB systems. Section IV is devoted to numerical simulations, demonstrating protruding advantages compared to existing results. Concluding remarks are made in Section V.

Notations: $P(x) > 0$ denotes a symmetric matrix function $P(x) : R^n \to R^{n\times n}$ which is positive definite for all $x \in R^n$. Column vector $\nabla V(x) = \frac{\partial V}{\partial x}(x)$ denotes the derivative of V with respect to x and $\nabla^2 V(x)$ denotes the Hessian of V. The time derivative of V along the vector field $f : R^n \to R^n$ is denoted by $\nabla V(x)' f(x)$. Also, $\cdot'$ denotes transpose operation. The notation Y_μ stands for $\sum_{i=1}^{s} \mu_i Y_i$ where $\mu_i \geq 0$ and $\sum_{i=1}^{s} \mu_i = 1$ where s is the number of fuzzy rules. $(A+B)+\star = (A+B)+(A+B)'$.

II. Preliminaries

To rehearse classical TS models, related working lemmas and theorems, we consider a nonlinear T-S fuzzy model which is obtained from a nonlinear system using techniques [1]:

$$\dot{x}(t) = A_\mu x(t) + B_\mu u(t) \tag{1}$$

where $x = [x_1, \cdots, x_n]'$ is the state vector, $u = [u_1, \cdots, u_m]'$ is the control input. All system matrices A_μ, B_μ are of real constant matrices, depending on the time-varying membership vector μ and having appropriate dimensions.

A fuzzy PDC controller is considered here and displayed below

$$u(t) = (\sum_{i=1}^{s} \mu_i K_i) x(t) = K_\mu x(t) \tag{2}$$

where the controller gain matrix $K_i \in R^{m\times n}$ is the state feedback gain to be determined.

Substituting (2) into (1) yields a closed-loop fuzzy system

$$\dot{x}(t) = (A_\mu + B_\mu K_\mu) x(t) = \bar{A}_{\mu\mu} x(t) \tag{3}$$

where at any instant the fuzzy system matrices are given by the convex combination of local T-S models and the time-varying parameter vector μ belongs to the unit simplex $\Omega = \{\mu \in R_+^s(\texttt{positive real}) | \mu_i \geq 0 \text{ and } \sum_{i=1}^{s} \mu_i = 1\}$.

Recall [14], [15], [16] and reference therein where the definition for quadratical stability is given and repeated here for convenience.

Definition 1: (Common P) The forced, disturbance free fuzzy system (3) is said to be quadratically stable if there exists a symmetric matrix $0 < P \in R^{n\times n}$ such that the following parameter-dependent LMIs (PD-LMIs) are satisfied for continuous-time systems:

$$\begin{aligned} M_{\mu\mu} &= \bar{A}'_{\mu\mu} P + P \bar{A}_{\mu\mu} \\ &= (A_\mu + B_\mu K_\mu)' P + \star. \end{aligned}$$

By congruence transformation, we have

$$M_{\mu\mu} = (Q A'_\mu + F'_\mu B'_\mu) + \star < 0 \tag{4}$$

where $F_\mu = K_\mu Q$ and thus $K_\mu = F_\mu Q^{-1}$. The result is readily obtained if the quadratic Lyapunov function $V(x) = x' Q^{-1} x, P = Q^{-1} > 0$ is used.

Now the question arises if one wants to extend the quadratic Lyapunov function to non-quadratic Lyapunov function of the form $V(x) = x' Q^{-1}(x) x$. One has to ensure that $\nabla V(x) = Q^{-1}(x) x$ is a gradient (vector) function of a positive definite function [17], [18]. This equality $\nabla V(x) = Q^{-1}(x) x$ usually does not hold true due to chain rule operation in derivation. Yet, paper [19] implements this condition by adding a set of equality constraints into SOS. This paper takes a different direction from the current result [19] and that is the focus of this paper. Motivated by the result [19], [20], we investigate homogeneous Lyapunov functions.

Definition 2: A function $V(x) : R^n \to R$ is said to be a (positive) homogeneous Lyapunov function of degree r, if V is a Lyapunov function and if

$$V(\lambda x) = \lambda^r V(x) \tag{5}$$

holds for all $x \in R^n$ and all $\lambda \geq 0$.

The definition can be easily verified by a simple example shown below.

Example 1: Let $V(x) = a x_1^2 + b x_1 x_2 + c x_2^2$ be a homogeneous function of degree 2, we have

$$V(3x) = a(3x_1)^2 + b(3x_1)(3x_2) + c(3x_2)^2 = 9V(x).$$

An important property of homogeneous functions is expressed by an appealing property, namely, by Euler's homogeneity relation (Euler's identity).

Theorem 1: (Euler's homogeneity relation, [21]) $V(x)$ is a homogeneous function of degree r, if and only if $V(x)$ satisfies

$$\nabla V(x)' x = r V(x). \tag{6}$$

Proof: The proof is quite simple and follows by differentiation of (5) w.r.t. λ

$$\frac{\partial V(\lambda x)}{\partial \lambda x_1} \frac{\partial \lambda x_1}{\partial \lambda} + \cdots + \frac{\partial V(\lambda x)}{\partial \lambda x_n} \frac{\partial \lambda x_n}{\partial \lambda} = r \lambda^{r-1} V(x)$$

and setting $\lambda = 1$ yields

$$x' \begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \\ \vdots \\ \frac{\partial V}{\partial x_n} \end{bmatrix} = x' \nabla V(x) = \nabla V(x)' x = rV(x).$$

Another useful relation is:

Corollary 1: Let V be a homogeneous function of degree r, then V satisfies

$$x' \nabla^2 V(x) x = r(r-1)V(x). \tag{7}$$

Proof: The proof is quite simple again and follows by differentiation of (6) w.r.t. x

$$\nabla^2 V(x) x(t) + \nabla V(x) = r \nabla V(x).$$

Multiplying the equation above by x' yields

$$x' \nabla^2 V(x) x + x' \nabla V(x) = r x' \nabla V(x)$$

which leads to

$$x' \nabla^2 V(x) x = (r-1) x' \nabla V(x). \tag{8}$$

Since (6), a simple substitution yields

$$x' \nabla^2 V(x) x = r(r-1)V(x).$$

Thus, the connection between $V(x)$ and Hessian of $V(x)$ is shown below

$$V(x) = \frac{1}{r(r-1)} x' \nabla^2 V(x) x.$$

The equation above confirms that given a homogeneous Lyapunov function of degree r, the function can be formed into a quadratic form via $\nabla^2 V(x)$ having x, constituting a non-quadratic Lyapunov function as opposed to fixed/common Lyapunov functions $V(x) = x'Qx$ where $Q > 0$ is constant matrix of appropriate dimension.

Using the relations (6) and (7), the main result of this paper can be established:

III. Polynomial Fuzzy Systems

Having established the machinery behind this investigation, we, to apply the analysis shown above, consider the following polynomial fuzzy system using technique [4].

$$\begin{aligned} \dot{x}(t) &= f(x) + g(x)u(t) \\ &= A_\mu(x)\hat{\mathtt{x}}(t) + B_\mu(x)u(t) \end{aligned} \tag{9}$$

where

- $x(t) \in R^n$ is the state vector, and $u(t) \in R^m$ is the control input vector.
- System matrices are defined as $A_\mu(x) = \sum_{i=1}^{s} \mu_i A_i(x)$ and $B_\mu(x) = \sum_{i=1}^{s} \mu_i B_i(x)$ where $A_i(x)$ and $B_i(x)$ are of compatible dimensions and are function of x.
- $\hat{\mathtt{x}}(t) = [\hat{x}_1, \ldots, \hat{x}_N]' \in R^N$ is a vector of monomials in $x(t)$ and N is the number of monomial terms of a certain degree, say, d. It is assumed that $\hat{\mathtt{x}}(t)$ = 0 if and only if $x(t) = 0$. (When $d = 1$, we have $N = n$.)
- $\mu_i(x(t)), i = 1, 2, \ldots, s$, are the normalized grades of membership and exhibit the following properties: $\mu_i(x(t)) \geq 0 \ \forall i$, and $\sum_{i=1}^{s} \mu_i(x(t)) = 1$.

In this paper, we also consider the class of PFMB control systems in which the polynomial fuzzy controller has the following form

$$u(t) = K_\mu(x)\hat{\mathtt{x}}(t) \tag{10}$$

where $K_\mu(x) = \sum_{i=1}^{s} \mu_i K_i(x)$.

The closed-loop system becomes

$$\dot{x}(t) = \sum_{i=1}^{s} \sum_{j=1}^{s} \mu_i \mu_j \Big(A_i(x) + B_i(x) K_j(x) \Big) \hat{\mathtt{x}}(t) \tag{11}$$

where $\hat{\mathtt{x}}(t) \in R^N$ is a vector of monomial of degree d.

For stability analysis, we apply Lyapunov-based analysis which utilizes the following homogeneously polynomial Lyapunov function of degree r in x.

$$V(\hat{\mathtt{x}}) = \hat{\mathtt{x}}' Q^{-1}(x) \hat{\mathtt{x}} \tag{12}$$

where $Q^{-1}(x)$ is a positive definite polynomial matrix whose entries are all of degree $(r - 2d)$. It is noted that for quadratic Lyapunov functions $r = 2$, then we have $\hat{\mathtt{x}} = x$ and $Q(x) = Q$ because $r - 2d = 2 - 2 = 0$.

Theorem 2: The closed-loop fuzzy system (11) is non-quadratically stabilizable by the controller (10), if there exits a symmetric matrix $Q(x) = Q'(x) \in R^{N \times N}$, $T(x) \in R^{N \times n}$, $F_i(x) \in R^{m \times N}$ such the following SOS are satisfied

$$\begin{aligned} \hat{\mathtt{x}}' \Big(Q(x) - \epsilon I \Big) \hat{\mathtt{x}} \quad &\text{is} \quad \text{SOS} \\ -\hat{\mathtt{x}}' \Big(M_{ii}(x) - \epsilon_1 I \Big) \hat{\mathtt{x}} \quad &\text{is} \quad \text{SOS} \\ -\hat{\mathtt{x}}' \Big(M_{ij}(x) + M_{ji}(x) - \epsilon_2 I \Big) \hat{\mathtt{x}} \quad &\text{is} \quad \text{SOS} \end{aligned}$$

where $i, j = 1, \dots s, j < i$,

$$\begin{aligned}
M_{ij} &= T(x)\Big(A_i(x)Q(x) + B_i(x)F_j(x)\Big) + \star \\
V(\hat{\mathtt{x}}) &= \hat{\mathtt{x}}'(t)Q^{-1}(x)\hat{\mathtt{x}}(t) \\
Q^{-1}(x) &= \nabla^2 V(\hat{\mathtt{x}}) \\
&= \begin{bmatrix} \frac{\partial^2 V}{\partial \hat{x}_1^2} & \frac{\partial^2 V}{\partial \hat{x}_1 \partial \hat{x}_2} & \cdots & \frac{\partial^2 V}{\partial \hat{x}_1 \partial \hat{x}_N} \\ & \frac{\partial^2 V}{\partial \hat{x}_2^2} & \cdots & \frac{\partial^2 V}{\partial \hat{x}_2 \partial \hat{x}_N} \\ & & \ddots & \vdots \\ \star & & & \frac{\partial^2 V}{\partial \hat{x}_N^2} \end{bmatrix} \\
T(x) &= [T_{ij}(x)] = \left[\frac{\partial \hat{x}_i}{\partial x_j}\right] \\
F_i(x) &= K_i(x)Q(x).
\end{aligned}$$

Proof: Consider a homogeneously polynomial Lyapunov function $V(\hat{\mathtt{x}})$ of degree r in x, as shown in (12). The time derivative of $V(\hat{\mathtt{x}})$ along the trajectory is

$$\begin{aligned}
\dot{V}(\hat{\mathtt{x}}) &= \frac{\partial V}{\partial t} \\
&= \frac{\partial V}{\partial \hat{x}_1}\frac{\partial \hat{x}_1}{\partial t} + \frac{\partial V}{\partial \hat{x}_2}\frac{\partial \hat{x}_2}{\partial t} + \cdots + \frac{\partial V}{\partial \hat{x}_N}\frac{\partial \hat{x}_N}{\partial t} \\
&= \begin{bmatrix} \dot{\hat{x}}_1 & \dot{\hat{x}}_2 & \cdots & \dot{\hat{x}}_N \end{bmatrix} \begin{bmatrix} \frac{\partial V}{\partial \hat{x}_1} \\ \frac{\partial V}{\partial \hat{x}_2} \\ \vdots \\ \frac{\partial V}{\partial \hat{x}_N} \end{bmatrix} \\
&= \dot{\hat{\mathtt{x}}}'(\mathtt{t})\nabla \mathtt{V}(\hat{\mathtt{x}}).
\end{aligned}$$

Therefore

$$\begin{aligned}
0 &> \dot{V} \\
&= \frac{1}{(r-1)}\dot{\hat{\mathtt{x}}}'(\mathtt{t})\nabla^2 \mathtt{V}(\hat{\mathtt{x}})\hat{\mathtt{x}}(\mathtt{t}), \text{ due to (8)} \\
&= \frac{1}{(r-1)}\dot{\hat{\mathtt{x}}}'(\mathtt{t})\mathtt{Q}^{-1}(\mathtt{x})\hat{\mathtt{x}}(\mathtt{t}) \\
&= \frac{1}{(r-1)}\dot{x}'(t)T'(x)Q^{-1}(x)\hat{\mathtt{x}}(t) \\
&= \frac{1}{(r-1)}\hat{\mathtt{x}}'(t)\Big(A_\mu(x) + B_\mu(x)K_\mu(x)\Big)' \\
&\quad T'(x)Q^{-1}(x)\hat{\mathtt{x}}(t) \\
&= \frac{1}{2(r-1)}\hat{\mathtt{x}}'(t)\bigg(\Big(A_\mu(x) + B_\mu(x)K_\mu(x)\Big)' \\
&\quad T'(x)Q^{-1}(x) + \star\bigg)\hat{\mathtt{x}}(t).
\end{aligned}$$

Notice that the constant factor $\frac{1}{2(r-1)}$ is irrelevant and can be removed from stability point of view. To continue,

$$\begin{aligned}
0 &> \hat{\mathtt{x}}'\bigg(\Big(A_\mu(x) + B_\mu(x)K_\mu(x)\Big)' T'(x)Q^{-1}(x) \\
&\quad + Q^{-1}(x)T(x)\Big(A_\mu(x) + B_\mu(x)K_\mu(x)\Big)\bigg)\hat{\mathtt{x}} \\
&= \hat{\mathtt{x}}'\Big(A'_\mu(x)T'(x)Q^{-1}(x) + Q^{-1}(x)T(x)A_\mu(x) \\
&\quad + K'_\mu(x)B'_\mu(x)T'(x)Q^{-1}(x) \\
&\quad + Q^{-1}(x)T(x)B_\mu(x)K_\mu(x)\Big)\hat{\mathtt{x}}.
\end{aligned}$$

Multiplying both sides of the equation above by $Q(x)$ leads to

$$\begin{aligned}
0 &> \hat{\mathtt{x}}'\Big(Q(x)A'_\mu(x)T'(x) + T(x)A_\mu(x)Q(x) \\
&\quad + F'_\mu(x)B'_\mu(x)T'(x) + T(x)B_\mu(x)F_\mu(x)\Big)\hat{\mathtt{x}}
\end{aligned}$$

where $F_\mu(x) = K_\mu(x)Q(x)$. Thus, we have

$$\begin{aligned}
0 &> \sum_{i=1}^{s}\mu_i^2\hat{\mathtt{x}}'\Big(T(x)\Big(A_i(x)Q(x) + B_i(x)F_i(x)\Big) \\
&\quad + \star\Big)\hat{\mathtt{x}} + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s}\mu_i\mu_j \\
&\quad \hat{\mathtt{x}}'\Big(T(x)\Big(A_i(x)Q(x) + B_i(x)F_j(x)\Big) + \star \\
&\quad + T(x)\Big(A_j(x)Q(x) + B_j(x)F_i(x)\Big) + \star\Big)\hat{\mathtt{x}} \\
&= \sum_{i=1}^{s}\mu_i^2\hat{\mathtt{x}}'M_{ii}(x)\hat{\mathtt{x}} \\
&\quad + \sum_{i=1}^{s-1}\sum_{j=i+1}^{s}\mu_i\mu_j\hat{\mathtt{x}}'\Big(M_{ij}(x) + M_{ji}(x)\Big)\hat{\mathtt{x}}.
\end{aligned}$$

Thus, a set of sufficient condition is readily obtained as

$$\begin{aligned}
\hat{\mathtt{x}}'\Big(Q(x) - \epsilon I\Big)\hat{\mathtt{x}} &\text{ is SOS} \\
-\hat{\mathtt{x}}'\Big(M_{ii}(x) - \epsilon_1 I\Big)\hat{\mathtt{x}} &\text{ is SOS} \\
-\hat{\mathtt{x}}'\Big(M_{ij}(x) + M_{ji}(x) - \epsilon_2 I\Big)\hat{\mathtt{x}} &\text{ is SOS}
\end{aligned}$$

where $\epsilon, \epsilon_1, \epsilon_2 > 0$ are sufficient small numbers.

It is readily seen that no derivative of $Q(x)$ terms are involved, and no assumption on input matrix $B(x)$ are needed. These non-convex assumptions were assumed/needed in [2]. Furthermore, $Q^{-1}(x)$ is known to be Hessian matrix of the Lyapunov $V(\hat{\mathtt{x}})$.

IV. Illustrative Examples

In this section, two examples are demonstrated. Theorem 2 is tested via SOSTOOLS to show that homogeneously polynomial Lyapunov matrices $Q(x)$ exist for different degrees of 2, 4, and 6.

Example 2: Consider a two-rule T-S fuzzy system [19] whose system matrices are listed below:

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -2-k & -1 \end{bmatrix}$$

where a varying parameters k is inserted to generate different fuzzy system matrices for testing stability via the proposed methods such that the stability can be guaranteed.

- Second-order quadratic Lyapunov function, stability is guaranteed for $k \leq 3.82$. When $k = 3.82$, the Lyapunov matrix

  $$Q = \begin{bmatrix} 0.7784 & 0.1980 \\ 0.1980 & 0.1996 \end{bmatrix}$$

  is found.
- Homogeneously polynomial Lyapunov function of degree 4 can be obtained for $k \leq 5.74$. When $k = 5.74$, the following Lyapunov is found.

  $$Q(x) = \begin{bmatrix} q_1(x) & q_2(x) \\ q_2(x) & q_3(x) \end{bmatrix}$$

  where

  $$\begin{aligned} q_1(x) &= 1.7433x_1^2 + 0.2681x_1x_2 + 0.1681x_2^2 \\ q_2(x) &= 0.1340x_1^2 + 0.3362x_1x_2 + 0.0621x_2^2 \\ q_3(x) &= 0.1681x_1^2 + 0.1243x_1x_2 + 0.0809x_2^2. \end{aligned}$$
- Homogeneously polynomial Lyapunov function of degree 6 can be obtained for $k \leq 6.2$. Particularly, when $k = 6.2$, the following Lyapunov is found.

  $$Q(x) = \begin{bmatrix} q_1(x) & q_2(x) \\ q_2(x) & q_3(x) \end{bmatrix}$$

  where

  $$\begin{aligned} q_1(x) &= 1.9140x_1^4 + 0.4225x_1^3x_2 + 0.6369x_1^2x_2^2 \\ &\quad +0.1451x_1x_2^3 + 0.0299x_2^4 \\ q_2(x) &= 0.1056x_1^4 + 0.4246x_1^3x_2 + 0.2177x_1^2x_2^2 \\ &\quad +0.1199x_1x_2^3 + 0.0147x_2^4 \\ q_3(x) &= 0.1061x_1^4 + 0.1451x_1^3x_2 + 0.1799x_1^2x_2^2 \\ &\quad +0.05886x_1x_2^3 + 0.0190x_2^4. \end{aligned}$$

It is noted that although the simulation example is borrowed from [19], the solving technique are totally different. The former requires a set of equality constraints $\nabla V(x) = x'P$ be written into the SOS condition whilst the latter does not require such equality constraints.

Example 3: Consider a two-rule T-S fuzzy system [2] whose system matrices are listed below:

$$A_1 = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ 0.2172 & -1 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

For quadratic Lyapunov function, we have

$$Q = \begin{bmatrix} 0.4170 & .16205e^{-6} \\ .16205e^{-6} & 0.3831 \end{bmatrix}$$

and the controller gains are

$$K_1 = [k_{11} \ k_{12}], \quad K_2 = [k_{21} \ k_{22}]$$

where $e^{-a} = 10^{-a}, a > 0$

$$\begin{aligned} k_{11} &= -0.9 + .2888e^{-7}x_1x_2 + .3656e^{-6}x_2^2 \\ &\quad -1.766x_1 - .5598x_2 - .1221e^{-13}x_1^2 \\ k_{12} &= .4311e^{-2} + .3979e^{-6}x_1x_2 - .1141e^{-5}x_2^2 \\ &\quad -.3157x_1 + .3144x_2 + .3143e^{-7}x_1^2 \\ k_{21} &= -0.9 - .1515e^{-6}x_1x_2 - .1089e^{-7}x_2^2 \\ &\quad -1.7697x_1 - .5598x_2 + .6410e^{-13}x_1^2 \\ k_{22} &= -.4921e^{-1} - .1185e^{-7}x_1x_2 - .3324e^{-6}x_2^2 \\ &\quad -.2935x_1 + .3604x_2 - .1649e^{-6}x_1^2. \end{aligned}$$

For homogeneously polynomial Lyapunov function of order 4, we have $Q(x)$ shown in (13) on page 6. As to the controller gains, it is omitted here for brevity since monomial terms are huge for order 4 and higher.

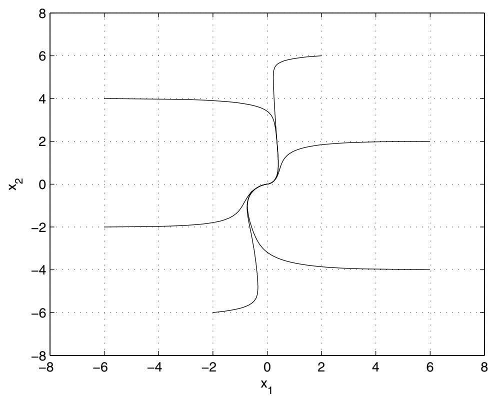
Furthermore, to show that the controller gain does stabilize the underlying system. Figure 3 shows the convergence of state trajectories at different initial condition.

V. Conclusion

By introducing the homogeneous Lyapunov function of degree r in x, we remove the two constraints inherited in the existing SOS method where $P(\tilde{x})$ only depends on state $\tilde{x}$ and whose corresponding rows in $B(x)$ are zero. Furthermore, to utilize SOS relaxations, we avoid the NP-hard cooperativity problems, providing a computationally tractable SOS-based scheme searching for existence of SOS decomposition, thus leading to characterization of solution to non-quadratic stability regarding

$$Q = \begin{bmatrix} 0.3215x_1^2 + 0.2070e^{-15}x_2x_1 + 0.8867x_2^2 & -0.0254x_1^2 - 0.6431x_2x_1 - 0.1035e^{-15}x_2^2 \\ -0.0254x_1^2 - 0.6431x_2x_1 - 0.1035e^{-15}x_2^2 & 0.7549x_1^2 + 0.0508x_2x_1 + 0.3216x_2^2 \end{bmatrix} \quad (13)$$

where $e^{-a} = 10^{-a}, a > 0$.



to fuzzy polynomial-based-model control systems. This is a new result on all the previous SOS in literature. Numerical examples show promising for the proposed methods.

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