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On three types of covering-based rough sets via definable sets

Yanfang Liu and William Zhu

***Abstract*—The study of definable sets in various generalized rough set models would provide better understanding to these models. Some algebraic structures of all definable sets have been investigated, and the relationships among the definable sets, the inner definable sets and the outer definable sets have been presented. In this paper, we further study the definable sets in three types of covering-based rough sets and present several necessary and sufficient conditions of definable sets. These three types of covering-based rough sets are based on three kinds of neighborhoods: the neighborhood, the complementary neighborhood and the indiscernible neighborhood, respectively. Some necessary and sufficient conditions of definable sets are presented through these three types of neighborhoods, and the relationships among the definable sets are investigated. Moreover, we study the relationships among these three types of neighborhoods, and present certain conditions that the union of the neighborhood and the complementary neighborhood is equal to the indiscernible neighborhood.**

***Keywords*-Rough set; Covering approximation space; Inner and outer definable sets; Complementary and indiscernible neighborhoods.**

I. INTRODUCTION

Two important viewpoints form a foundation for Pawlak's rough set theory [15], [16]: one is that "knowledge is based on the ability to classify objects"; the other is that "uncertain, or rough concepts and knowledge can be defined "approximately" by determined, or definable concepts and knowledge". And the advantage of Pawlak's rough set theory is that it does not need any additional information about data, it has been successfully applied to various fields such as process control, economics, medical diagnosis, biochemistry, environmental science, biology, chemistry, psychology, conflict analysis, and other fields abound in [2], [7], [8], [10], [14], [19], [23], [26], [31].

In the development of the theory of rough sets, approximation operators are typically defined by using equivalence relations or partitions. Researchers have considered various generalized forms of Pawlak's model, such as relation-based rough sets [1], [22], [27], [28], [29], covering-based rough sets [20], [21], [24], [30], and rough-fuzzy sets and fuzzy-rough sets [3], [4], [11], [17]. The properties and applications of various generalized rough set models have been extensively discussed. However, little attention has been paid to investigate the definable sets.

In fact, systematic study of definable sets in various generalized rough set models would provide better understanding to these models. D. Pei [18] investigated the mathematical structure of the definable sets in several generalized rough set models such as relation-based models, covering-based models, and fuzzy-based models. X. Ge and Z. Li [6] presented the relationships among the definable sets, the inner definable sets and the outer definable sets in covering approximation spaces. G. Liu and Y. Sai [12] studied the algebraic structures of the definable sets of the covering-based rough set model defined by W. Xu and W. Zhang [25]. In this paper, we further study the definable sets of three types of covering-based rough sets and give some necessary and sufficient conditions of definable sets.

Here we will continue to use the marks of the literature [6] to represent these three types of covering-based rough set models [20], [21], [24]: $\mathbf{C}_6$, $\mathbf{C}_7$ and $\mathbf{C}_{10}$. In fact, these models are based on the neighborhood, the complementary neighborhood and the indiscernible neighborhood, respectively. We present some necessary and sufficient conditions of definable sets in these models. We have that 1) a set is an inner definable set of $\mathbf{C}_6$ if and only if it is an outer definable set of $\mathbf{C}_{10}$; 2) a set is an outer definable set of $\mathbf{C}_6$ if and only if it is an inner definable set of $\mathbf{C}_{10}$; 3) a set is a definable set of $\mathbf{C}_6$ if and only if it is a definable set of $\mathbf{C}_{10}$; 4) if a set is a definable set of $\mathbf{C}_7$, then it is a definable set of $\mathbf{C}_6$ (or $\mathbf{C}_{10}$). In addition, we study the relationships among the neighborhood, the complementary neighborhood and the indiscernible neighborhood. We first prove the union of neighborhood and complementary neighborhood belongs to the indiscernible neighborhood. Moreover, a condition of the union of neighborhood and complementary neighborhood to be equal to the indiscernible neighborhood is given.

Yanfang Liu and William Zhu are with Lab of Granular Computing, Minnan Normal University, Zhangzhou 363000, China (email: williamfengzhu@gmail.com (William Zhu)).

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The paper is structured as follows: in Section II, we present some definitions and properties of generalized rough sets induced by coverings. Section III, we study some necessary and sufficient conditions of definable sets in covering approximation spaces. In Section IV, we investigate the relationships among the neighborhood, the complementary neighborhood and the indiscernible neighborhood. Finally, we conclude the paper in Section V.

II. Preliminaries

In this section, we introduce the fundamental ideas about Pawlak's rough sets and the existing three types of covering-based rough sets.

We start to recall Pawlak's rough sets [15]. Let U be a non-empty finite set and $\mathbf{P} = \{P_1, \cdots, P_m\}$ a partition of U where $P_1, \cdots, P_m$ are the equivalence classes. In the rough set theory, the equivalence classes are also called elementary sets of $\mathbf{P}$. Let $\emptyset$ denote the empty set. For any $X \subseteq U$, we can describe X by the elementary sets of $\mathbf{P}$ and the two sets:

$$\underline{\mathbf{P}}(X) = \cup\{P \in \mathbf{P} : P \subseteq X\},$$
$$\overline{\mathbf{P}}(X) = \cup\{P \in \mathbf{P} : P \cap X \neq \emptyset\}$$

are called the lower and the upper approximations of X, respectively.

For any $X \subseteq U$, it is called a definable set if $\underline{\mathbf{P}}(X) = \overline{\mathbf{P}}(X)$, otherwise, it is called a rough set.

Proposition 1: Let $-X$ be the complement of X in U, we have the following properties of Pawlak's rough sets:

(L1) $\underline{\mathbf{P}}(U) = U$
(H1) $\overline{\mathbf{P}}(U) = U$
(L2) $\underline{\mathbf{P}}(\emptyset) = \emptyset$
(H2) $\overline{\mathbf{P}}(\emptyset) = \emptyset$
(L3) $\underline{\mathbf{P}}(X) \subseteq X$
(H3) $X \subseteq \overline{\mathbf{P}}(X)$
(L4) $\underline{\mathbf{P}}(X \cap Y) = \underline{\mathbf{P}}(X) \cap \underline{\mathbf{P}}(Y)$
(H4) $\overline{\mathbf{P}}(X \cup Y) = \overline{\mathbf{P}}(X) \cup \overline{\mathbf{P}}(Y)$
(L5) $\underline{\mathbf{P}}(\underline{\mathbf{P}}(X)) = \underline{\mathbf{P}}(X)$
(H5) $\overline{\mathbf{P}}(\overline{\mathbf{P}}(X)) = \overline{\mathbf{P}}(X)$
(L6) $X \subseteq Y \Rightarrow \underline{\mathbf{P}}(X) \subseteq \underline{\mathbf{P}}(Y)$
(H6) $X \subseteq Y \Rightarrow \overline{\mathbf{P}}(X) \subseteq \overline{\mathbf{P}}(Y)$
(L7) $\underline{\mathbf{P}}(-\underline{\mathbf{P}}(X)) = -\underline{\mathbf{P}}(X)$
(H7) $\overline{\mathbf{P}}(-\overline{\mathbf{P}}(X)) = -\overline{\mathbf{P}}(X)$
(L8) $\forall P \in \mathbf{P}, \underline{\mathbf{P}}(P) = P$
(H8) $\forall P \in \mathbf{P}, \overline{\mathbf{P}}(P) = P$
(LH9) $\underline{\mathbf{P}}(-X) = -\overline{\mathbf{P}}(X)$
(HL9) $\overline{\mathbf{P}}(-X) = -\underline{\mathbf{P}}(X)$

We present basic concepts of covering-based rough sets used in this paper. They are a covering of a set, the neighborhood of a point, the complementary neighborhood of a point, the indiscernible neighborhood of a point with respect to a covering, and the existing three types of covering-based rough sets.

Definition 1: (Covering) Let U be a universe of discourse and $\mathbf{C}$ a family of subsets of U. $\mathbf{C}$ is called a covering of U if none of subsets in $\mathbf{C}$ is empty and $\cup\mathbf{C} = U$. The ordered pair $(U, \mathbf{C})$ is called a covering approximation space if $\mathbf{C}$ is a covering of U.

It is clear that a partition of U is certainly a covering of U, so the concept of a covering is an extension to the concept of a partition. Unless stated, for any covering $\mathbf{C}$ of U, for all $K_1, K_2 \in \mathbf{C}$, we have $K_1 \neq K_2$ in this paper.

Definition 2: (Three types of neighborhoods [13], [32], [36]) Let $(U, \mathbf{C})$ be a covering approximation space and $x \in U$.
$\cap\{K \in \mathbf{C} : x \in K\}$ is called the neighborhood of x and denoted as $N_{\mathbf{C}}(x)$.
$\cup\{K \in \mathbf{C} : x \in K\}$ is called the indiscernible neighborhood of x and denoted as $I_{\mathbf{C}}(x)$.
$\{y \in U : x \in N_{\mathbf{C}}(y)\}$ is called the complementary neighborhood of x and denoted as $M_{\mathbf{C}}(x)$.
When there is no confusion, we omit $\mathbf{C}$ at the lowercase.

Based on the neighborhoods, different types of covering rough sets have been defined. In this paper, we recall and further consider three types of them. Here we will continue to use the marks of the literature [6].

Definition 3: ([20], [21], [24]) Let $(U, \mathbf{C})$ be a covering approximation space. For each $n \in \{6, 7, 10\}$, $\underline{\mathbf{C}_n}$ and $\overline{\mathbf{C}_n}$ are defined as follows and are called n-th lower covering approximation operator and n-th upper covering approximation operator on $(U, \mathbf{C})$ respectively.

(1) $\underline{\mathbf{C}_6}(X) = \{x \in U : N(x) \subseteq X\}$;
$\overline{\mathbf{C}_6}(X) = \{x \in U : N(x) \cap X \neq \emptyset\}$.
(2) $\underline{\mathbf{C}_7}(X) = \{x \in U : \forall K \in \mathbf{C}(x \in K \rightarrow K \subseteq X)\}$;
$\overline{\mathbf{C}_7}(X) = \cup\{K \in \mathbf{C} : K \cap X \neq \emptyset\}$.
(3) $\underline{\mathbf{C}_{10}}(X) = \{x \in U : \forall u \in U(x \in N(u) \rightarrow u \in X)\}$;
$\overline{\mathbf{C}_{10}}(X) = \cup\{N(x) : x \in X\}$.

Through the literatures [9], [32], [33], we see the above three types of covering approximation operators $\underline{\mathbf{C}_6}$, $\underline{\mathbf{C}_7}$ and $\underline{\mathbf{C}_{10}}$ all satisfy the properties $(L1) - (L6)$ and $(LH9)$, and $\overline{\mathbf{C}_6}$, $\overline{\mathbf{C}_7}$ and $\overline{\mathbf{C}_{10}}$ all satisfy the properties $(H1) - (H6)$ and $(HL9)$.

Recently, D. Pei generalized definable sets of approximation spaces to inner definable sets and outer definable sets.

Definition 4: ([18]) Let $(U, \mathbf{C})$ be a covering approximation space with approximation operators $\underline{\mathbf{C}}$ and $\overline{\mathbf{C}}$. A set X of U is called an inner (resp. outer) definable set of $(U, \mathbf{C})$ if $\underline{\mathbf{C}}(X) = X$ (resp. $\overline{\mathbf{C}}(X) = X$).

X. Ge and Z. Li [6] established some relationships among definable sets, inner definable sets and outer definable sets in covering approximation spaces, which deepened some results on definable sets in approximation spaces.

***Theorem** 1:* ([6]) Let $(U, \mathbf{C})$ be a covering approximation space. For any $X \subseteq U$,

(1) $\underline{\mathbf{C}_6}(X) = \overline{\mathbf{C}_6}(X) \Leftrightarrow \underline{\mathbf{C}_6}(X) = X$, $\overline{\mathbf{C}_6}(X) = X$;
$\underline{\mathbf{C}_6}(X) = X \nRightarrow \overline{\mathbf{C}_6}(X) = X$;
$\overline{\mathbf{C}_6}(X) = X \nRightarrow \underline{\mathbf{C}_6}(X) = X$.

(2) $\underline{\mathbf{C}_7}(X) = \overline{\mathbf{C}_7}(X) \Leftrightarrow \underline{\mathbf{C}_7}(X) = X$;
$\underline{\mathbf{C}_7}(X) = \overline{\mathbf{C}_7}(X) \Leftrightarrow \overline{\mathbf{C}_7}(X) = X$;
$\underline{\mathbf{C}_7}(X) = X \Leftrightarrow \overline{\mathbf{C}_7}(X) = X$.

(3) $\underline{\mathbf{C}_{10}}(X) = \overline{\mathbf{C}_{10}}(X) \Leftrightarrow \underline{\mathbf{C}_{10}}(X) = X$, $\overline{\mathbf{C}_{10}}(X) = X$;
$\underline{\mathbf{C}_{10}}(X) = X \nRightarrow \overline{\mathbf{C}_{10}}(X) = X$;
$\overline{\mathbf{C}_{10}}(X) = X \nRightarrow \underline{\mathbf{C}_{10}}(X) = X$.

Unary covering is an important concept of covering-based rough sets, and it has played an important role to investigate some properties of several types of covering-based rough sets. Therefore, we introduce the notion of unary covering as follows.

***Definition** 5:* (Unary covering [35]) Let $(U, \mathbf{C})$ be a covering approximation space. Let $Md(x) = \{K \in \mathbf{C} : x \in K \wedge \forall S \in \mathbf{C}(x \in S \wedge S \subseteq K \rightarrow K = S)\}$. $\mathbf{C}$ is called unary if $|Md(x)| = 1$ for all $x \in U$.

W. Zhu and F. Wang [34] introduced a notion of reducible element, and used it to solve some problems of covering-based rough sets. We present this notion as follows.

***Definition** 6:* Let $(U, \mathbf{C})$ be a covering approximation space and $K \in \mathbf{C}$. If K is a union of some sets in $\mathbf{C} - \{K\}$, we say K is a reducible element of $\mathbf{C}$, otherwise K is an irreducible element of $\mathbf{C}$. If every element of $\mathbf{C}$ is an irreducible element, we say $\mathbf{C}$ is irreducible; otherwise $\mathbf{C}$ is reducible.

III. NECESSARY AND SUFFICIENT CONDITIONS FOR A SET TO BE DEFINABLE, INNER DEFINABLE OR OUTER DEFINABLE

X. Ge et al. [5] obtained another representations of the 7-th lower and upper covering approximation operators. That is,

$\underline{\mathbf{C}_7}(X) = \{x \in U : I(x) \subseteq X\}$;
$\overline{\mathbf{C}_7}(X) = \{x \in U : I(x) \cap X \neq \emptyset\}$.

L. Ma [13] also obtained another expressions of the 10-th lower and upper covering approximation operators through introducing a notion of the complementary neighborhood. That is,

$\underline{\mathbf{C}_{10}}(X) = \{x \in U : M(x) \subseteq X\}$;
$\overline{\mathbf{C}_{10}}(X) = \{x \in U : M(x) \cap X \neq \emptyset\}$.

X. Ge and Z. Li [6] established some relationships among definable sets, inner definable sets and outer definable sets in covering approximation spaces. In this section, we further present some necessary and sufficient conditions for a set to be a definable one, an inner definable one or an outer definable one.

***Lemma** 1:* ([13], [33]) Let $(U, \mathbf{C})$ be a covering approximation space and $x, y \in U$.
(1) $y \in N(x) \Rightarrow N(y) \subseteq N(x)$;
(2) $y \in M(x) \Rightarrow M(y) \subseteq M(x)$.

***Corollary** 1:* Let $(U, \mathbf{C})$ be a covering approximation space and $x, y \in U$.
(1) $y \in N(x) \Leftrightarrow N(y) \subseteq N(x)$;
(2) $y \in M(x) \Leftrightarrow M(y) \subseteq M(x)$.

Proof: According to Definition 2, we have $z \in N(z)$ and $z \in M(z)$ for any $z \in U$. Then according to Lemma 1, we can easily prove that $y \in N(x) \Leftrightarrow N(y) \subseteq N(x)$ and $y \in M(x) \Leftrightarrow M(y) \subseteq M(x)$. ■

***Lemma** 2:* Let $(U, \mathbf{C})$ be a covering approximation space. For any $X \subseteq U$, we have
(1) $X \subseteq \bigcup_{x \in X} N(x)$;
(2) $X \subseteq \bigcup_{x \in X} M(x)$;
(3) $X \subseteq \bigcup_{x \in X} I(x)$.

The following two theorems present necessary and sufficient conditions of inner and outer definable sets in the 6-th type of covering-based rough sets from the viewpoint of the neighborhood and the complementary neighborhood.

***Theorem** 2:* Let $(U, \mathbf{C})$ be a covering approximation space. For any $X \subseteq U$, we have $\underline{\mathbf{C}_6}(X) = X$ if and only if $X = \bigcup_{x \in X} N(x)$.

Proof: ($\Rightarrow$): Suppose $X \neq \bigcup_{x \in X} N(x)$. According to Lemma 2, there exists $y \in U$ such that $y \in \bigcup_{x \in X} N(x) - X$. That is, there exists $x \in X$ such that $y \in N(x)$ and $y \notin X$. According to Lemma 1, we have $N(y) \subseteq N(x)$. Since $\underline{\mathbf{C}_6}(X) = X$, that is, $x \in \underline{\mathbf{C}_6}(X)$, i.e., $N(x) \subseteq X$. Then $N(y) \subseteq X$. So $y \in \underline{\mathbf{C}_6}(X)$ which is contradictory with $\underline{\mathbf{C}_6}(X) = X$. Therefore, if $\underline{\mathbf{C}_6}(X) = X$, then $X = \bigcup_{x \in X} N(x)$.

($\Leftarrow$): Since $X = \bigcup_{x \in X} N(x)$, then for any $x \in X$, $N(x) \subseteq X$. According to Definition 3, we have $x \in \underline{\mathbf{C}_6}(X)$ for any $x \in X$, i.e., $X \subseteq \underline{\mathbf{C}_6}(X)$. Since $\underline{\mathbf{C}_6}(X) \subseteq X$ for all $X \subseteq U$. Therefore, if $X = \bigcup_{x \in X} N(x)$, then $\underline{\mathbf{C}_6}(X) = X$. ■

***Theorem** 3:* Let $(U, \mathbf{C})$ be a covering approximation space. For any $X \subseteq U$, we have $\overline{\mathbf{C}_6}(X) = X$ if and only if $X = \bigcup_{x \in X} M(x)$.

Proof: ($\Rightarrow$): Suppose $X \neq \bigcup_{x \in X} M(x)$. According to Lemma 2, there exists $y \in U$ such that $y \in \bigcup_{x \in X} M(x) - X$. That is, there exists $x \in X$ such that $y \in M(x)$, i.e., $x \in N(y)$. Then $N(y) \cap X \neq \emptyset$. According to Definition 3, we see $y \in \overline{\mathbf{C}_6}(X)$. Since $y \notin X$, so $y \in \overline{\mathbf{C}_6}(X)$ is contradictory with $\overline{\mathbf{C}_6}(X) = X$. Therefore, if $\overline{\mathbf{C}_6}(X) = X$, then $X = \underset{x\in X}{\cup} M(x)$.

($\Leftarrow$): Suppose $\overline{\mathbf{C}_6}(X) \neq X$. According to the properties of 6-th covering upper approximation operator, we have $X \subseteq \overline{\mathbf{C}_6}(X)$ for all $X \subseteq U$. Then there exists $y \in U$ such that $y \in \overline{\mathbf{C}_6}(X) - X$. According to Definition 2, we see $N(y) \cap X \neq \emptyset$. That is, there exists $x \in X$ such that $x \in N(y)$, i.e., $y \in M(x)$. Then $y \in \underset{x\in X}{\cup} M(x)$. Since $y \notin X$, so $y \in \underset{x\in X}{\cup} M(x)$ is contradictory with $X = \underset{x\in X}{\cup} M(x)$. Therefore, if $X = \underset{x\in X}{\cup} M(x)$, then $\overline{\mathbf{C}_6}(X) = X$. ■

In the following two theorems, necessary and sufficient conditions of inner and outer definable sets in the 10-th type of covering-based rough sets are given.

Theorem 4: Let $(U, \mathbf{C})$ be a covering approximation space. For any $X \subseteq U$, we have $\underline{\mathbf{C}_{10}}(X) = X$ if and only if $X = \underset{x\in X}{\cup} M(x)$.

Proof: ($\Rightarrow$): Suppose $X \neq \underset{x\in X}{\cup} M(x)$. According to Lemma 2, there exists $y \in U$ such that $y \in \underset{x\in X}{\cup} M(x) - X$. That is, there exists $x \in X$ such that $y \in M(x)$. According to Lemma 1, we see $M(y) \subseteq M(x)$. Since $x \in \underline{\mathbf{C}_{10}}(X)$, we have $M(x) \subseteq X$. Then $M(y) \subseteq X$, i.e., $y \in \underline{\mathbf{C}_{10}}(X)$. Since $y \notin X$, so $y \in \underline{\mathbf{C}_{10}}(X)$ is contradictory with $\underline{\mathbf{C}_{10}}(X) = X$. Therefore, if $\underline{\mathbf{C}_{10}}(X) = X$, then $X = \underset{x\in X}{\cup} M(x)$.

($\Leftarrow$): Since $X = \underset{x\in X}{\cup} M(x)$, we have for any $x \in X$, $M(x) \subseteq X$. According to anther expressions of 10-th covering lower approximation operator, we see $x \in \underline{\mathbf{C}_{10}}(X)$ for any $x \in X$, i.e., $X \subseteq \underline{\mathbf{C}_{10}}(X)$. Since $\underline{\mathbf{C}_{10}}(X) \subseteq X$ for all $X \subseteq U$. Therefore, we have if $X = \underset{x\in X}{\cup} M(x)$, then $\underline{\mathbf{C}_{10}}(X) = X$. ■

Theorem 5: Let $(U, \mathbf{C})$ be a covering approximation space. For any $X \subseteq U$, we have $\overline{\mathbf{C}_{10}}(X) = X$ if and only if $X = \underset{x\in X}{\cup} N(x)$.

Proof: ($\Rightarrow$): Suppose $X \neq \underset{x\in X}{\cup} N(x)$. Then there exists $y \in U$ such that $y \in \underset{x\in X}{\cup} N(x) - X$. That is, there exists $x \in X$ such that $y \in N(x)$, i.e., $x \in M(y)$. Then $M(y) \cap X \neq \emptyset$, i.e., $y \in \overline{\mathbf{C}_{10}}(X)$. Since $y \notin X$. So $\overline{\mathbf{C}_{10}}(X) \neq X$ which is contradictory with $\overline{\mathbf{C}_{10}}(X) = X$. Therefore, if $\overline{\mathbf{C}_{10}}(X) = X$, then $X = \underset{x\in X}{\cup} N(x)$.

($\Leftarrow$): Suppose $\overline{\mathbf{C}_{10}}(X) \neq X$. We see $X \subseteq \overline{\mathbf{C}_{10}}(X)$, then there exists $y \in U$ such that $y \in \overline{\mathbf{C}_{10}}(X) - X$. According to another representation of 10-th covering approximation operators, we have $M(y) \cap X \neq \emptyset$, i.e., there exists $x \in X$ such that $x \in M(y)$. According to Definition 2, we have $y \in N(x)$. Since $y \notin X$, we have $X \neq \underset{x\in X}{\cup} N(x)$. Therefore, if $X = \underset{x\in X}{\cup} N(x)$, then $\overline{\mathbf{C}_{10}}(X) = X$. ■

Based on the above results, we can easily obtain the following two corollaries: a set is an inner definable set of $\mathbf{C}_6$ if and only if it is an outer definable set of $\mathbf{C}_{10}$; a set is an outer definable set of $\mathbf{C}_6$ if and only if it is an inner definable set of $\mathbf{C}_{10}$.

Corollary 2: Let $(U, \mathbf{C})$ be a covering approximation space. For any $X \subseteq U$, we have $\underline{\mathbf{C}_6}(X) = X$ if and only if $\overline{\mathbf{C}_{10}}(X) = X$.

Corollary 3: Let $(U, \mathbf{C})$ be a covering approximation space. For any $X \subseteq U$, we have $\overline{\mathbf{C}_6}(X) = X$ if and only if $\underline{\mathbf{C}_{10}}(X) = X$.

From the above results, a necessary and sufficient condition of definable sets in 6-th and 10-th covering-based rough sets is obtained. Moreover, we see a set is a definable set of $\mathbf{C}_6$ if and only if it is a definable set of $\mathbf{C}_{10}$.

Theorem 6: Let $(U, \mathbf{C})$ be a covering approximation space. For any $X \subseteq U$, the following statements are equivalent:
(1) $\underline{\mathbf{C}_6}(X) = \overline{\mathbf{C}_6}(X)$;
(2) $\underline{\mathbf{C}_{10}}(X) = \overline{\mathbf{C}_{10}}(X)$;
(3) $X = \underset{x\in X}{\cup} (N(x) \cup M(x))$.

Proof: According to Theorems 1, 2 and 3, and according to Corollaries 2 and 3, it is straightforward. ■

We study the definable sets, the inner definable sets and the outer definable sets of 7-th covering-based rough sets in the following theorem.

Theorem 7: Let $(U, \mathbf{C})$ be a covering approximation space. For any $X \subseteq U$, the following statements are equivalent:
(1) $\underline{\mathbf{C}_7}(X) = X$;
(2) $\overline{\mathbf{C}_7}(X) = X$;
(3) $\underline{\mathbf{C}_7}(X) = \overline{\mathbf{C}_7}(X)$;
(4) $X = \underset{x\in X}{\cup} I(x)$.

Proof: According to Theorem 1, we see (1), (2) and (3) are equivalent. Therefore, we need to prove (1) $\Leftrightarrow$ (4).

(1) $\Rightarrow$ (4): Suppose $X \neq \underset{x\in X}{\cup} I(x)$. According to Lemma 2, there exists $y \in \underset{x\in X}{\cup} I(x) - X$. Then there exists $x \in X$ such that $y \in I(x)$. Hence $I(x) \nsubseteq X$. According to another representation of 7-th covering lower approximation operator, we see $x \notin \underline{\mathbf{C}_7}(X)$, which is contradictory with $\underline{\mathbf{C}_7}(X) = X$. Therefore, if $\underline{\mathbf{C}_7}(X) = X$, then $X = \underset{x\in X}{\cup} I(x)$.

(4) $\Rightarrow$ (1): Since $X = \underset{x\in X}{\cup} I(x)$, we have for any $x \in X$, $I(x) \subseteq X$. That is, $x \in \underline{\mathbf{C}_7}(X)$ for any $x \in X$, i.e., $X \subseteq \underline{\mathbf{C}_7}(X)$. Since $\underline{\mathbf{C}_7}(X) \subseteq X$ for all $X \subseteq U$. Therefore, we have if $X = \underset{x\in X}{\cup} I(x)$, then $\underline{\mathbf{C}_7}(X) = X$. ■

We will investigate the relationship between definable sets of $\mathbf{C}_6$ and ones of $\mathbf{C}_7$. A lemma is presented to solve this issue.

***Lemma** 3:* Let $(U, \mathbf{C})$ be a covering approximation space and $x \in U$. Then $M(x) \subseteq I(x)$.

Proof: According to Definition 2, we have
$y \in M(x) \Leftrightarrow x \in N(y) \Leftrightarrow \forall K \in \mathbf{C}(y \in K \rightarrow x \in K) \Rightarrow y \in I(x)$.
Therefore, $M(x) \subseteq I(x)$. ■

***Theorem** 8:* Let $(U, \mathbf{C})$ be a covering approximation space. For any $X \subseteq U$, if $\underline{\mathbf{C}_7}(X) = \overline{\mathbf{C}_7}(X)$, then $\underline{\mathbf{C}_6}(X) = \overline{\mathbf{C}_6}(X)$.

Proof: According to Theorems 6 and 7, we have $\underline{\mathbf{C}_7}(X) = \overline{\mathbf{C}_7}(X)$ if and only if $X = \bigcup\limits_{x \in X} I(x)$, $\underline{\mathbf{C}_6}(X) = \overline{\mathbf{C}_6}(X)$ if and only if $X = \bigcup\limits_{x \in X}(N(x) \cup M(x))$. Then we need to prove if $X = \bigcup\limits_{x \in X} I(x)$, then $X = \bigcup\limits_{x \in X}(N(x) \cup M(x))$.

According to Definition 2, it is easy to obtain $N(x) \subseteq I(x)$ for all $x \in U$. According to Lemma 3, we have $M(x) \subseteq I(x)$ for all $x \in U$. Therefore, for any $x \in U$, $N(x) \cup M(x) \subseteq I(x)$. Then $X \subseteq \bigcup\limits_{x \in X}(N(x) \cup M(x)) \subseteq \bigcup\limits_{x \in X} I(x)$. Since $X = \bigcup\limits_{x \in X} I(x)$, we have $X = \bigcup\limits_{x \in X}(N(x) \cup M(x))$.

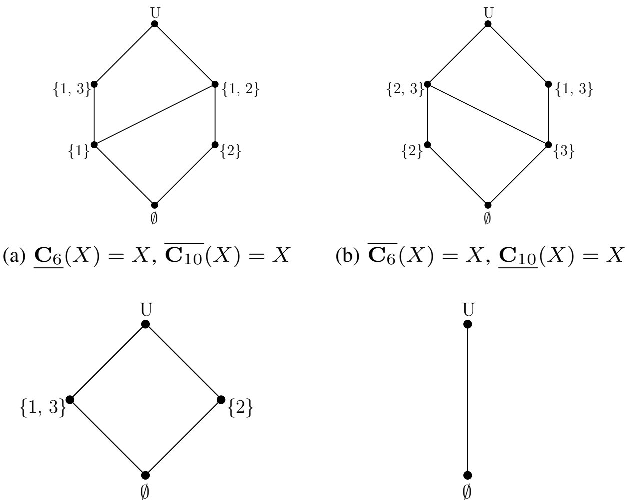
Therefore, if $\underline{\mathbf{C}_7}(X) = \overline{\mathbf{C}_7}(X)$, then $\underline{\mathbf{C}_6}(X) = \overline{\mathbf{C}_6}(X)$. ■

***Corollary** 4:* Let $(U, \mathbf{C})$ be a covering approximation space. For any $X \subseteq U$, if $\underline{\mathbf{C}_7}(X) = \overline{\mathbf{C}_7}(X)$, then $\underline{\mathbf{C}_{10}}(X) = \overline{\mathbf{C}_{10}}(X)$.

***Example** 1:* Let $U = \{1, 2, 3\}$ and $\mathbf{C} = \{\{1, 2\}, \{2\}, \{1, 3\}\}$ be a covering of U. Then, we have

$N(1) = \{1\}$	$M(1) = \{1, 3\}$	$I(1) = \{1, 2, 3\}$
$N(2) = \{2\}$	$M(2) = \{2\}$	$I(2) = \{1, 2\}$
$N(3) = \{1, 3\}$	$M(3) = \{3\}$	$I(3) = \{1, 3\}$

We use the following figure to represent the definable sets, the inner and outer definable sets of these three types of covering-based rough sets.



(a) $\underline{\mathbf{C}_6}(X) = X$, $\overline{\mathbf{C}_{10}}(X) = X$ (b) $\overline{\mathbf{C}_6}(X) = X$, $\underline{\mathbf{C}_{10}}(X) = X$

(c) $\underline{\mathbf{C}_6}(X) = \overline{\mathbf{C}_6}(X)$ $\underline{\mathbf{C}_{10}}(X) = \overline{\mathbf{C}_{10}}(X)$ (d) $\underline{\mathbf{C}_7}(X) = X$, $\overline{\mathbf{C}_7}(X) = X$, $\underline{\mathbf{C}_7}(X) = \overline{\mathbf{C}_7}(X)$

Fig. 1: Definable sets, inner and outer definable sets of $\mathbf{C}_6$, $\mathbf{C}_7$, $\mathbf{C}_{10}$

IV. Relationships among $N(x)$, $M(x)$ and $I(x)$

Above the two sections, we see $\mathbf{C}_6$, $\mathbf{C}_7$, $\mathbf{C}_{10}$ and their definable sets are all closely linked with the neighborhood, the complementary neighborhood and the indiscernible neighborhood. It is necessary to study the relationships among these three types of neighborhoods.

In the literature [13], L. Ma has presented the relationships between the neighborhood and the complementary neighborhood: $N(x) \nsubseteq M(x)$ and $M(x) \nsubseteq N(x)$. We present an example to illustrate this situation.

***Example** 2:* Let $U = \{1, 2, 3\}$ and $\mathbf{C} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$ be a covering of U. We have

$N(1) = \{1\}$	$M(1) = \{1, 2, 3\}$
$N(2) = \{1, 2\}$	$M(2) = \{2, 3\}$
$N(3) = \{1, 2, 3\}$	$M(3) = \{3\}$

We present the following proposition to describe the connections between the neighborhood and the complementary neighborhood.

***Proposition** 2:* Let $(U, \mathbf{C})$ be a covering approximation space and $x, y \in U$. Then $N(x) = N(y)$ if and only if $M(x) = M(y)$.

Proof: $(\Rightarrow)$: Since $N(x) = N(y)$, according to Definition 2 and Corollary 1, we have for any $z \in U$, $z \in M(x) \Leftrightarrow x \in N(z) \Leftrightarrow N(x) \subseteq N(z) \Leftrightarrow N(y) \subseteq N(z) \Leftrightarrow y \in N(z) \Leftrightarrow z \in M(y)$. Therefore, if $N(x) = N(y)$, then $M(x) = M(y)$.

$(\Leftarrow)$: Since $M(x) = M(y)$, according to Definition 2 and Corollary 1, we have for any $z \in U$, $z \in N(x) \Leftrightarrow x \in M(z) \Leftrightarrow M(x) \subseteq M(z) \Leftrightarrow M(y) \subseteq M(z) \Leftrightarrow y \in M(z) \Leftrightarrow z \in N(y)$. Therefore, if $M(x) = M(y)$, then $N(x) = N(y)$. ■

L. Ma has presented the relationships between the neighborhood and the complementary neighborhood: $N(x) \nsubseteq M(x)$ and $M(x) \nsubseteq N(x)$. A condition is given in the following proposition, under which $N(x)$ is equal to $M(x)$.

***Lemma** 4:* [13] Let $(U, \mathbf{C})$ be a covering approximation space. If $N(x) = M(x)$ for any $x \in U$, then $\{N(x) : x \in U\}$ forms a partition of U.

***Proposition** 3:* Let $(U, \mathbf{C})$ be a covering approximation space. Then $N(x) = M(x)$ for any $x \in U$ if and only if $\{N(x) : x \in U\}$ forms a partition of U.

Proof: According to Lemma 4, we need to prove if

$\{N(x) : x \in U\}$ forms a partition of U, then $N(x) = M(x)$ for any $x \in U$.

Since $\{N(x) : x \in U\}$ forms a partition of U, we have $y \in N(x)$ if and only if $x \in N(y)$ for any $x, y \in U$. According to Definition 2, we see $x \in N(y)$ if and only if $y \in M(x)$. Therefore, for any $x \in U$, $N(x) = M(x)$. ∎

In the following part, the relationships among the neighborhood, the complementary neighborhood and the indiscernible neighborhood are presented.

Proposition 4: Let $(U, \mathbf{C})$ be a covering approximation space. For all $x \in U$,

$$N(x) \cup M(x) \subseteq I(x).$$

Proof: According to Definition 2, it is easy to obtain $N(x) \subseteq I(x)$ for all $x \in U$. Based on Lemma 3, $M(x) \subseteq I(x)$ for all $x \in U$. That is, $N(x) \cup M(x) \subseteq I(x)$ for all $x \in U$. ∎

An issue "for any element of a universe, does the indiscernible neighborhood of the element belong to the union of the neighborhood of the element and the complementary neighborhood of the element?" is naturally put forward. We give a counter example to solve this issue.

Example 3: Let $U = \{1,2,3\}$ and $\mathbf{C} = \{\{1\},\{2\},\{1,2,3\}\}$ be a covering of U. According to Definition 2, we have

$N(1) = \{1\}$ $\quad M(1) = \{1,3\}$ $\quad I(1) = \{1,2,3\}$
$N(2) = \{2\}$ $\quad M(2) = \{2,3\}$ $\quad I(2) = \{1,2,3\}$
$N(3) = \{1,2,3\}$ $\quad M(3) = \{3\}$ $\quad I(3) = \{1,2,3\}$

Then, $N(1) \cup M(1) \subset I(1)$ and $N(2) \cup M(2) \subset I(2)$.

According to Proposition 4, we see $N(x) \cup M(x) \subseteq I(x)$ for each $x \in U$. We present certain conditions that the union of the neighborhood and the complementary neighborhood is equal to the indiscernible neighborhood.

Proposition 5: Let $(U, \mathbf{C})$ be a covering approximation space. If $\mathbf{C}$ is a partition of U, then $I(x) = N(x) \cup M(x)$ for each $x \in U$.

For a covering approximation space $(U, \mathbf{C})$, $N(x) \cup M(x) \subseteq I(x)$ for each $x \in U$, we ask the following questions:

1. When $I(x) = N(x) \cup M(x)$ for each $x \in U$, is $\mathbf{C}$ a partition of U?
2. When $I(x) = N(x) \cup M(x)$ for each $x \in U$, is $\{N(x) : x \in U\}$ a partition of U? And what about the converse?
3. When $I(x) = N(x) \cup M(x)$ for each $x \in U$, is $\mathbf{C}$ a unary covering? And what about the converse?

The answers to these questions are no. The following are counterexamples.

Example 4: Let $U = \{1,2\}$ and $\mathbf{C} = \{\{1\},\{1,2\}\}$ be a covering of U. We have

$N(1) = \{1\}$ $\quad M(1) = \{1,2\}$ $\quad I(1) = \{1,2\}$
$N(2) = \{1,2\}$ $\quad M(2) = \{2\}$ $\quad I(2) = \{1,2\}$

Then we see for any $x \in U$, $I(x) = N(x) \cup M(x)$, but $\mathbf{C}$ is not a partition of U. And $\{N(x) : x \in U\}$ is not a partition of U.

Example 5: Let $U = \{1,2\}$ and $\mathbf{C} = \{\{1\},\{2\},\{1,2\}\}$ be a covering of U. We have

$N(1) = \{1\}$ $\quad M(1) = \{1\}$ $\quad I(1) = \{1,2\}$
$N(2) = \{2\}$ $\quad M(2) = \{2\}$ $\quad I(2) = \{1,2\}$

Then we see $\{N(x) : x \in U\}$ forms a partition of U and $\mathbf{C}$ is a unary covering, but $I(1) \neq N(1) \cup M(1)$ and $I(2) \neq N(2) \cup M(2)$.

Example 6: Let $U = \{1,2,3\}$ and $\mathbf{C} = \{\{1,2\},\{2,3\}\}$ be a covering of U. We have

$N(1) = \{1,2\}$ $\quad M(1) = \{1\}$ $\quad I(1) = \{1,2\}$
$N(2) = \{2\}$ $\quad M(2) = \{1,2,3\}$ $\quad I(2) = \{1,2,3\}$
$N(3) = \{2,3\}$ $\quad M(2) = \{3\}$ $\quad I(3) = \{2,3\}$

Since $|Md(2)| = 2$, we have $\mathbf{C}$ is not a unary covering, but for any $x \in U$, $I(x) = N(x) \cup M(x)$.

In the following part, we present a condition of covering, under which $I(x)$ is equal to the union of $N(x)$ and $M(x)$. A proposition is introduced to obtain the condition as follows.

Proposition 6: Let $(U, \mathbf{C})$ be a covering approximation space and K a reducible element of $\mathbf{C}$. Let $\mathcal{C} = \{\mathbf{C}' \subseteq \mathbf{C} - \{K\} : \cup\mathbf{C}' = K\}$. Then there exists $\mathbf{C}' \in \mathcal{C}$ such that $K' \in \mathbf{C}', K' - \cup(\mathbf{C}' - \{K'\}) \neq \emptyset$ and $\cup(\mathbf{C}' - \{K'\}) - K' \neq \emptyset$.

Proof: We prove it using reduction to absurdity.

(1) Suppose for all $\mathbf{C}' \in \mathcal{C}$, $K_1 \in \mathbf{C}'$, we have $K_1 - \cup(\mathbf{C}' - \{K_1\}) = \emptyset$. Then $K_1 \subseteq \cup(\mathbf{C}' - \{K_1\})$, that is, $\cup(\mathbf{C}' - \{K_1\}) = K$, i.e., $\mathbf{C}' - \{K_1\} \in \mathcal{C}$. Therefore, for all $K_2 \in \mathbf{C}' - \{K_1\}$, $K_2 - \cup(\mathbf{C}' - \{K_1 \cup K_2\}) = \emptyset$, i.e., $\cup(\mathbf{C}' - \{K_1 \cup K_2\}) = K$. Hence $\mathbf{C}' - \{K_1 \cup K_2\} \in \mathcal{C}$. Let $|\mathbf{C}'| = m$. Similarly, we have $K_m - \cup(\mathbf{C}' - \{K_1 \cup K_2 \cdots \cup K_{m-1}\}) = \emptyset$, that is, $\cup(\mathbf{C}' - \{K_1 \cup K_2 \cdots \cup K_{m-1}\}) = K$ i.e., $K_m = K$ which is contradictory with $K_m \neq K$.

(2) Suppose for all $\mathbf{C}' \in \mathcal{C}$, $K_1 \in \mathbf{C}'$, we have $\cup(\mathbf{C}' - \{K_1\}) - K_1 = \emptyset$. We have $K_1 \subseteq \cup(\mathbf{C}' - \{K_1\})$, that is $K_1 = K$ which is contradictory with $K_1 \neq K$.

To sum up, this completes the proof. ∎

In order to illustrate the above proposition, we present the following example.

Example 7: Let $U = \{1,2,3\}$, $K_1 = \{1,2\}$, $K_2 = \{1,3\}$, $K_3 = \{2,3\}$, $K_4 = \{1,2,3\}$, $\mathbf{C} = \{K_1, K_2, K_3, K_4\}$. $\mathbf{C}$ is a covering of U. Since K_4 is a reducible element of $\mathbf{C}$, we have $K_4 = K_1 \cup K_2$, where $K_1 - K_2 = \{1\}$ and $K_2 - K_1 = \{2\}$.

***Proposition** 7:* Let $(U, \mathbf{C})$ be a covering approximation space. If $I(x) = N(x) \cup M(x)$ for each $x \in U$, then $\mathbf{C}$ is irreducible.

Proof: Suppose $\mathbf{C}$ is reducible, and K is a reducible element of $\mathbf{C}$. Let $\mathcal{C} = \{\mathbf{C}' \subseteq \mathbf{C} - \{K\} : \cup\mathbf{C}' = K\}$. According to Proposition 6, there exists $\mathbf{C}' \in \mathcal{C}$ such that $K' \in \mathbf{C}'$, $K' - \cup(\mathbf{C}' - \{K'\}) \neq \emptyset$ and $\cup(\mathbf{C}' - \{K'\}) - K' \neq \emptyset$. Then, let $x \in K' - \cup(\mathbf{C}' - \{K'\})$ and $y \in \cup(\mathbf{C}' - \{K'\}) - K'$.
(1) $|K'| = 1$
That is, $K' = \{x\}$. Then $N(x) = \{x\}$ and $K \subseteq I(x)$.

(1a) If there exists $K_1 \in \mathbf{C}$ such that $\{x, y\} \subseteq K_1$. Since $y \in \cup(\mathbf{C}' - \{K'\}) - K'$, we have $K_2 \in \mathbf{C}' - \{K'\}$ such that $y \in K_2$ and $x \notin K_2$. Therefore, $x \notin N(y)$, i.e., $y \notin M(x)$. That is, $y \notin N(x) \cup M(x)$. We see $y \in K \subseteq I(x)$. Hence $N(x) \cup M(x) \neq I(x)$ which is contradictory with $I(z) = N(z) \cup M(z)$ for each $z \in U$.

(1b) If $\{x, y\} \nsubseteq K_1$ for all $K_1 \in \mathbf{C} - \{K\}$, then $x \notin N(y)$, i.e., $y \notin M(x)$. Hence $y \notin N(x) \cup M(x)$. Since $y \in K \subseteq I(x)$, we have $N(x) \cup M(x) \neq I(x)$ which is contradictory with $I(z) = N(z) \cup M(z)$ for each $z \in U$.
(2) $|K'| \geq 2$
Let $z \in K' - \{x\}$. Then we have $z \in K$ and $N(z) \subseteq K'$.

(2a) If $z \notin \cup(\mathbf{C}' - \{K'\})$, then $z \notin N(y)$, i.e., $y \notin M(z)$. Since $y \in \cup(\mathbf{C}' - \{K'\}) - K'$, we have $y \notin N(z)$. Then $y \notin N(z) \cup M(z)$. However, $K \subseteq I(z)$ and $y \in K$. Therefore, $I(z) \neq N(z) \cup M(z)$ which is contradictory with $I(x) = N(x) \cup M(x)$ for each $x \in U$.

(2b) If $z \in \cup(\mathbf{C}' - \{K'\})$, then $N(z) \subseteq K' \cap (\cup(\mathbf{C}' - \{K'\}))$. So we have $x, y \notin N(z)$.

(2b') If $z \notin N(x)$, i.e., $x \notin M(z)$, then $x \notin N(z) \cup M(z)$. Since $z \in K$, we have $K \subseteq I(z)$. We see $x \in K$, therefore, $I(z) \neq N(z) \cup M(z)$ which is contradictory with $I(x) = N(x) \cup M(x)$ for each $x \in U$.

(2b'') If $z \in N(x)$, we see $N(x) \subseteq K'$, then $y \notin N(x)$. Since $N(y) \subseteq \cup(\mathbf{C}' - \{K'\})$, we have $x \notin N(y)$, i.e., $y \notin M(x)$. Therefore, $y \notin N(x) \cup M(x)$. We see $y \in K \subseteq I(x)$, then $N(x) \cup M(x) \neq I(x)$ which is contradictory with $I(z) = N(z) \cup M(z)$ for each $z \in U$. ■

Based on the above results, we easily obtain the relationships among the three types of covering-based rough sets.

***Theorem** 9:* Let $(U, \mathbf{C})$ be a covering approximation space. For any $X \subseteq U$, we have,
(1) $\underline{\mathbf{C}_7}(X) \subseteq \underline{\mathbf{C}_6}(X) \cap \underline{\mathbf{C}_{10}}(X)$;
(2) $\overline{\mathbf{C}_7}(X) \supseteq \overline{\mathbf{C}_6}(X) \cup \overline{\mathbf{C}_{10}}(X)$.

The proof of this theorem is simple, so we omit it. We use Figure 2 to further illustrate the relationships among the three types of covering-based rough sets.

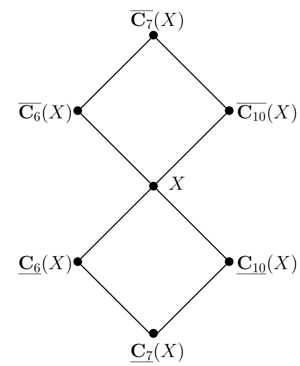


Fig. 2: Relationships among three types of covering-based rough sets

V. Conclusions

Definable sets play an important role in various generalized rough set models. In this paper, we presented some necessary and sufficient conditions of definable sets in three types of covering-based rough sets which were proved to be expressed by the neighborhood, the complementary neighborhood and the indiscernible neighborhood, respectively. Furthermore, the relationships among these three types of neighborhoods were investigated.

Some researchers have constructed a one-to-one correspondence between coverings and binary relations. For example, the 6-th type of covering-based rough sets and dominance relations are corresponding to each other, and the 7-th type of covering-based rough sets and tolerance relations are corresponding to each other. Dominance relations and tolerance relations have been defined in set-valued information systems. Based on these results, we will investigate the matroidal structures of all definable sets in covering-based rough sets, and apply them to attribute reduction in set-valued information systems in future works.

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