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A Mathematical Programming Method for the Multiple Attribute Decision Making with Interval Intuitionistic Fuzzy Values

Junfeng Chu, Xinwang Liu
School of Economics and Management
Southeast University
Nanjing 210096, China
Email: chujunfeng1987@gmail.com, xwliu@seu.edu

Abstract—**In this paper, we investigate the multiple attribute decision making problems where the decision-making information and attribute weight vector are both given by the interval-valued intuitionistic fuzzy number (IVIFN). We introduce a mathematical model to obtain the comprehensive value of each alternative by the form of IVIFN. Then we utilize the TOPSIS method to rank all the alternatives. Finally, an illustrative example is used to illustrate applicability of the proposed method.**

Keywords—multiple attribute decision making; interval-valued intuitionistic fuzzy number; Mathematical Programming Method

I. INTRODUCTION

Since the theory of fuzzy set(FS) was proposed by Zadeh [1] in 1965, because of its effective description about the vagueness and imprecise of information, it has been attracting much attention of researchers all over the world. Atanassov [2] extends the theory of fuzzy set to the intuitionistic fuzzy set(IFS), which is characterized by a membership function, a non-membership function and a hesitancy function. It is proved that the IFS can describe the imprecise and uncertainty decision-making information more suitable. Atanassov and Gargov [3] propose the concept of the interval-valued intuitionistic fuzzy set(IVIFS) which is the extensive form of IFS. The IVIFS is characterized by the membership function and non-membership function with intervals rather than the crisp numbers. Multiple attribute decision-making (MADM) can be characterized as a process of choosing the best one from a set of alternatives with respect to some attributes or ranking the order of the alternatives. Its theory and methods are widely applied to various domains, such as economy, administration and military. A large amount of methods have been introduced to tackle the MADM problems under interval-valued intuitionistic fuzzy environment [4-10]. Xu [11] utilizes the Choquet integral to develop some intuitionistic fuzzy aggregation operators. The operators not only consider the importance of the elements or their ordered positions, but also can reflect the correlations among the elements or their ordered positions. Ye [12] proposes a fuzzy cross-entropy of the interval-valued intuitionistic fuzzy set to derive the optimal evaluation for the weight of each alternative. Yu et al. [13] proposes the interval-valued intuitionistic fuzzy prioritized weighted average (IVIFPWA) operator, the interval-valued intuitionistic fuzzy prioritized weighted geometric (IVIFPWG) operator to capture the prioritization phenomenon among the aggregated arguments. Yue [14] develops an approach for aggregating attribute satisfactory interval and attribute dissatisfactory interval into the collective attribute interval-valued intuitionistic fuzzy number. Li [15] develops a methodology for solving MADM problems with both ratings of alternatives on attributes and weights being expressed with IVIF sets by constructing a pair of nonlinear fractional programming models. Lakshmana [16] introduces and studies a new method for ranking interval-valued intuitionistic fuzzy sets. Wang et al. [17] propose an approach to multiple attribute decision making with incomplete attribute weight information under interval-valued intuitionistic fuzzy set environment. There are few researches about how to tackle this type of MADM where the decision-making information and attribute weight vector are both given by IVIFS. Zhang and Yu [18] presents an optimization model to determine attribute weights for MADM problems with incomplete weight information of criteria under IVIFS environment. A series of mathematical programming models based on cross-entropy are constructed and eventually transformed into a single mathematical programming model to determine the weights of attributes. Tan [19] develops an extension of TOPSIS to investigate the group decision-making problem in interval-valued intuitionistic fuzzy environment where inter-dependent or interactive characteristics among criteria and preference of decision makers are taken into account. Wang and Liu [20] introduce some Einstein geometric operators on interval-valued intuitionistic fuzzy sets, such as Einstein product, Einstein exponentiation etc., to investigate thedecision-making problem where individual assessments are provided as IVIFN. Ye [21] proposes an extended technique for order preference by similarity to ideal solution (TOPSIS) method for group decision making with interval-valued intuitionistic fuzzy numbers to solve the partner selection problem under incomplete and uncertain information environment. Chen et al. [22] propose the interval-valued

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intuitionistic fuzzy weighted average operator based on the traditional weighted average method and the Karnik–Mendel algorithms. Then, a fuzzy ranking method for intuitionistic fuzzy values based on likelihood-based comparison relations between intervals is proposed. Park et al. [23] extend the TOPSIS method to solve multiple attribute group decision making (MAGDM) problems in interval-valued intuitionistic fuzzy environment. We try to propose one feasible method to deal with it according to the former researches' results. A series of mathematical goal programmings are given to aggregate the decision-making information of each alternative into an interval-valued intuitionistic fuzzy number. Then we apply the TOPSIS method to rank all the alternatives according to their corresponding comprehensive values .

The paper is organized as follows：In section Ⅱ, we briefly present the basic concepts such as the IVIFS , the distance measure of the IVIFS. In section Ⅲ, we firstly describe the multiple attribute decision making problems under interval-valued intuitionistic fuzzy environment. Then we introduce a mathematical model and a method tacking the multiple attribute decision-making problems based on interval-valued intuitionistic fuzzy set. In section Ⅳ, we apply an illustrative example to show the process of the proposed method. Finally, section Ⅴ summarizes this paper.

II. PRELIMINARIES

Definition 1 [3]. Let $D[0,1]$ be the collection of all the closed subintervals in [0, 1], and X is an ordinary finite non-empty set, then the object of the form $\tilde{A}=\{x,\mu_{\tilde{A}}(x),\nu_{\tilde{A}}(x)\,|\,x\in X\}$ in X is a interval-valued intuitionistic fuzzy sets. Where $\mu_{\tilde{A}}(x):X\to D[0,1]$ and $\nu_{\tilde{A}}(x)\to D[0,1]$ are the membership function in X satisfying $0<\sup_x \mu_{\tilde{A}}(x)+\sup_x \nu_{\tilde{A}}(x)\le 1$ and where $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ are the degree of membership and the degree of non-membership. For each $x\in X$, $\mu_{\tilde{A}}(x)$ and $\nu_{\tilde{A}}(x)$ are the closed interval in [0, 1], their lower and upper boundaries are expressed as $[\mu_{\tilde{A}}(x)]^L$, $[\mu_{\tilde{A}}(x)]^U$, $[\nu_{\tilde{A}}(x)]^L$, $[\nu_{\tilde{A}}(x)]^U$. All the interval-valued intuitionistic fuzzy sets in X is denoted by IVIFS[X].

For each interval-valued intuitionistic fuzzy sets in X, the following function $\pi_{\tilde{A}}(x)$ denotes the degree of non-determinacy about $x\in\tilde{A}$, i.e., the degree of hesitation:

$$\pi_{\tilde{A}}(x):X\to D[0,1].$$

Let the lower and upper boundary of $\pi_{\tilde{A}}(x)$ be $[\pi_{\tilde{A}}(x)]^L$ and $[\pi_{\tilde{A}}(x)]^U$

where

$$[\pi_{\tilde{A}}(x)]^L=1-[\mu_{\tilde{A}}(x)]^U-[\nu_{\tilde{A}}(x)]^U$$
$$[\pi_{\tilde{A}}(x)]^U=1-[\mu_{\tilde{A}}(x)]^L-[\nu_{\tilde{A}}(x)]^L$$
$$\left[[\pi_{\tilde{A}}(x)]^L,\ [\pi_{\tilde{A}}(x)]^U\right]\in D[0,1]$$

The concept of interval-valued intuitionistic fuzzy sets is the extension of the intuitionistic fuzzy set.

Definition 2 Let the ordinary finite non-empty sets $X=\{x_1,x_2,x_3\cdots x_n\}$, $\tilde{A},\tilde{B}\in IVIFS[X]$

$$\tilde{A}=\left\{\left\langle x,\left[[\mu_{\tilde{A}}(x)]^L,[\mu_{\tilde{A}}(x)]^U\right],\left[[\nu_{\tilde{A}}(x)]^L,[\nu_{\tilde{A}}(x)]^U\right]\right\rangle |\, x\in X\right\}$$
$$\tilde{B}=\left\{\left\langle x,\left[[\mu_{\tilde{B}}(x)^L,[\mu_{\tilde{B}}(x)]^U\right],\left[[\nu_{\tilde{B}}(x)^L,[\nu_{\tilde{B}}(x)]^U\right]\right\rangle |\, x\in X\right\}$$
$$[\pi_{\tilde{A}}(x)]^L=1-[\mu_{\tilde{A}}(x)]^U-[\nu_{\tilde{A}}(x)]^U$$
$$[\pi_{\tilde{A}}(x)]^U=1-[\mu_{\tilde{A}}(x)]^L-[\nu_{\tilde{A}}(x)]^L$$
$$[\pi_{\tilde{B}}(x)]^L=1-[\mu_{\tilde{B}}(x)]^U-[\nu_{\tilde{B}}(x)]^U$$
$$[\pi_{\tilde{B}}(x)]^U=1-[\mu_{\tilde{B}}(x)]^L-[\nu_{\tilde{B}}(x)]^L$$

Xu and Chen [24] defines the normalized Euclidean distance between any two interval-valued intuitionistic fuzzy sets as follows:

$$d(\tilde{A},\tilde{B})=$$
$$\left[\frac{1}{4n}\sum_{j=1}^{n}\left(\begin{array}{c}\left(\mu_{\tilde{A}}^L(x_j)-\mu_{\tilde{B}}^L(x_j)\right)^2+\left(\mu_A^U(x_j)-\mu_{\tilde{B}}^U(x_j)\right)^2\\+\left(\nu_{\tilde{A}}^L(x_j)-\nu_{\tilde{B}}^L(x_j)\right)^2+\left(\nu_{\tilde{A}}^U(x_j)-\nu_{\tilde{B}}^U(x_j)\right)^2\end{array}\right)\right]^{1/2} \quad (1)$$

III. AN APPROACH TO MULTIPLE ATTRIBUTE DECISION MAKING WITH INTERVAL INTUITIONISTIC FUZZY INFORMATION

A. The Description of the Multiple Attribute Decision Making Problem under Interval-valued Intuitionistic Fuzzy Environment

The multiple attribute decision making problems under interval-valued intuitionistic fuzzy environment are that of the decision-making information is given by the form of the intuitionistic fuzzy sets or the intuitionistic fuzzy numbers. It can be described as follows. Let $X=\{x_1,x_2,\cdots,x_n\}$ be a finite set of non-inferior alternatives. The purpose of the multiple attribute decision making problems is to choose the best alternative or rank them according to the given objectives. Suppose every alternative have m independent attributes and the set of attributes is denoted by the form of $\tilde{C}=\{\tilde{c}_1,\tilde{c}_2,\cdots,\tilde{c}_m\}$. Let the ratings of each alternative with respect to the m attributes are expressed as an interval-valued intuitionistic fuzzy numbers. We assume that $[\tilde{\mu}_{ij}^{\,L},\tilde{\mu}_{ij}^{\,U}]$ and $[\tilde{\nu}_{ij}^{\,L},\tilde{\nu}_{ij}^{\,U}]$ are the satisfaction and the dissatisfaction degree of the alternative $x_i\in X$ （$i=1,2,\cdots,n$） on the attribute $\tilde{c}_j\in\tilde{C}$ ($j=1,2,\cdots,m$) to the fuzzy concept "excellence", respectively. We also call them the membership degree and the non-membership, where $[\tilde{\mu}_{ij}^{\,L},\tilde{\mu}_{ij}^{\,U}]\in D[0,1]$, $[\tilde{\nu}_{ij}^{\,L},\tilde{\nu}_{ij}^{\,U}]\in D[0,1]$ satisfying $\tilde{\mu}_{ij}^{\,U}+\tilde{\nu}_{ij}^{\,U}\le 1$, $D[0,1]$ is the collection of the subset of unit interval $[0,1]$.

Let $\tilde{X}_{ij}=\{<x_j,[\tilde{\mu}_{ij}^{L},\tilde{\mu}_{ij}^{U}],[\tilde{\nu}_{ij}^{L},\tilde{\nu}_{ij}^{U}]>\}$, ($i=1,2,\cdots,n$, $j=1,2,\cdots,m$), then an interval-valued intuitionistic fuzzy decision matrix can be denoted by $\tilde{A}=\left[\tilde{X}_{ij}\right]_{n\times m}$, whose elements are interval-valued intuitionistic fuzzy number. Obviously, according to the theory of interval-valued intuitionistic fuzzy set, the minimum value of the alternative $x_i\in X$ ($i=1,2,\cdots,n$) on the attribute $\tilde{c}_j\in\tilde{C}$ ($j=1,2,\dots,m$) is within the closed interval $[\tilde{\mu}_{ij}^{L},\tilde{\mu}_{ij}^{U}]$ and the maximum value is within the closed interval $[1-\tilde{\nu}_{ij}^{U},1-\tilde{\nu}_{ij}^{L}]$. If the satisfaction interval degree of the alternative $x_i\in X$ ($i=1,2,\cdots,n$) on the attribute $\tilde{c}_j\in\tilde{C}$ ($j=1,2,\cdots,m$) to the fuzzy concept "excellence" is denoted as $[\tilde{\xi}_{ij},\tilde{\eta}_{ij}]$, then there is a condition that $[\tilde{\mu}_{ij}^{L},\tilde{\mu}_{ij}^{U}]\le[\tilde{\xi}_{ij},\tilde{\eta}_{ij}]\le[1-\tilde{\nu}_{ij}^{U},1-\tilde{\nu}_{ij}^{L}]$.

Let $\tilde{W}=(\omega_1,\omega_2,\cdots,\omega_m)^T$ be the attribute weight vector. Suppose the weight of the j-th attribute $\omega_j\,(j=1,2,\cdots,m)$ is also an interval-valued intuitionistic fuzzy number denoted as $([\tilde{\rho}_j^{L},\tilde{\rho}_j^{U}],[\tilde{\tau}_j^{L},\tilde{\tau}_j^{U}])$, where $[\tilde{\rho}_j^{L},\tilde{\rho}_j^{U}]\in D[0,1]$, $[\tilde{\tau}_j^{L},\tilde{\tau}_j^{U}]\in D[0,1]$, $D[0,1]$ is the collection of the subset of $[0,1]$. Furthermore, $[\tilde{\rho}_j^{L},\tilde{\rho}_j^{U}]$, $[\tilde{\tau}_j^{L},\tilde{\tau}_j^{U}]$ are the degree of membership and degree of non-membership of the j-th attribute to the concept "importance", respectively. Similarly, let the best satisfaction degree of the j-th attribute to the concept "importance" be $\tilde{\omega}_j$. Therefore, the minimum satisfaction degree is within the interval $[\tilde{\rho}_j^{L},\tilde{\rho}_j^{U}]$ and the maximum one is within the interval $[1-\tilde{\tau}_j^{U},1-\tilde{\tau}_j^{L}]$. Let the best interval satisfaction degree of the j-th attribute to the concept "importance" be $[\tilde{\omega}_j^{L},\tilde{\omega}_j^{U}]$, then we have $[\tilde{\rho}_j^{L},\tilde{\rho}_j^{U}]\le[\tilde{\omega}_j^{L},\tilde{\omega}_j^{U}]\le[1-\tilde{\tau}_j^{U},1-\tilde{\tau}_j^{L}]$. In order to find the best satisfaction degree of the j-th attribute, as is same to Li[25], we assume that $\sum_{j=1}^{m}\tilde{\rho}_j^{L}\le 1,\sum_{j=1}^{m}\tilde{\tau}_j^{L}\le 1,\sum_{j=1}^{m}\tilde{\rho}_j^{U}\ge 1,\sum_{j=1}^{m}\tilde{\tau}_j^{U}\ge 1$ and $\tilde{\omega}_j^{L}\le\tilde{\omega}_j\le\tilde{\omega}_j^{U}$, $\sum_{j=1}^{m}\tilde{\omega}_j=1$, $j=1,2,\cdots,m$.

The primary purpose of the following proposed method for the MADM under interval-valued intuitionistic fuzzy environment is to obtain the comprehensive value of each alternative by aggregating the decision-making information and the attribute vector. Then we could rank all the alternative or choose the best one according to the comprehensive values.

B. A Mathematical Programming for Aggregating the Decision Making Information and the Attribute Weight

In the multiple attribute decision-making under interval-valued intuitionistic fuzzy environment, there are numbers of methods aggregating the decision-making information to obtain a crisp comprehensive value. In the process of aggregation, a lot of significant information will be lost to some extent. In order to avoid this problem, we ought to propose a method to obtain the comprehensive value by the form of IVIFN.

By the analysis mentioned above, the satisfaction interval degree of the alternative $x_i\in X$ ($i=1,2,\cdots,n$) on the attribute $\tilde{c}_j\in\tilde{C}$ ($j=1,2,\cdots,m$) to the fuzzy concept "excellence" is $[\tilde{\xi}_{ij},\tilde{\eta}_{ij}]$. We aggregate all the attribute values of each alternative to obtain its weighted comprehensive value which can be expressed as follows:

$$[z_i^{L},z_i^{U}]=[\sum_{j=1}^{m}\tilde{\xi}_{ij}\tilde{\omega}_j,\sum_{j=1}^{m}\tilde{\eta}_{ij}\tilde{\omega}_j]\quad(i=1,2,\cdots,n)。$$

The interval value $[z_i^{L},z_i^{U}]$ can show the strengths and weakness of alternative satisfying the decision-maker's objective. Under normal circumstances, we would like to obtain the maximum comprehensive value of each alternative. Hence, we can establish the following mathematical programming models for $x_i\in X$ ($i=1,2,\cdots,n$) to calculate the maximum comprehensive value:

$$\max\left\{\tilde{z}_i^{L}=\sum_{j=1}^{m}\tilde{\xi}_{ij}\tilde{\omega}_j\right\}$$
$$s.t.\begin{cases}\tilde{\mu}_{ij}^{L}\le\tilde{\xi}_{ij}\le 1-\tilde{\nu}_{ij}^{U}\\ \tilde{\omega}_j^{L}\le\tilde{\omega}_j\le\tilde{\omega}_j^{U}\\ \rho_j^{L}\le\tilde{\omega}_j^{L}\le 1-\tau_j^{U}\\ \rho_j^{U}\le\tilde{\omega}_j^{U}\le 1-\tau_j^{L}\\ \sum_{j=1}^{m}\tilde{\omega}_j=1\\ j=1,2,\cdots,m\end{cases}\tag{2}$$

$$\max\left\{\tilde{z}_i^{U}=\sum_{j=1}^{m}\tilde{\eta}_{ij}\tilde{\omega}_j\right\}$$
$$s.t.\begin{cases}\tilde{\mu}_{ij}^{U}\le\tilde{\eta}_{ij}\le 1-\tilde{\nu}_{ij}^{L}\\ \tilde{\omega}_j^{L}\le\tilde{\omega}_j\le\tilde{\omega}_j^{U}\\ \rho_j^{L}\le\tilde{\omega}_j^{L}\le 1-\tau_j^{U}\\ \rho_j^{U}\le\tilde{\omega}_j^{U}\le 1-\tau_j^{L}\\ \sum_{j=1}^{m}\tilde{\omega}_j=1\\ j=1,2,\cdots,m\end{cases}\tag{3}$$

For each alternative $x_i\in X$ ($i=1,2,\cdots,n$), we should establish the optimal model shown above. This needs to be stressed that the optimal model is a bicriterial problem.

To solve the above model, we can deal with it as is the same to Li [25] and Wang et al. [17]. That is to say, we can convert the mathematical programming in equation (2) to the following programs for each alternative $x_i \in X$ ($i = 1, 2, \cdots, n$):

$$\min\left\{\tilde{z}_i^{LL} = \sum_{j=1}^{m} \tilde{\mu}_{ij}^{L} \tilde{\omega}_j\right\}$$

$$s.t.\begin{cases} \tilde{\omega}_i^{L} \le \tilde{\omega}_i \le \tilde{\omega}_i^{U} \\ \tilde{\omega}_j^{L} \le \tilde{\omega}_j \le \tilde{\omega}_j^{U} \\ \rho_j^{L} \le \tilde{\omega}_j^{L} \le 1-\tau_j^{U} \\ \rho_j^{U} \le \tilde{\omega}_j^{U} \le 1-\tau_j^{L} \\ \sum_{j=1}^{m} \tilde{\omega}_j = 1 \\ j = 1, 2, \cdots, m \end{cases} \tag{4}$$

$$\max\left\{\tilde{z}_i^{LU} = \sum_{j=1}^{m} (1-\tilde{\nu}_{ij}^{U}) \tilde{\omega}_j\right\}$$

$$s.t.\begin{cases} \tilde{\omega}_j^{L} \le \tilde{\omega}_j \le \tilde{\omega}_j^{U} \\ \rho_j^{L} \le \tilde{\omega}_j^{L} \le 1-\tau_j^{U} \\ \rho_j^{U} \le \tilde{\omega}_j^{U} \le 1-\tau_j^{L} \\ \sum_{j=1}^{m} \tilde{\omega}_j = 1 \\ j = 1, 2, \cdots, m \end{cases} \tag{5}$$

Where $i = 1, 2, \cdots, n$

Similarly, we can convert the mathematical programming in Eq. (3) to the following programmings for each alternative $x_i \in X$ ($i = 1, 2, \cdots, n$):

$$\min\left\{\tilde{z}_i^{UL} = \sum_{j=1}^{m} \tilde{\mu}_{ij}^{U} \tilde{\omega}_j\right\}$$

$$s.t.\begin{cases} \tilde{\omega}_j^{L} \le \tilde{\omega}_j \le \tilde{\omega}_j^{U} \\ \rho_j^{L} \le \tilde{\omega}_j^{L} \le 1-\tau_j^{U} \\ \rho_j^{U} \le \tilde{\omega}_j^{U} \le 1-\tau_j^{L} \\ \sum_{j=1}^{m} \tilde{\omega}_j = 1 \\ j = 1, 2, \cdots, m \end{cases} \tag{6}$$

$$\max\left\{\tilde{z}_i^{UU} = \sum_{j=1}^{m} (1-\tilde{\nu}_{ij}^{L}) \tilde{\omega}_j\right\}$$

$$s.t.\begin{cases} \tilde{\omega}_j^{L} \le \tilde{\omega}_j \le \tilde{\omega}_j^{U} \\ \rho_j^{L} \le \tilde{\omega}_j^{L} \le 1-\tau_j^{U} \\ \rho_j^{U} \le \tilde{\omega}_j^{U} \le 1-\tau_j^{L} \\ \sum_{j=1}^{m} \tilde{\omega}_j = 1 \\ j = 1, 2, \cdots, m \end{cases} \tag{7}$$

Where $i = 1, 2, \cdots, n$.

By solving the above linear programming from Eq. (4), (5), (6) and (7), we can obtain the corresponding optimal $\tilde{z}_i^{LL}$, $\tilde{z}_i^{LU}$, $\tilde{z}_i^{UL}$ and $\tilde{z}_i^{UU}$, respectively. Simultaneously, four attribute weight vectors $\tilde{W}_i^{LL} = \left(\tilde{\omega}_{i1}^{LL}, \tilde{\omega}_{i2}^{LL}, \cdots, \tilde{\omega}_{im}^{LL}\right)^T$, $\tilde{W}_i^{LU} = \left(\tilde{\omega}_{i1}^{LU}, \tilde{\omega}_{i2}^{LU}, \cdots, \tilde{\omega}_{im}^{LU}\right)^T$ $\tilde{W}_i^{UL} = \left(\tilde{\omega}_{i1}^{UL}, \tilde{\omega}_{i2}^{UL}, \cdots, \tilde{\omega}_{im}^{UL}\right)^T$ $\tilde{W}_i^{UU} = \left(\tilde{\omega}_{i1}^{UU}, \tilde{\omega}_{i2}^{UU}, \cdots, \tilde{\omega}_{im}^{UU}\right)^T$ with respect to the four linear programming can be obtained as well, but the four attribute weight vectors are not identical in most cases. However, the purpose of multiple attribute decision-making is to choose the best alternative or rank them. Therefore we should uniform the four attribute weight vectors. Obviously, the constraint condition of each linear programming is all the same. Hence, we combine the four linear programming models to obtain the consistent attribute weight vector denoted by $\tilde{W}^0 = (\tilde{\omega}_1^0, \tilde{\omega}_2^0, \cdots, \tilde{\omega}_m^0)^T$. The combined programming is expressed as follows:

$$\begin{aligned} &\min && \left\{\tilde{z}_i^{LL} = \sum_{j=1}^{m} \tilde{\mu}_{ij}^{L} \tilde{\omega}_j\right\} \\ &\max && \left\{\tilde{z}_i^{LU} = \sum_{j=1}^{m} (1-\tilde{\nu}_{ij}^{U}) \tilde{\omega}_j\right\} \\ &\min && \left\{\tilde{z}_i^{UL} = \sum_{j=1}^{m} \tilde{\mu}_{ij}^{U} \tilde{\omega}_j\right\} \\ &\max && \left\{\tilde{z}_i^{UU} = \sum_{j=1}^{m} (1-\tilde{\nu}_{ij}^{L}) \tilde{\omega}_j\right\} \end{aligned}$$

$$s.t.\begin{cases} \tilde{\omega}_j^{L} \le \tilde{\omega}_j \le \tilde{\omega}_j^{U} \\ \rho_j^{L} \le \tilde{\omega}_j^{L} \le 1-\tau_j^{U} \\ \rho_j^{U} \le \tilde{\omega}_j^{U} \le 1-\tau_j^{L} \\ \sum_{j=1}^{m} \tilde{\omega}_j = 1 \\ j = 1, 2, \cdots, m \end{cases} \tag{8}$$

Where $i = 1, 2, \cdots, n$.

Obviously, that is the multiple objective programming problems for each alternative $x_i \in X$. It is easy to prove that the minimization problem can be transformed to the corresponding maximization problem and they are the same optimal solution. After transforming the minimization problems to the maximization ones, we add all the four objective functions to obtain a single goal programming problem for all the alternatives as follows:

$$\max \quad \left\{ \tilde{z} = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}(2-\tilde{\mu}_{ij}{}^{L}-\tilde{\mu}_{ij}{}^{U}-\tilde{\nu}_{ij}{}^{L}-\tilde{\nu}_{ij}{}^{U})\tilde{\omega}_j}{n} \right\}$$

$$s.t.\begin{cases} \tilde{\omega}_j{}^{L} \le \tilde{\omega}_j \le \tilde{\omega}_j{}^{U} \\ \rho_j{}^{L} \le \tilde{\omega}_j{}^{L} \le 1-\tau_j{}^{U} \\ \rho_j{}^{U} \le \tilde{\omega}_j{}^{U} \le 1-\tau_j{}^{L} \\ \sum_{j=1}^{m}\tilde{\omega}_j = 1 \\ j=1,2,\cdots,m \end{cases} \tag{9}$$

We can obtain the consistent attribute weight vector $\widetilde{W}^0=\left(\tilde{\omega}_1{}^0,\tilde{\omega}_2{}^0,\cdots,\tilde{\omega}_m{}^0\right)^T$ from solving the above linear programming model. Then, We plug the $\widetilde{W}^0$ into the Eq. (4), (5), (6) and (7) in order to calculate the corresponding optimal values, i.e.

$$\min \tilde{z}_i{}^{LL} = \sum_{j=1}^{m}\tilde{\mu}_{ij}{}^{L}\tilde{\omega}_j{}^{0};$$

$$\max \tilde{z}_i{}^{LU} = \sum_{j=1}^{m}(1-\tilde{\nu}_{ij}{}^{U})\tilde{\omega}_j{}^{0} = 1-\sum_{j=1}^{m}\tilde{\nu}_{ij}{}^{U}\tilde{\omega}_j{}^{0} \tag{10}$$

$$\min \tilde{z}_i{}^{UL} = \sum_{j=1}^{m}\tilde{\mu}_{ij}{}^{U}\tilde{\omega}_j{}^{0};$$

$$\max \tilde{z}_i{}^{UU} = \sum_{j=1}^{m}(1-\tilde{\nu}_{ij}{}^{L})\tilde{\omega}_j{}^{0} = 1-\sum_{j=1}^{m}\tilde{\nu}_{ij}{}^{L}\tilde{\omega}_j{}^{0} \tag{11}$$

Obviously, we have

$$\sum_{j=1}^{m}\tilde{\mu}_{ij}{}^{L}\tilde{\omega}_j{}^{0} \le \sum_{j=1}^{m}\tilde{\mu}_{ij}{}^{U}\tilde{\omega}_j{}^{0},$$

$$\left(1-\sum_{j=1}^{m}\tilde{\nu}_{ij}{}^{U}\tilde{\omega}_j{}^{0}\right) \le \left(1-\sum_{j=1}^{m}\tilde{\nu}_{ij}{}^{L}\tilde{\omega}_j{}^{0}\right),$$

$$0 \le \sum_{j=1}^{m}\tilde{\mu}_{ij}{}^{U}\tilde{\omega}_j{}^{0} + \sum_{j=1}^{m}\tilde{\nu}_{ij}{}^{U}\tilde{\omega}_j{}^{0} \le 1.$$

Hence, the comprehensive value of each alternative with respect to all the attributes can be expressed as the following form of the interval-valued intuitionistic fuzzy number :

$$\tilde{A}_i^0 = \left(\tilde{x}_i, \left\langle \left[\sum_{j=1}^{m}\tilde{\mu}_{ij}{}^{L}\tilde{\omega}_j{}^{0}, \sum_{j=1}^{m}\tilde{\mu}_{ij}{}^{U}\tilde{\omega}_j{}^{0}\right], \left[\sum_{j=1}^{m}\tilde{\nu}_{ij}{}^{L}\tilde{\omega}_j{}^{0}, \sum_{j=1}^{m}\tilde{\nu}_{ij}{}^{U}\tilde{\omega}_j{}^{0}\right] \right\rangle \right)$$

$$(i=1,2,\cdots,n) \tag{12}$$

C. *The Calculation Procedures of the Multiple Attribute Decision-Making Method Based on Interval-valued Intuitionistic Fuzzy Set*

Being faced with the multiple attribute decision-making problems that of the decision-making information given by the form of interval-valued intuitionistic fuzzy number, we can obtain the comprehensive value $\tilde{A}_i^0$ ($i=1,2,\cdots,n$) of each alternative by applying the above models in the Section IV. The next critical step is how we choose the best alternative(s) or rank them. There are a lot of proposed methods to deal with this issue. Hwang and Yoon [26] propose an effective method that is called TOPSIS(Technique for order preference by similarity to ideal solution) to choose the best alternative(s) or rank them. It calculates the relative closeness coefficient to the positive ideal solution. The bigger the relative closeness coefficient is, the better the corresponding alternative is.

The relative closeness coefficient K_i of each alternative $x_i \in X$ ($i=1,2,\cdots,n$) can be calculated by the following formula:

$$K_i = \frac{d(\tilde{A}_i^0, A^-)}{d(\tilde{A}_i^0, A^+)+d(\tilde{A}_i^0, A^-)} \quad (i=1,2,\cdots,n) \tag{13}$$

Where $d(\cdot)$ is the distance measure of the interval-valued intuitionistic fuzzy number. The $d(\tilde{A}_i^0, A^-)$ is the distance measure between the alternative $x_i \in X$ ($i=1,2,\cdots,n$) and the negative ideal solution. The $d(\tilde{A}_i^0, A^+)$ is the distance measure between the alternative $x_i \in X$ ($i=1,2,\cdots,n$) and the positive ideal solution. In this case, we choose the intuitionistic fuzzy number $([1,1],[0,0])$ as the positive ideal solution, i.e., $\tilde{A}^+=([1,1],[0,0])$ and $\tilde{A}^-=([0,0],[1,1])$ as the negative ideal solution. There are amount of proposed distance measure of the interval-valued intuitionistic fuzzy number have been proposed in recent years[27-29]. We utilize the common distance measure the normalized Euclidean distance expressed by equation (1) here. Obviously, $K_i \in [0,1]$.

We can briefly describe the calculation procedures of the multiple attribute decision-making method based on interval-valued intuitionistic fuzzy set, which involves the following steps.

Step 1: Normalize the decision-making information to the cost-type interval-valued intuitionistic fuzzy numbers.

Step 2: Apply the proposed method in section Ⅲ-B to obtain the comprehensive value of each alternative.

Step 3: Utilize the TOPSIS method to rank the alternatives or choose the best one.

Step 4: End.

IV. ILLUSTRATIVE EXAMPLE

We apply an illustrative example to show the process of the proposed multiple attribute decision-making method under interval-valued intuitionistic fuzzy environment in this section. A company intends to purchase several portable computers to improve the efficiency of working by invitation for tender. There are five PC makers to be allowed to bid on this project. This company evaluates the alternative makers x_i ($i=1,2,\cdots,5$) according to the following principles: the price(c_1), processing rate of cpu (c_2), hard drive capacity(c_3), appearance design(c_4), battery life(c_5). The decision-making information is given by the form of interval-valued intuitionistic fuzzy set, as listed in the decision matrix $\tilde{A}=\left[\tilde{A}_1,\tilde{A}_2,\tilde{A}_3,\tilde{A}_4,\tilde{A}_5\right]^T=\left[\tilde{X}_{ij}\right]_{n\times m}$, which is listed in the Table Ⅰ.

The attribute weight vector is also expressed by the following form of interval-valued intuitionistic fuzzy number :

$$\tilde{\omega}_1=([\rho_1^L,\rho_1^U],[\tau_1^L,\tau_1^U])=([0.1,0.5],[0.3,0.5]),$$
$$\tilde{\omega}_2=([\rho_2^L,\rho_2^U],[\tau_2^L,\tau_2^U])=([0.2,0.6],[0.2,0.3]),$$
$$\tilde{\omega}_3=([\rho_3^L,\rho_3^U],[\tau_3^L,\tau_3^U])=([0.1,0.8],[0.1,0.2])$$
$$\tilde{\omega}_4=([\rho_4^L,\rho_4^U],[\tau_4^L,\tau_4^U])=([0.3,0.7],[0.1,0.2])$$
$$\tilde{\omega}_5=([\rho_5^L,\rho_5^U],[\tau_5^L,\tau_5^U])=([0.2,0.6],[0.1,0.3]).$$

Step 1: The decision matrix is given in Table Ⅰ. In particular, the attribute value with different types of each alternative $x_i\in X$ ($i=1,2,\cdots,5$) on corresponding attribute $\tilde{C}=\left\{\tilde{c}_1,\tilde{c}_2,\cdots,\tilde{c}_5\right\}$ has been normalized in some way. In this paper we omit the process of normalization.

TABLE Ⅰ
EVALUATION INFORMATION

	$\tilde{c}_1$	$\tilde{c}_2$	$\tilde{c}_3$	$\tilde{c}_4$	$\tilde{c}_5$
$\tilde{A}_1$	([0.6,0.7], [0.2,0.3])	([0.4,0.5], [0.3,0.4])	([0.7,0.8], [0.1,0.2])	([0.3,0.4], [0.2,0.5])	([0.5,0.7], [0.1,0.3])
$\tilde{A}_2$	([0.4,0.7], [0.2,0.3])	([0.5,0.8], [0.1,0.2])	([0.3,0.5], [0.2,0.4])	([0.5,0.6], [0.2,0.3])	([0.3,0.5], [0.1,0.4])
$\tilde{A}_3$	([0.3,0.4], [0.2,0.5])	([0.7,0.8], [0.0,0.1])	([0.4,0.5], [0.2,0.2])	([0.4,0.7], [0.2,0.3])	([0.3,0.6], [0.1,0.3])
$\tilde{A}_4$	([0.6,0.8], [0.1,0.2])	([0.4,0.5], [0.3,0.4])	([0.7,0.8], [0.0,0.1])	([0.4,0.7], [0.1,0.2])	([0.5,0.7], [0.2,0.3])
$\tilde{A}_5$	([0.3,0.4], [0.4,0.5])	([0.5,0.7], [0.1,0.2])	([0.2,0.4], [0.2,0.3])	([0.3,0.5], [0.2,0.4])	([0.5,0.6], [01,0.2])

Step 2: According to the linear programming model in equation (9), we can obtain the following mathematical programming:

$$\max Z=0.38\omega_1+0.42\omega_2+0.56\omega_3+0.52\omega_4+0.54\omega_5$$
$$s.t.\begin{cases}\tilde{\omega}_1^L\le\omega_1\le\tilde{\omega}_1^U;0.1\le\tilde{\omega}_1^L\le0.5;0.5\le\tilde{\omega}_1^U\le0.6;\\ \tilde{\omega}_2^L\le\omega_2\le\tilde{\omega}_2^U;0.2\le\tilde{\omega}_2^L\le0.7;0.6\le\tilde{\omega}_2^U\le0.8;\\ \tilde{\omega}_3^L\le\omega_3\le\tilde{\omega}_3^U;0.1\le\tilde{\omega}_3^L\le0.8;0.8\le\tilde{\omega}_3^U\le0.9;\\ \tilde{\omega}_4^L\le\omega_4\le\tilde{\omega}_4^U;0.3\le\tilde{\omega}_4^L\le0.8;0.7\le\tilde{\omega}_4^U\le0.9;\\ \tilde{\omega}_5^L\le\omega_5\le\tilde{\omega}_5^U;0.2\le\tilde{\omega}_5^L\le0.7;0.6\le\tilde{\omega}_5^U\le0.9;\\ \sum_{j=1}^{5}\tilde{\omega}_j=1\end{cases}\qquad(14)$$

Solving the above programming model, we can obtain the optimal attribute weight vector denoted as follows:

$$\tilde{W}^0=(\tilde{\omega}_1^0,\tilde{\omega}_2^0,\cdots,\tilde{\omega}_m^0)^T=(0.1,0.2,0.2,0.3,0.2)^T\quad(15)$$

We combine the information in the decision matrix in Table Ⅰ and the equation (15), then we can obtain the comprehensive value of each alternative $x_i\in X$ ($i=1,2,\cdots,5$) utilizing the equation (12) as follows:

$\tilde{A}_1=([0.47,0.59],[0.18,0.36]);\tilde{A}_2=([0.41,0.61],[0.16,0.32]);$
$\tilde{A}_3=([0.43,0.63],[0.14,0.26]);\tilde{A}_4=([0.5,0.69],[0.14,0.24]);$
$\tilde{A}_5=([0.36,0.53],[0.18,0.31]);$

Step 3: Rank the five alternatives by calculating the corresponding relative closeness coefficient C_i of each alternative $x_i\in X$ ($i=1,2,\cdots,5$) using the equation (13). The computed results are expressed as follows:

$$K_1=0.6218,\ K_2=0.6225,\ K_3=0.6485,$$
$$K_4=0.6872,\ K_5=0.5890$$

Therefore, the optimal and worse order to the five different alternatives is obtained applying the idea of TOPSIS method.

Since $K_4>K_3>K_2>K_1>K_5$, the ranking order result of the five alternatives is denoted as $\tilde{A}_4\succ\tilde{A}_3\succ\tilde{A}_2\succ\tilde{A}_1\succ\tilde{A}_5$, where symbol "$\succ$"means "superior to". Therefore, the most appropriate PC maker is x_4.

In addition，if we utilize the approach to comparing two IVIFNs proposed by Xu [30], we can obtain the same ranking of all the alternatives about this illustrative example. That is to say, it reflects the efficiency and rationality the method proposed in this paper to some extent.

V. CONCLUSION

In this paper, we investigate a mathematical model to tackle the multiple attribute decision-making problems under interval-valued intuitionistic fuzzy environment in which the decision-making information is given by IVIFN. What is more, the attribute weight vector is expressed by IVIFN as well. It can describe the uncertainty more accurately and more appropriately then the fuzzy set and the intuitionistic fuzzy set in the real-world environment. By solving a series of optimal programming problems, we obtain the comprehensive value of each alternative by form of IVIFN. Then we rank the alternatives by utilizing the TOPSIS method. Finally, an example has been used to illustrate the application and the validity of the proposed method.

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REFERENCES

[1]L. A. Zadeh, "fuzzy sets," *Information and Control* vol. 8, pp. 338~353, 1965.

[2]K. T. Atanassov, " Intuitionistic fuzzy sets," *Fuzzy Sets and Systems,* vol. 20, pp. 89-97, 1986.

[3]K. Atanassov and G. Gargov, "Interval valued intuitionistic fuzzy sets," *Fuzzy Sets and Systems,* vol. 31, pp. 343-349, 1989.

[4]X. D. Liu, S. H. Zheng, and F. L. Xiong, "Entropy and Subsethood for General Interval-Valued Intuitionistic Fuzzy Sets Fuzzy Systems and Knowledge Discovery." vol. 3613, L. Wang and Y. Jin, Eds., ed: Springer Berlin / Heidelberg, 2005, pp. 481-481.

[5]Z. Xu and R. R. Yager, "Dynamic intuitionistic fuzzy multi-attribute decision making," *International Journal of Approximate Reasoning,* vol. 48, pp. 246-262, 2008.

[6]Z. S. Xu and J Chen, "On Geometric Aggregation over Interval-Valued Intuitionistic Fuzzy Information," in *Fuzzy Systems and Knowledge Discovery, 2007. FSKD 2007. Fourth International Conference on*, 2007, pp. 466-471.

[7]C. P. Wei, P. Wang, and Y. Z. Zhang, "Entropy, similarity measure of interval-valued intuitionistic fuzzy sets and their applications," *Information Sciences,* vol. 181, pp. 4273-4286, 2011.

[8]D. G. Park, Y. C. Kwun, J. H. Park, and I. Y. Park, "Correlation coefficient of interval-valued intuitionistic fuzzy sets and its application to multiple attribute group decision making problems," *Mathematical and Computer Modelling,* vol. 50, pp. 1279-1293, 2009.

[9]V. Lakshmana Gomathi Nayagam and S. Geetha, "Ranking of interval-valued intuitionistic fuzzy sets," *Applied Soft Computing,* vol. 11, pp. 3368-3372, 2011.

[10]H. W. Liu and G. J. Wang, "Multi-criteria decision-making methods based on intuitionistic fuzzy sets," *European Journal of Operational Research,* vol. 179, pp. 220-233, 2007.

[11]Z. S. Xu, "Choquet integrals of weighted intuitionistic fuzzy information," *Information Sciences,* vol. 180, pp. 726-736, 2010.

[12]J. Ye, "Fuzzy cross entropy of interval-valued intuitionistic fuzzy sets and its optimal decision-making method based on the weights of alternatives," *Expert Systems with Applications,* vol. 38, pp. 6179-6183, 2011.

[13]D. Yu, Y. Wu, and T. Lu, "Interval-valued intuitionistic fuzzy prioritized operators and their application in group decision making," *Knowledge-Based Systems,* vol. 30, pp. 57-66, 2012.

[14]Z. L Yue, "An approach to aggregating interval numbers into interval-valued intuitionistic fuzzy information for group decision making," *Expert Systems with Applications,* vol. 38, pp. 6333-6338, 2011.

[15]D. F. Li, "Linear programming method for MADM with interval-valued intuitionistic fuzzy sets," *Expert Systems with Applications,* vol. 37, pp. 5939-5945, 2010.

[16]V. Lakshmana Gomathi Nayagam, S. Muralikrishnan, and G. Sivaraman, "Multi-criteria decision-making method based on interval-valued intuitionistic fuzzy sets," *Expert Systems with Applications,* vol. 38, pp. 1464-1467, 2011.

[17]Z. J. Wang, K. W. Li, and W. Z. Wang, "An approach to multiattribute decision making with interval-valued intuitionistic fuzzy assessments and incomplete weights," *Information Sciences,* vol. 179, pp. 3026-3040, 2009.

[18]H. Zhang and L. Yu, "MADM method based on cross-entropy and extended TOPSIS with interval-valued intuitionistic fuzzy sets," *Knowledge-Based Systems,* vol. 30, pp. 115-120, 2012.

[19]C. Q. Tan, "A multi-criteria interval-valued intuitionistic fuzzy group decision making with Choquet integral-based TOPSIS," *Expert Systems with Applications,* vol. 38, pp. 3023-3033, 2011.

[20]W. Z. Wang and X. W. Liu, "The multi-attribute decision making method based on interval-valued intuitionistic fuzzy Einstein hybrid weighted geometric operator," *Computers & Mathematics with Applications,* 2013.

[21]F. Ye, "An extended TOPSIS method with interval-valued intuitionistic fuzzy numbers for virtual enterprise partner selection," *Expert Systems with Applications,* vol. 37, pp. 7050-7055, 2010.

[22]S. M. Chen, L. W. Lee, H. C. Liu, and S. W. Yang, "Multiattribute decision making based on interval-valued intuitionistic fuzzy values," *Expert Systems with Applications,* vol. 39, pp. 10343-10351, 2012.

[23]J. H. Park, I. Y. Park, Y. C. Kwun, and X. Tan, "Extension of the TOPSIS method for decision making problems under interval-valued intuitionistic fuzzy environment," *Applied Mathematical Modelling,* vol. 35, pp. 2544-2556, 2011.

[24]Z. S. Xu and J. Chen, "An overview of distance and similarity measures of intuitionistic fuzzy sets," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems,* vol. 16, pp. 529-555, 2008.

[25]D. F. Li, "Multiattribute decision making models and methods using intuitionistic fuzzy sets," *Journal of Computer and System Sciences,* vol. 70, pp. 73-85, 2005.

[26]C. L. Hwang and K. Yoon,*"Multiple Attribute Decision Making: Methods and Applications", A State of the Art Survey*. Berlin, Germany: SpringerVerlag, 1981.

[27]E. Szmidt and J. Kacprzyk, "Distances between intuitionistic fuzzy sets," *Fuzzy Sets and Systems,* vol. 114, pp. 505-518, 2000.

[28]W. Wang and X. Xin, "Distance measure between intuitionistic fuzzy sets," Pattern *Recognition Letters,* vol. 26, pp. 2063-2069, 2005.

[29]Z. S. Xu, "A method based on distance measure for interval-valued intuitionistic fuzzy group decision making," *Information Sciences,* vol. 180, pp. 181-190, 2010.

[30]Z. S. Xu, "Methods for aggregating interval-valued intuitionistic fuzzy information and their application to decision making", *Control and Decision*, vol. 22(2007), pp. 215–219 (in Chinese), 2007.