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Fuzzy Disturbance Observer for a Class of Polynomial Fuzzy Control Systems

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***Abstract*—Disturbance observer-based control provides a promising approach to handle system disturbance and improve robustness. In this paper, a new fuzzy disturbance observer (FDO) is proposed into the SOS-based approach, where the polynomial fuzzy model is used to develop the system controller. Compared with other works published so far, the FDO mainly features two things: 1) the estimation error between the FDO and disturbance shrinks asymptotically to zero if the disturbance has a constant steady-state value; 2) parameters involved in the FDO is adjusted on the basis of the polynomial fuzzy model which is basically nonlinear. Finally, computer simulations are provided to illustrate the effectiveness of the proposed approach.**

I. Introduction

There have been significant advances in the study of the stability analysis and controller synthesis based on the so-called Takagi-Sugeno (T-S) fuzzy model [1] since the beginning of 1990s. The T-S fuzzy model is a set of fuzzy rules. In each fuzzy rule, the antecedent represents a local state space region whereas the local dynamics is represented by a linear model such as the form of state-space representation in the corresponding consequent. The control design is developed on the basis of the fuzzy model by the so-called parallel distributed compensation (PDC) scheme, where sufficient conditions for system stability in the sense of Lyapunov are provided by solving certain linear matrix inequalities (LMIs) [2], which is referred to as LMI-based approach in this paper.

Recently, the T-S fuzzy model has been extended to a polynomial fuzzy model [3], which allows the local dynamics in the consequent of each rule to be represented by the form of state-dependent linear-like representation involving polynomial matrices and vectors of monomials in the control state. A polynomial fuzzy controller is thus developed based on the PDC-like scheme, where sufficient conditions for system stability in the sense of Lyapunov are provided by solving certain sums of squares (SOS) [3], which is referred to as SOS-based approach in this paper. The solution to the stability conditions can be found numerically using the third-party Matlab toolbox [4].

It is clear that the SOS-based approach is more comprehensive than the LMI-based one, particularly from the standpoint of the modelling precision. That is, compared the regular T-S fuzzy model used in the LMI-based approach that is a linear model in each local state space region, the one used in the SOS-based approach is a polynomial model that is basically nonlinear. This means that the polynomial model is more effective than the T-S fuzzy model to represent the dynamics of the plant to be controlled that is usually nonlinear; nevertheless, there does exist modelling error between the dynamics of the plant to be controlled and its polynomial T-S fuzzy model. Moreover, external disturbance imposed to the plant is due to be taken account in the model. Here in this paper we use the term, lumped disturbance, to express the modelling error including the external disturbance, unmodelled dynamics and parameter perturbations. Generally speaking, the control performance depends considerably on the quantity of the lumped disturbance. That is, if the quantity of the lumped disturbance is significantly large and is not considered in the model that is used for the controller design, then the control performance is no longer to be satisfactory as expected.

Disturbance observer-based control provides a promising approach to handle system disturbance and improve robustness [5]. In this framework, a baseline controller is first designed under the assumption that there is not the lumped disturbance, and then the compensation is added to counteract the influence of the disturbance that is estimated by a properly designed disturbance observer (DOB) [6], [7]. In this paper, a new fuzzy disturbance observer (FDO), motivated by the works in [7], is proposed into the SOS-based approach. This implies that the FDO is basically nonlinear because the basis of the FDO is the fuzzy polynomial model which as such is nonlinear. In fact, there are a lot of existing works of FDOs ([8], [9], [10], [11], [12] and references within), which basically use the fuzzy universal approximator [13] that is linear with respect to an adjustable parameter vector, to approximate the disturbance. The parameter vector in the fuzzy approximator is adjusted not on the basis of the disturbance as such but on the basis of system stability, and as a result, the fuzzy approximator may not march the disturbance at all. Compared with the existing works, the FDO in this paper mainly features two things: 1) the estimation error between the FDO and disturbance shrinks asymptotically to zero if the disturbance has a constant steady-state value; 2) parameters involved in the FDO is adjusted on the basis of the polynomial fuzzy model which is basically nonlinear.

II. Polynomial Fuzzy Model with Disturbance Observer

Consider the following nonlinear system:

$$\dot{x}(t) = f(x(t), u(t), d(t)) \tag{1}$$

where f is a nonlinear function; $x(t) \in R^n$, the state vector; $u(t) \in R^{m_1}$, the input vector; $d(t) \in R^m$, the lumped disturbance vector that is supposed to be bounded. The system can be expressed in terms of the polynomial fuzzy model with disturbance observer as follows.

Rule i :

If $\theta_1(t)$ is M_1^i and $\cdots$ and $\theta_p(t)$ is M_p^i, Then

$$\dot{x}(t) = A_i(x(t))X(x(t)) + B_i(x(t))\left(u(t) + d_i(t)\right) \tag{2}$$

$$\left.\begin{aligned} \hat{d}_i(t) &= q_i(t) + \beta_i(x(t)) \\ \dot{q}_i(t) &= -\ell_i(x(t))\Big(A_i(x(t))X(x(t)) \\ &\qquad + B_i(x(t))(u(t) + \hat{d}_i(t))\Big) \end{aligned}\right\} \tag{3}$$

where $\theta_j(j = 1, 2, \cdots, p)$ is a variable in the antecedent; $M_j^i(i = 1, 2, \ldots r)$, a fuzzy term corresponding to ith rule; $A_i(x(t)) \in R^{n\times N}, B_i(x(t)) \in R^{n\times m}$, polynomial matrices in $x(t)$; $X(x(t)) \in R^N$, a vector of monomials in $x(t)$ with assumption that $X(x(t)) = 0$ iff $x(t) = 0$; $d_i(t) \in R^m$, the lumped disturbance including modelling error, external disturbance, unmodelled dynamics and parameter perturbations; $\hat{d}_i(t) \in R^m$, the estimate of the unknown disturbance; $q_i(t) \in R^m$, the internal state vector of the (nonlinear) disturbance observer; $\beta_i(x(t) \in R^m$, polynomial vector in $x(t)$ to be designed; $\ell_i(x(t)) \in R^{m\times n}$, the observer gain which is defined as

$$\ell_i(x(t) = \frac{\partial \beta_i(x(t))}{\partial x^T(t)} \tag{4}$$

satisfying

$$\ell_i(x(t)B_i(x(t)) > 0. \tag{5}$$

Note that for a given $B_i(x(t)$, it is easy to find a $\beta_i(x(t))$ which upholds (4) and (5) by using some software packages such as the third-party MATLAB toolbox, SOSOPT, which supports the operation of differentiation as well.

It is clear that if the lumped disturbance $d_i(t)$ is not considered, i.e., $d_i(t) = 0$, then the disturbance estimate and observer (3) are not needed anymore, and (2) becomes

$$\dot{x}(t) = A_i(x(t))X(x(t)) + B_i(x(t))u(t) \tag{6}$$

which is the regular form in the related studies [3]. In general, the disturbance does not admit the mathematical expressions. Therefore, the model proposed in this paper is a general form that includes the regular case such as (6).

In the polynomial fuzzy model (2), the control input $u(t)$ and the lumped disturbance $d_i(t)$ shared the same control matrix $B_i(x(t))$. In the case of different matrices, i.e.,

$$\begin{aligned} \dot{x}(t) = A_i(x(t))X(x(t)) + B_i(x(t))u(t) \\ + B_{0i}(x(t))d_{0i}(t) \end{aligned} \tag{7}$$

where $B_{0i}(x(t)) \in R^{n\times m}$, and $d_{0i}(t) \in R^m$ denotes the (original) lumped disturbance. If we replace $d_{0i}(t)$ by $d_i(t)$ such that

$$B_i(x(t))d_i(t) = B_{0i}(x(t))d_{0i}(t) \tag{8}$$

thus, (7) becomes (2). The disturbance is expressed in the form of $d_{0i}(t)$ in (7), whereas the one is expressed in the form of $d_i(t) = B_i^\dagger(x(t))B_{0i}(x(t))d_{0i}$, where $(\cdot)^\dagger$ denotes the Moore-Penrose generalized inverse. No matter what form the disturbance is expressed in, the purpose we try to address it is no change. Therefore, the polynomial fuzzy model (2) also covers the more general case (7). Regarding the disturbance, we give the following assumption:

Assumption 1: The disturbance has a constant steady-state value, i.e.,

$$\lim_{t\to\infty} \dot{d}_i(t) = 0. \tag{9}$$

Define the estimation error

$$e_i(t) = d_i(t) - \hat{d}_i(t) \tag{10}$$

then we have

$$\begin{aligned} \dot{e}_i(t) &= \dot{d}_i(t) - \dot{\hat{d}}_i(t) \\ &= -\dot{q}_i(t) - \frac{\partial \beta_i(x(t))}{\partial x(t)^T}\dot{x}(t) + \dot{d}_i(t) \\ &= -\dot{q}_i(t) - \ell_i(x(t))\dot{x}(t) + \dot{d}_i(t) \\ &= -\ell_i(t)B_i(x(t))e_i(t) + \dot{d}_i(t) \end{aligned} \tag{11}$$

which implies that the estimation error $e_i(t)$ shrinks asymptotically to zero due to $\ell_i(t)B_i(x(t)) > 0$ according to (5) and $\lim_{t\to\infty} \dot{d}_i(t) = 0$ from *Assumption 1*, i.e.,

$$\lim_{t\to\infty} e_i(t) \to 0. \tag{12}$$

Therefore, it is reasonable to assume that the estimation error upholds the following inequality all the time.

$$||e_i(x(t))||^2 \leq \delta_{ei}^*||X(x(t)||^2 \tag{13}$$

where $||\cdot||$ denotes the Euclidean norm, and δ_{ei}^* is a potitive constant.

Remark 1: If the lumped disturbance $d_i(t)$ is slowly time varing, i.e., $\dot{d}_i(t) \simeq 0$, then (11) leads to (12) immediately without the assumption (9).

Remark 2: Because of $\ell_i(x(t) = \frac{\partial\beta_i(x(t))}{\partial x(t)^T}$ in which $\beta_i(x(t))$ is a polynomial vector in $x(t)$ to be designed, $\beta_i(x(t))$ can be chosen so that $\ell_i(t)B_i(x(t))$ is positively large enough compared to the magnitude of $\dot{d}_i(t)$, then we still have that $e_i(t)$ can become very small quickly in case of $\dot{d}_i(t) \neq 0$.

Define

$$\Delta_{ij}(t) = \hat{d}_i(t) - \hat{d}_j(t). \tag{14}$$

Because $d_i(t)$ is the lumped disturbance which is supposed to be bounded, $\hat{d}_i(t)$ should be bounded as well due to $\lim_{t\to\infty} \hat{d}_i(t) \to d_i(t)$ from (12). In consideration of both $g_i(x(t))$ and $\hat{d}_i(t)$ being bounded, we put the following assumption on $\Delta_{ij}(x(t))$.

Assumption 2: $\Delta_{ij}(x(t))$ is norm-bounded as follows.

$$||\Delta_{ij}(t)||^2 \leq \delta_{ij}^* ||X(x(t)||^2 \tag{15}$$

where δ_{ij}^* is a potitive constant.

As a result of defuzzification, $\dot{x}(t)$ in the polynomial fuzzy model (2) can be calculated by

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r} \alpha_i(\theta(t)) \Big(A_i(x(t))X(x(t)) \\ &\quad + B_i(x(t))\left(u(t) + d_i(t)\right)\Big), \end{aligned} \tag{16}$$

where

$$\theta(t) = [\theta_1(t)\ \theta_2(t) \cdots \theta_p(t)]$$
$$\alpha_i(\theta(t)) = \frac{\omega_i(\theta(t))}{\sum_{i=1}^{r} w_i(\theta(t))} \geq 0, \quad \sum_{i=1}^{r} \alpha_i(\theta(t)) = 1$$
$$\omega_i(\theta(t)) = \prod_{j=1}^{p} M_j^i(\theta_j(t)).$$

III. Adaptive Polynomial Fuzzy Controller

Based on the polynomial fuzzy model with disturbance observer shown in the previous section, an adaptive polynomial fuzzy controller is proposed as follows.

Control Rule i :

If $\theta_1(t)$ is M_1^i and $\cdots$ and $\theta_p(t)$ is M_p^i, Then

$$u(t) = F_i(x(t))X(x(t)) - \hat{d}_i(t) \tag{17}$$

where $F_i(x(t)) \in R^{m \times N}$ is the baseline feedback gain which is a polynomial vector in $x(t)$ to be designed, $\hat{d}(t)$ is obtained from (3), which is the compensation to counteract the influence of the disturbance.

The overall controller is calculated by

$$u(t) = \sum_{i=1}^{r} \alpha_i(\theta(t)) \Big(F_i(x(t))X(x(t)) - \hat{d}_i(t) \Big). \tag{18}$$

From here, unless confusion arises arguments such as t will be omitted just for notational convenience.

Substituting (18) into (16), it follows that

$$\begin{aligned}\dot{x} &= \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i\alpha_j \Big(A_i(x)X(x) + B_i(x)F_j(x)X(x) \\ &\quad - B_i(x)\hat{d}_j + B_i(x)d_i \Big) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i\alpha_j \Big(\big(A_i(x) + B_i(x)F_j(x)\big)X(x) \\ &\quad + B_i(x)\big(-\hat{d}_i + \Delta_{ij}(x)\big) + B_i(x)d_i \Big) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i\alpha_j \Big(\big(A_i(x) + B_i(x)F_j(x)\big)X(x) \\ &\quad - B_i(x)\hat{d}_i + B_i(x)\Delta_{ij}(x) + B_i(x)d_i \Big) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} \alpha_i\alpha_j \Big(\big(A_i(x) + B_i(x)F_j(x)\big)X(x) \\ &\quad + B_i(x)e_i + B_i(x)\Delta_{ij}(x) \Big) \end{aligned} \tag{19}$$

where (14) is used.

The stability of the control system that contains the polynomial fuzzy model with disturbance observer (2) and (3), and the adaptive polynomial fuzzy controller (18) is investigated based on the Lyapunov stability theory. Let us consider the following polynomial quadratic Lyapunov function candidate [3]:

$$V(x) = X(x)^T P^{-1}(z) X(x) \tag{20}$$

where $P^{-1}(z) \in R^{N \times N}$ is a symmetric positive-definite polynomial matrix in z, and $z = [x_{k_1}\ x_{k_2}\ \ldots\ x_{k_d}]$ is a vector to be chosen such that $K = \{k_1, k_2, \ldots, k_d\}$ denotes the indices of the corresponding zero rows in $B_i(x)$ for all i. Denoting the kth row in $A_i(x)$ and $B_i(x)$ as $A_{i,k}(x)$ and $B_{i,k}(x)$, respectively, we have

$$B_{i,k}(x) = 0 \tag{21}$$

for all $k \in K$, and

$$\begin{aligned}\frac{dP^{-1}(z(t))}{dt} &= -P^{-1}(z(t))\dot{P}(z(t))P^{-1}(z(t)) \\ &= -P^{-1}(z(t)) \left(\sum_{k=1}^{k_d} \frac{\partial P(z(t))}{\partial x_k(t)} \dot{x}_k(t) \right) P^{-1}(z(t)) \end{aligned} \tag{22}$$

where the relation $\dot{P}(z)P^{-1}(z) + P(z)\dot{P}^{-1}(z) = 0$ is used due to the facts that $P(z)P^{-1}(z) = I$ and $\frac{dP(z)P^{-1}(z)}{dt} = 0$, and

$$\dot{x}_k = \sum_{i=1}^{r} \alpha_i A_{i,k}(x)X(x) \tag{23}$$

for all $k \in K$. The time derivative of $V(x)$ is given by

$$\begin{aligned}\dot{V}(x) &= \dot{X}^T(x)P^{-1}(z)X(x) + X^T(x)P^{-1}(z)\dot{X}(x) \\ &\quad + X^T(x)\dot{P}^{-1}(z)X(x). \end{aligned} \tag{24}$$

Since

$$\begin{aligned}\dot{X}(x) &= \sum_{i=1}^{n} \frac{\partial X(x)}{\partial x_i} \dot{x}_i \\ &= T(x)\dot{x} \end{aligned} \tag{25}$$

where $x^T = [x_1\ x_2\ \cdots x_n]$, and

$$T(x) = \left[\frac{\partial X(x)}{\partial x_1}\ \frac{\partial X(x)}{\partial x_2}\ \cdots \frac{\partial X(x)}{\partial x_n} \right] \in R^{N \times n}, \tag{26}$$

we have

$$\begin{aligned}\dot{V}(x) &= \dot{x}^T T^T(x) P^{-1}(z) X(x) + X^T(x) P^{-1}(z) T(x)\dot{x} \\ &\quad - X^T(x) P^{-1}(z) \left(\sum_{k=1}^{k_d} \frac{\partial P(z)}{\partial x_k} \dot{x}_k \right) P^{-1}(z) X(x) \end{aligned} \tag{27}$$

where (22) is used. Substituting (19) and (23) into (27), we have

$$\begin{aligned}\dot{V}(x) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j\Big\{X^T(x)\left(G_{ij}(x)+G_{ij}^T(x)\right)X(x)\\ &\quad +2X^T(x)P^{-1}(z)T(x)B_i(x)e_i\\ &\quad +2X^T(x)P^{-1}(z)T(x)B_i(x)\Delta_{ij}(x)\Big\}\\ &\quad -\sum_{1=1}^{r}\alpha_i X^T(x)P^{-1}(z)\\ &\quad \times\sum_{k=1}^{k_d}\frac{\partial P(z)}{\partial x_k}\big(A_{i,k}(x)X(x)\big)P^{-1}(z)X(x)\\ &= \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j\Big\{X^T(x)\left(G_{ij}(x)+G_{ij}^T(x)\right)X(x)\\ &\quad +2X^T(x)P^{-1}(z)T(x)B_i(x)e_i\\ &\quad +2X^T(x)P^{-1}(z)T(x)B_i(x)\Delta_{ij}(x)\\ &\quad -X^T(x)P^{-1}(z)\\ &\quad \times\sum_{k=1}^{k_d}\frac{\partial P(z)}{\partial x_k}\big(A_{i,k}(x)X(x)\big)P^{-1}(z)X(x)\Big\}\end{aligned}\tag{28}$$

where $G_{ij}(x)=\big(A_i(x)+B_i(x)F_j(x)\big)^T T^T(x)P^{-1}(z)$, and the fact that $\sum_{j=1}^{r}\alpha_j=1$ is used.
Since

$$\begin{aligned}&2X^T(x)P^{-1}(z)T(x)B_i(x)\Delta_{ij}(x)\\ \le\ &\kappa_1(x)X^T(x)P^{-1}(z)T(x)B_i(x)\\ &\times B_i^T(x)T^T(x)P^{-1}(z)X(x)+\frac{1}{\kappa_1(x)}\Delta_{ij}^T(x)\Delta_{ij}(x)\\ \le\ &\kappa_1(x)X^T(x)P^{-1}(z)T(x)B_i(x)\\ &\times B_i^T(x)T^T(x)P^{-1}(z)X(x)\\ &+\frac{1}{\kappa_1(x)}\sigma_{ij}^*X^T(x)X(x)\end{aligned}\tag{29}$$

$$\begin{aligned}&2X^T(x)P^{-1}(z)T(x)B_i(x)e_i\\ \le\ &\kappa_2(x)X^T(x)P^{-1}(z)T(x)B_i(x)\\ &\times B_i^T(x)T^T(x)P^{-1}(z)X(x)+\frac{1}{\kappa_2(x)}e_i^Te_i\\ \le\ &\kappa_2(x)X^T(x)P^{-1}(z)T(x)B_i(x)\\ &\times B_i^T(x)T^T(x)P^{-1}(z)X(x)\\ &+\frac{1}{\kappa_2(x)}\sigma_{ei}^*X^T(x)X(x)\end{aligned}\tag{30}$$

and where (13) and (15) are used, and $\kappa_1(x)>0,\kappa_2(x)>0$ are polynomials in x; therefore,

$$\dot{V}(x) = \sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j X^T(x)\Omega_{ij}(x)X(x)\tag{31}$$

where

$$\begin{aligned}\Omega_{ij}(x) &= G_{ij}(x)+G_{ij}^T(x)\\ &\quad +\kappa_1(x)P^{-1}(z)T(x)B_i(x)B_i^T(x)T^T(x)P^{-1}(z)\\ &\quad +\kappa_2(x)P^{-1}(z)T(x)B_i(x)B_i^T(x)T^T(x)P^{-1}(z)\\ &\quad +\frac{1}{\kappa_1(x)}\sigma_{ij}^*I+\frac{1}{\kappa_2(x)}\sigma_{ei}^*I-P^{-1}(z)\\ &\qquad \times\sum_{k=1}^{k_d}\frac{\partial P(z)}{\partial x_k}\big(A_{i,k}(x)X(x)\big)P^{-1}(z).\end{aligned}\tag{32}$$

According to the Lyapunov stability theory, the control system concerned is asymptotically stable if the following inequality is satisfied:

$$-\Omega_{ij}>0, \qquad i,j=1,2,\ldots,r.\tag{33}$$

Pre- and post-multiplying both sides of (33) by $P(z)$, we have

$$\begin{aligned}&-T(x)\big\{A_i(x)P(z)+B_i(x)M_j(x)\big\}\\ &-\big\{A_i(x)P(z)+B_i(x)M_j(x)\big\}^TT^T(x)\\ &-\kappa_1(x)T(x)B_i(x)B_i^T(x)T^T(x)\\ &-\kappa_2(x)T(x)B_i(x)B_i^T(x)T^T(x)\\ &-P(z)\frac{1}{\kappa_1(x)}\sigma_{ij}^*P(z)-P(z)\frac{1}{\kappa_2(x)}\sigma_{ei}^*P(z)\\ &+\sum_{k=1}^{k_d}\frac{\partial P(z)}{\partial x_k}\big(A_{i,k}(x)X(x)\big)>0\end{aligned}\tag{34}$$

where $M_j(x)=F_j(x)P(z)$. By the Schur complement, (34) holds if the following conditions are SOS:

$$v_2^T\begin{bmatrix}(*) & P(z) & P(z)\\ P(z) & \frac{\kappa_1(x)}{\sigma_{ij}^*} & 0\\ P(z) & 0 & \frac{\kappa_2(x)}{\sigma_{ei}^*}\end{bmatrix}v_2, \quad i,j=1,2\cdots,r\tag{35}$$

where $v_2\in R^{3N}$ is an arbitrary vector independent of x,

$$\begin{aligned}(*) &= -T(x)\big\{A_i(x)P(z)+B_i(x)M_j(x)\big\}\\ &\quad -\big\{A_i(x)P(z)+B_i(x)M_j(x)\big\}^TT^T(x)\\ &\quad -\kappa_1(x)T(x)B_i(x)B_i^T(x)T^T(x)\\ &\quad -\kappa_2(x)T(x)B_i(x)B_i^T(x)T^T(x)\\ &\quad +\sum_{k=1}^{k_d}\frac{\partial P(z)}{\partial x_k}\big(A_{i,k}(x)X(x)\big)-\epsilon_{2ij}(x)I\end{aligned}\tag{36}$$

and $\epsilon_{2ij}(x)\ge 0$ is a polynomial in x. If the SOS (35) with (36) are feasible, then the control feedback gain $F_i(x)$ in (17) is calculated by

$$F_i(x)=M_i(x)P^{-1}(x).\tag{37}$$

Summarizing the above result, we get a theorem as follows.

Theorem 1: Consider the polynomial fuzzy model with the lumped disturbance and its observer (2) and (3) subject to the Assumptions 1 and 2, under the adaptive polynomial fuzzy controller (17), where $F_i(x)$ is obtained from (37), the closed-loop control system will be asymptotically stable, if there exist symmetric polynomial matrix $P(z) \in R^{N\times N}$, polynomial matrix $M_i(x) \in R^{m\times N}$, and polynomials $\kappa_1(x) > 0, \kappa_2(x) > 0$, $\epsilon_1(x) > 0$, $\epsilon_{2ij}(x) \geq 0$ such that

$$v_1^T(P(z) - \epsilon_1(x)I)v_1 \text{ is SOS} \tag{38}$$

$$(35) \text{ is SOS} \tag{39}$$

where $v_1 \in R^N, v_2 \in R^{3N}$ are arbitrary vectors independent of x.

The configuration of the control system with the FDO is depicted in Fig.1.

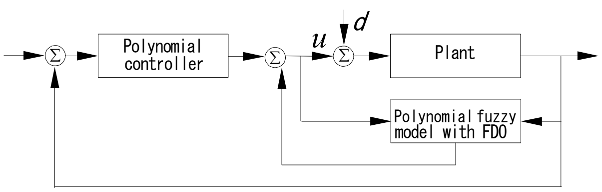

Fig. 1. Configuration of the control system with the FDO

Regarding the SOS in Theorem 1, we give the following remarks.

Remark 3: $\dot{V}(x)$ in (31) can be written as

$$\begin{aligned}\dot{V}(x) \quad = \quad & \sum_{i=1}^{r}\alpha_i^2 X^T(x)\Omega_{ii}(x)X(x) \\ & +\sum_{i=1}^{r}\sum_{j=i+1}^{r}\alpha_i\alpha_j X^T(x) \\ & \times\left(\Omega_{ij}(x)+\Omega_{ji}(x)\right)X(x). \end{aligned} \tag{40}$$

Therefore, after some similar manipulation as above, the control system stability condition in (39) can be replaced by

$$v_2^T\begin{bmatrix} (*1) & P(z) & P(z) \\ P(z) & \frac{\kappa_1(x)}{\sigma_{ii}^*} & 0 \\ P(z) & 0 & \frac{\kappa_2(x)}{\sigma_{ei}^*}\end{bmatrix}v_2 \quad \text{is SOS}, \quad i = 1,\cdots,r \tag{41}$$

$$v_2^T\begin{bmatrix} (*2) & P(z) & P(z) \\ P(z) & \frac{\kappa_1(x)}{\sigma_{ij}^*} & 0 \\ P(z) & 0 & \frac{\kappa_2(x)}{\sigma_{ei}^*}\end{bmatrix}v_2 \quad \text{is SOS}, \quad j > i \tag{42}$$

where

$$\begin{aligned}(*1) \quad = \quad & -T(x)\{A_i(x)P(z)+B_i(x)M_i(x)\} \\ & -\{A_i(x)P(z)+B_i(x)M_i(x)\}^T T^T(x) \\ & -\kappa_1(x)T(x)B_i(x)B_i^T(x)T^T(x) \\ & -\kappa_2(x)T(x)B_i(x)B_i^T(x)T^T(x) \\ & +\sum_{k=1}^{k_d}\frac{\partial P(z)}{\partial x_k}\big(A_{i,k}(x)X(x)\big) - \epsilon_{3i}(x)I \end{aligned}$$

$$\begin{aligned}(*2) \quad = \quad & -T(x)\{A_i(x)P(z)+B_i(x)M_j(x)\} \\ & -\{A_i(x)P(z)+B_i(x)M_j(x)\}^T T^T(x) \\ & -\kappa_1(x)T(x)B_i(x)B_i^T(x)T^T(x) \\ & -\kappa_2(x)T(x)B_i(x)B_i^T(x)T^T(x) \\ & +\sum_{k=1}^{k_d}\frac{\partial P(z)}{\partial x_k}\big(A_{i,k}(x)X(x)\big) \\ & -T(x)\{A_j(x)P(z)+B_{1j}(x)M_i(x)\} \\ & -\{A_j(x)P(z)+B_{1j}(x)M_i(x)\}^T T^T(x) \\ & -\kappa_1(x)T(x)B_{1j}(x)B_{1j}^T(x)T^T(x) \\ & -\kappa_2(x)T(x)B_{2j}(x)B_{2j}^T(x)T^T(x) \\ & +\sum_{k=1}^{k_d}\frac{\partial P(z)}{\partial x_k}\big(A_{j,k}(x)X(x)\big) - \epsilon_{2ij}(x)I \end{aligned}$$

and $\epsilon_{3i}(x) > 0$ is a polynomial in x.

Remark 4: If the lumped disturbance $d_i(t)$ in (2) is not considered, i.e., $d_i(t) = 0$, then from (34) the control system stable conditions become that

$$v_3\begin{pmatrix} -T(x)\{A_i(x)P(z)+B_i(x)M_j(x)\} \\ -\{A_i(x)P(z)+B_i(x)M_j(x)\}^T T^T(x) \\ +\sum_{k=1}^{k_d}\frac{\partial P(z)}{\partial x_k}\big(A_{i,k}(x)X(x)\big)-\epsilon_{2ij}(x)\end{pmatrix}v_3 \quad \text{is SOS}, \quad i,j = 1,\cdots,r \tag{43}$$

where $v_3 \in R^N$ is arbitrary vector that is independent of x. (43) is one of the regular SOS for the control system stability without the lumped disturbance.

IV. Simulation

To illustrate the proposed results, consider the following nonlinear system [3]:

$$\dot{x}_1 = -x_1 + x_1^2 + x_1^3 + x_1^2x_2 - x_1x_2^2 + x_2 + x_1(u+d) \tag{44}$$

$$\dot{x}_2 = -\sin(x_1) - x_2 \tag{45}$$

where $-\pi/2 \leq x_1 \leq \pi/2$. The $x_1 - x_2$ phase plane with $u = 0$ and $d = 0$ is shown in Fig. 2, which indicates the nonlinear system is unstable.

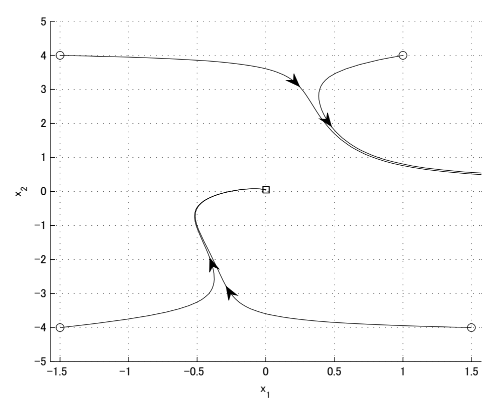

Fig. 2. $x_1 - x_2$ phase plane with $u = 0$ and $d = 0$

Based on the concept of sector nonlinearity [2], the non-polynomial function, $\sin(x_1)$, can be exactly expressed by

$$\sin(x_1) = N_1(x_1)x_1 + N_2(x_1)\frac{2}{\pi}x_1$$

where $N_1(x_1)+N_2(x_1) = 1$. Therefore, we have the following membership functions (Fig.3):

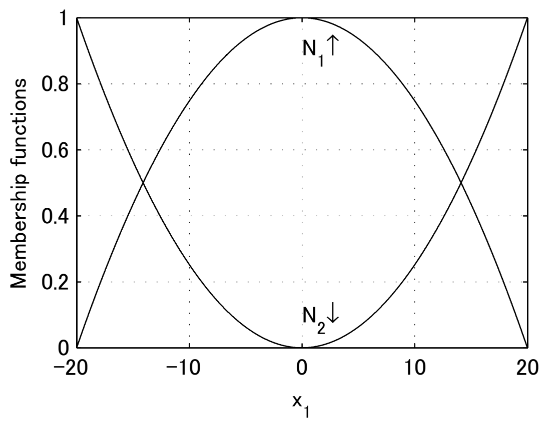

Fig. 3. Membership functions $N_1(x_1)$ and $N_2(x_1)$

$$N_1(x_1) = \frac{\sin(x_1) - \frac{2}{\pi}x_1}{\left(1-\frac{2}{\pi}\right)x_1} \quad (46)$$

$$N_2(x_1) = \frac{x_1 - \sin(x_1)}{\left(1-\frac{2}{\pi}\right)x_1} \quad (47)$$

and the nonlinear system can be represented by the following polynomial fuzzy model:

$$\text{Rule i : If } x_1 \text{ is } N_i(x_1), \text{Then}$$
$$\dot{x} = A_i(x)X(x) + B_i(x)(u + d_i)$$

where $i = 1, 2$, $x = X(x) = [x_1, x_2]^T, z = x_1$, and

$$A_1(x) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A_2(x) = \begin{bmatrix} -1 + x_1 + x_1^2 + x_1x_2 - x_2^2 & 1 \\ -\frac{2}{\pi} & 1 \end{bmatrix}$$

$$B_1(x) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, \quad B_2(x) = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}.$$

For disturbance observer (3), by setting $\beta_i(x) = 80 * x_1^2$, we have $\ell_i(x) = [160x_1 \ 0]$ satisfying $\ell_i(x)B_i(x) > 0$ in (5). In order to obtain $M_i(x)$ for $F_i(x)$ in (37), we have to solve SOS in Theorem 1. Setting the related parameters as $\sigma_{ei} = 0.1$, $\sigma_{ij} = 0.1$, $\epsilon_1(x) = 0.0001$, $\epsilon_{2ij}(x) = 0.0001$, $\kappa_1 = 1.0$, $\kappa_2 = 1.0$, we have

$$\left.\begin{array}{l} F_1(x) = [-0.25706x_1 - 0.99966x_2 - 1.1333, \\ \qquad 2.7698x_1 + 9.2919 \times 10^{-5}x_2 - 0.37828] \\ F_2(x) = [-1.2743x_1 - 1.0002x_2 - 1.1333, \\ \qquad 1.728x_1 - 5.0601 \times 10^{-5}x_2 - 0.37825] \end{array}\right\} \quad (48)$$

where the degrees of polynomials $M_i(x)$, and $P(z)$ are set to be 1, and 0, respectively. It is obvious that when the degrees are set differently from the preceding ones, different F_i will be obtained accordingly.

Using the fuzzy controller (17) where $F_1(x)$ and $F_2(x)$ are shown in (48), the $x_1 - x_2$ phase plane with $d = 0$ is shown in Fig. 4. The nonlinearly unstable system is stabilized via the controller.

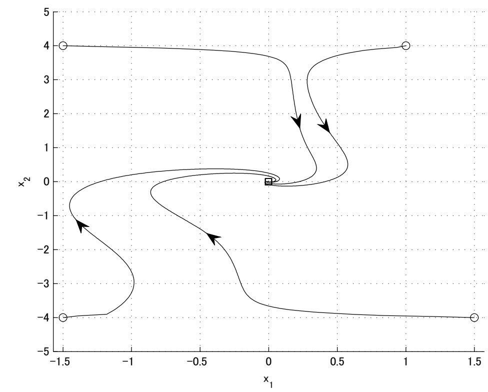

Fig. 4. $x_1 - x_2$ phase plane with fuzzy controller (17) and $d = 0$

Next, let us consider the following square wave disturbance $d(t)$ that is imposed on the nonlinear system:

$$d(t) = \left\{ \begin{array}{ll} 2, & 1.0 \le t \le 2.0 \\ -2, & 2.0 < t \le 3.0 \\ 0, & \text{otherwise} \end{array} \right. \quad (49)$$

For comparison purposes, we re-calculate $F_1(x)$, and $F_2(x)$ for the regular case where the lumped disturbance is not taken into consideration so that (43) is used instead of (39) in Theorem 1. We have

$$\left.\begin{array}{l} F_1(x) = [-0.954x_1 - 1.0004x_2 - 1.1266, \\ \qquad 1.8042x_1 - 8.3521 \times 10^{-5}x_2 - 0.40235] \\ F_2(x) = [-0.94551x_1 - 0.99992x_2 - 1.1261, \\ \qquad 1.8198x_1 + 1.7559 \times 10^{-5}x_2 - 0.40181] \end{array}\right\}. \quad (50)$$

The control results are depicted in Figs.5 and 6, where the solid lines are related to the proposed controller with (48) and the dotted lines are related to the regular controller with (50). It is shown that the proposed controller can stabilize the system with the disturbance (49), whereas the regular controller is no longer to make the system stable when the disturbance occurs at $t = 1$. Fig.7 shows the disturbance (49) and its estimate (dotted line) which is obtained by $\hat{d}(t) = \sum_{i=1}^{r} \alpha_i \hat{d}_i(t)$.

V. CONCLUSION

A new FDO was proposed into the SOS-based approach, where the polynomial fuzzy model is used to develop the system controller. The parameters involved in the FDO were adjusted on the basis of the polynomial fuzzy model, and the estimation error between the FDO and the disturbance can shrink asymptotically to zero under the assumption that the disturbance has a constant steady-state value. As one of future works, how to alleviate the assumption such as varying disturbance will be taken into consideration.

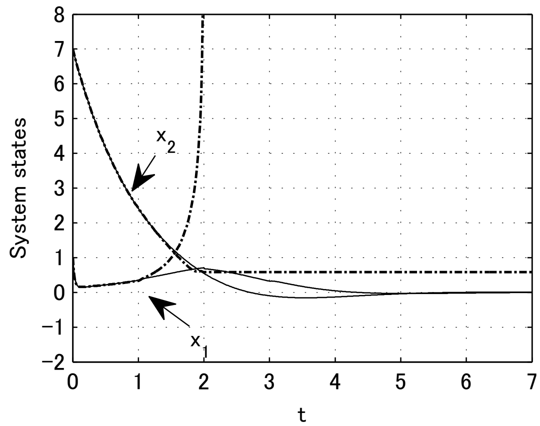

Fig. 5. System states driven by the proposed controller with (48) and the regular controller with (50)

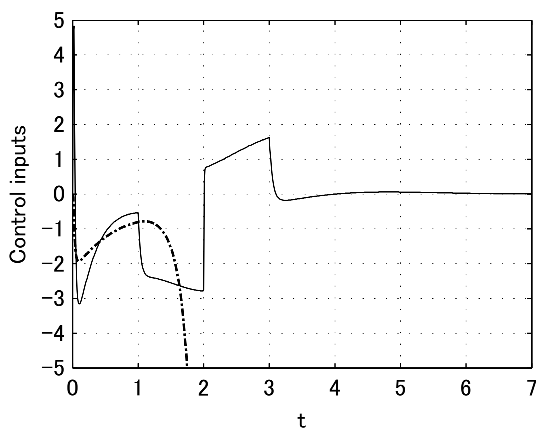

Fig. 6. Control inputs corresponding with the proposed controller with (48) and the regular controller with (50)

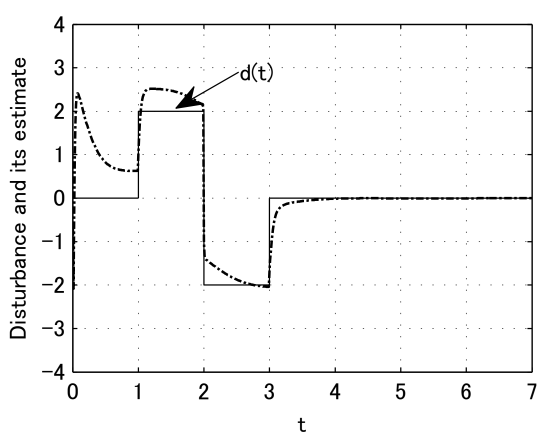

Fig. 7. Disturbance and its estimate

REFERENCES

[1] T. Takagi, and M. Sugeno, Fuzzy identification of systems and its applications to modeling and control, *IEEE Trans. Syst., Man, Cybern.,* vol.15, pp.116-132, 1985.

[2] K. Tanaka, and H.O. Wang, *Fuzzy Control System Design and Analysis – A Linear Matrix Inequality Approach,* Wiley, New York, 2001.

[3] K. Tanaka, H. Yoshida, H. Ohtake, H.O. Wang, "A sum-of-squares approach to modeling and control of nonlinear dynamical systems with polynomial fuzzy systems," *IEEE Trans. Fuzzy Syst.,* vol.17, no.4, pp.911-922, 2009.

[4] S. Prajna, A. Papachristodoulou, and P. A. Parrilo, "Introducing SOS-TOOLS: A general purpose sum of squares programming solver," in *Proc. 41st IEEE Conf. Decision Control,* 2002.

[5] B. Yao, M. A. Majed, M. Tomizuka, "High-performance robust motion control of machine tools: an adaptive robust control approach and comparative experiments," *IEEE/ASME Trans. Mechatronics,* vol.2, no.2, pp.63-76, 1997.

[6] W.-H. Chen, "Disturbance observer based control for nonlinear systems," *IEEE/ASME Trans. Mechatronics,* vol.9, no.4, pp.706-710, 2004.

[7] J. Yang, W.-H. Chen, S. Li, "Non-linear disturbance observer-based robust control for systems with mismatched disturbances/uncertainties," *IET Control Theory Appl.,* vol.5, Iss.18, pp.2053-2062, 2011.

[8] E. Kim, "A fuzzy disturbance observer and its application to control," *IEEE Trans. Fuzzy Syst.,* vol.10, no.1, pp.77-84, 2002.

[9] E. Kim, S. Lee, "Output feedback tracking control of MIMO systems using a fuzzy disturbance observer and its application to the speed control of a PM synchronous motor," *IEEE Trans. Fuzzy Syst.,* vol.13, no.6, pp.725-741, 2005.

[10] H. Han,"Adaptive fuzzy controller for a class of uncertain nonlinear systems," *J. of Japan Society for Fuzzy Theory and Intelligence Informatics,* vol. 21, no. 4, pp. 577-586, 2009.

[11] H. Han, T. Koshiro, "Adaptive T-S fuzzy controller using fuzzy approximators," *Proceedings of 2010 IEEE World Congress on Computational Intelligence,* Barcelona, Spain, 2010.

[12] Y.-Y. Cao and P. M. Frank, "Robust H_∞ disturbance attenuation for a class of uncertain discrete-time fuzzy systems," *IEEE Trans. on Fuzzy Systems*, vol. 8, no, 4, pp. 406-415, 2000.

[13] L.-X. Wang and J. M. Mendal, "Fuzzy basis functions, universal approximation, and orthogonal least-squares learning" *IEEE trans. Neural Networks,* vol. 3, pp. 807-814, 1992.