2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)
July 6-11, 2014, Beijing, China

A method for estimating criteria weights from interval-valued intuitionistic fuzzy preference relation

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***Abstract*—Interval-valued intuitionistic fuzzy preference relation is a useful tool to express decision maker's interval-valued intuitionistic fuzzy preference information over criteria in the process of multi-criteria decision making. How to derive the priority weights from an interval-valued intuitionistic fuzzy preference relation is an interesting and important issue in decision making with interval-valued intuitionistic fuzzy preference relation(s). In this paper, some new concepts such as interval-valued interval fuzzy sets, interval-valued interval fuzzy preference relation and consistent interval-valued intuitionistic fuzzy preference relation, are defined, and the equivalent interval-valued interval fuzzy preference relation of interval-valued intuitionistic fuzzy preference relation is given. Then a method for estimating criteria weights from interval- valued intuitionistic fuzzy preference relations is developed, two numerical examples are provided to illustrate the developed method.**

Keywords—interval-valued intuitionistic fuzzy set; preference relation; multiple criteria decision making; consistent; goal programming models

I. INTRODUCTION

The concept of fuzzy sets (FSs) was introduced by Zadeh [1]. Interval-valued fuzzy set (IVFS) theory [2] is an extension of fuzzy set theory in which to each element of the universe a closed subinterval of the unit interval is assigned which approximates the unknown membership degree. Another extension of fuzzy set theory is intuitionistic fuzzy set theory introduced by Atanassov [3]. Intuitionistic fuzzy sets assign to each element of the universe not only a membership degree, but also a non-membership degree which is less than or equal to 1 minus the membership degree (in fuzzy set theory the non-membership degree is always equal to 1 minus the membership degree). Later Atanassov and Gargov [4] introduced interval-valued intuitionistic fuzzy sets (IVIFSs). Clearly the IVIFSs are extensions of the IFSs as well as of the IVFSs.

Decision making is one of the most common activities in the real world, including multi-criteria decision making and multi-attribute decision making. In the process of multi-criteria decision making, a decision maker is usually asked to give his/her preferences over criteria. Preference relations (or called pairwise comparison matrices, judgment matrices) are very useful in expressing decision maker's preference information in decision problems of various fields, such as politics, social psychology, engineering, management, business and economics, etc. However, a decision maker may have vague knowledge about the preference degrees of one criteria over another, and cannot estimate his/her preference with exact numerical value, but with fuzzy value, such as triangular fuzzy number [5] or interval fuzzy number [2] or intuitionistic fuzzy number [3]. During the past years, the use of fuzzy preference relations is receiving increasing attention, and a number of studies have focused on this issue, and various types of fuzzy preference relations have been developed, including the triangular fuzzy preference relations [5-6], the interval fuzzy preference relations [7-11], the intuitionistic fuzzy preference relations [12-15] and interval- valued intuitionistic fuzzy preference relations [16]. In the process of multi-criteria decision making under interval- valued intuitionistic fuzzy information, the interval-valued intuitionistic fuzzy preference relation [16] is used by the decision maker to express his/her preference information over criteria, then the priority weights derived from the interval-valued intuitionistic fuzzy preference relation can be used as the weights of criteria. Consider that the estimation of the weights of criteria play an important roles in a multiple criteria decision making process, how to derive the priority weights from an interval-valued intuitionistic fuzzy preference relation is an interesting and important research topic. Up to date, however, no investigation has been devoted to this issue. This paper is focused on the study of interval-valued intuitionistic fuzzy preference relations.

To do that, the remainder of this paper is organized as follows: Section 2 reviews some basic concepts related to IVFSs, IFSs. Section 3 introduces interval-valued interval fuzzy sets, interval-valued interval fuzzy preference relations and the consistent interval-valued intuitionistic fuzzy preference relations. In Section 4, we develop a goal programming approach to deriving criteria weights based on interval-valued intuitionistic preference relations, and furnish two numerical examples to demonstrate how the approach can be applied. Finally, we conclude the paper in Section 5.

II. PRELIMINARIES

Some basic concepts on IVFSs and IFSs are introduced below to facilitate future discussions.

Definition 1[2]. Let a set X be fixed, an interval-valued fuzzy set (IVFS) $\bar{A}$ in X is defined as:

$$\bar{A} = \{[\bar{A}^L(x), \bar{A}^U(x)] \mid x \in X\} \quad (1)$$

where the functions $\bar{A}^L(x): X \to [0,1]$ and $\bar{A}^U(x)$: $X \to [0,1]$ satisfy the condition:

$$0 \le \bar{A}^L(x) \le \bar{A}^U(x) \le 1, \forall x \in X \quad (2)$$

where $\bar{A}^L(x)$ and $\bar{A}^U(x)$ denote the lower and upper degrees of membership of element $x \in X$ to set $\bar{A}$, respectively. That is, interval-valued fuzzy set (IVFS) $\bar{A}$ assign to each element $x \in X$ a closed interval membership degree $[\bar{A}^L(x), \bar{A}^U(x)]$ which approximates the "real", but unknown.

We denote by IVFSs(X) the set of all IVFSs on X, for any given x, the interval $\bar{A}(x) = [\bar{A}^L(x), \bar{A}^U(x)]$ is called an interval number (IVFN) [2].

Let $\bar{a} = [\bar{a}^L, \bar{a}^U]$ and $\bar{b} = [\bar{b}^L, \bar{b}^U]$ be two positive interval numbers, the interval arithmetic operations can be summarized as follows [17]:

Addition: $\bar{a} + \bar{b} = [\bar{a}^L + \bar{b}^L, \bar{a}^U + \bar{b}^U]$,

Subtraction: $\bar{a} - \bar{b} = [\bar{a}^L - \bar{b}^U, \bar{a}^U - \bar{b}^L]$,

Multiplication: $\bar{a} \times \bar{b} = [\bar{a}^L \bar{b}^L, \bar{a}^U \bar{b}^U]$,

Division: $\bar{a} \div \bar{b} = [\frac{\bar{a}^L}{\bar{b}^U}, \frac{\bar{a}^U}{\bar{b}^L}], \bar{b}^L > 0$,

Scalar Multiplication: $\lambda \bar{a} = [\lambda \bar{a}^L, \lambda \bar{a}^U], \lambda > 0$.

Similar to the score function used for comparing vague sets [18], a midpoint method was present to compare two interval numbers simply [19]:

Definition 2. Let $\bar{a} = [\bar{a}^L, \bar{a}^U]$ and $\bar{b} = [\bar{b}^L, \bar{b}^U]$ be two interval numbers, $m(\bar{a}) = (\bar{a}^L + \bar{a}^U)/2$ and $m(\bar{b}) = (\bar{b}^L + \bar{b}^U)/2$ are the midpoints of $\bar{a}$ and $\bar{b}$ respectively, then

1) If $m(\bar{a}) < m(\bar{b})$, then $\bar{a} \prec \bar{b}$,

2) If $m(\bar{a}) = m(\bar{b})$, then $\bar{a} \sim \bar{b}$.

Note that the midpoint method is one which roughly compares two any interval numbers, the comprehensive method refer to the literature [19-21].

Next, recall another relation comparing IVFSs. Let L([0, 1]) denote the family of all the closed subintervals of [0,1], that is

$$L([0,1]) = \{\bar{x} = [\bar{x}^L, \bar{x}^U] \mid (\bar{x}^L, \bar{x}^U) \in [0,1]^2, \bar{x}^L \le \bar{x}^U\} \quad (3)$$

Interval-valued fuzzy sets can be seen as L-fuzzy sets (i.e. mappings from the universe to a complete lattice [22]). In the case that a special lattice $L([0,1], \le_L)$ is defined in the following way, $\bar{x}, \bar{y} \in L([0,1])$, then

$$\bar{x} \le_L \bar{y} \Leftrightarrow \bar{x}^L \le \bar{y}^L \ \& \ \bar{x}^U \le \bar{y}^U \quad (4)$$

The relation $\le_L$ is transitive and anti-symmetric; it expresses the idea that $\bar{x}$ links strongly to $\bar{y}$ [23]. Indeed, $L([0,1], \le_L)$ forms a complete lattice of intervals, and the set $L([0,1])$ can be represented by the triangle in the two-dimensional Euclidean space with corner points (0, 0), (1, 1) and (0, 1). We denote the smallest and the biggest element of $L([0,1], \le_L)$ by $0_L = [0,0]$ and $1_L = [1,1]$ respectively, where 0_L and 1_L are the extremal elements of $L([0,1])$. Evidently, it is not a linear lattice, for there exist elements that are not comparable. Other relations comparing IVFSs can be found in [19, 23]. In this paper, we will mainly consider that $\bar{x} \le_L \bar{y} \Leftrightarrow \bar{x}^L \le \bar{y}^L \ \& \ \bar{x}^U \le \bar{y}^U$.

Definition 3 [3]. Let X be a fixed set, an intuitionistic fuzzy set (IFS) $\hat{A}$ in X is defined as:

$$\hat{A} = \{< x, \mu_{\hat{A}}(x), \nu_{\hat{A}}(x) > \mid x \in X\} \quad (5)$$

where the functions $\mu_{\hat{A}}: X \to [0,1]$ and $\nu_{\hat{A}}: X \to [0,1]$ satisfy the condition:

$$0 \le \mu_{\hat{A}}(x) + \nu_{\hat{A}}(x) \le 1, \forall x \in X. \quad (6)$$

And they denote the degrees of membership and non-membership of element $x \in X$ to set $\hat{A}$, respectively. $\pi_{\hat{A}}(x) = 1 - \mu_{\hat{A}}(x) - \nu_{\hat{A}}(x)$ is usually called the intuitionistic fuzzy index of $x \in \hat{A}$, representing the degree of indeterminacy or hesitation of x to $\hat{A}$. It is obvious that $0 \le \pi_{\hat{A}}(x) \le 1$ for every $x \in X$.

Based on intuitionistic fuzzy set, Xu [14, 16] defined the concept of intuitionistic preference relation:

Definition 4. An intuitionistic fuzzy preference relation $\hat{\mathbf{R}}$ on the set X is represented by a matrix $\hat{R} = (\hat{r}_{ij})_{n \times n} \in X \times X$ with $\hat{r}_{ij} =< (x_i, x_j), \ \mu(x_i, x_j), \nu(x_i, x_j) >$, for all $i, j = 1, 2, \cdots, n.$ For convenience, let $\hat{r}_{ij} = (\mu_{ij}, \nu_{ij})$, for all $i, j = 1, 2, \cdots, n$, where r_{ij} is an intuitionistic fuzzy number, indicating the degree μ_{ij} to which x_i is preferred to x_j and the degree $\tilde{\nu}_{ij}$ to which x_i is not preferred to x_j, and $\pi_{ij} = 1 - \mu_{ij} - \nu_{ij}$ is interpreted as the indeterminacy or hesitation degree to which x_i is preferred to x_j. Furthermore, μ_{ij} and ν_{ij} satisfy the following characteristics:

$$0 \le \mu_{ij} + \nu_{ij} \le 1, \mu_{ij} = \nu_{ji}, \nu_{ij} = \mu_{ji}, \mu_{ii} = \nu_{ii} = 0.5 \quad (7)$$

Note that an intuitionistic preference relation $R = (r_{ij})_{n \times n}$ is equivalent to an interval fuzzy preference relation [20, 30] $\bar{R} = (\bar{r}_{ij})_{n \times n}$, where $\bar{r}_{ij} = [\mu_{ij}, 1 - \nu_{ij}], \ i, j = 1, 2, \cdots, n.$ Xu [15-16] defined the consistent intuitionistic preference relation.

Definition 5 [15-16]. Let $\omega=(\omega_1,\omega_2,\cdots,\omega_n)^T$ be the vector of priority weights, where ω_i reflects the importance degree of criterion x_i, $\omega_i \ge 0, i=1,2,\cdots,n, \sum_{i=1}^{n}\omega_i=1$, then an intuitionistic preference relation $R=(r_{ij})_{n\times n}$ is called a consistent intuitionistic preference relation if the following condition is satisfied ($i=1,2,\cdots,n-1; j=i+1,\cdots,n$):

$$\mu_{ij} \le 0.5(\omega_i-\omega_j+1) \le 1-v_{ij}. \quad (8)$$

III. CONSISTENT INTERVAL-VALUED INTUITIONISTIC FUZZY PREFERENCE RELATIONS

First, we introduce the concept of the interval-valued intuitionistic fuzzy set.

Definition 6 [4]. Let X be a non-empty set of the universe, an interval-valued intuitionistic fuzzy set (IVIFS) $\tilde{A}$ in X is defined as:

$$\tilde{A}=\{<x,\tilde{\mu}_{\tilde{A}}(x),\tilde{v}_{\tilde{A}}(x)>|\, x\in X\} \quad (9)$$

where $\tilde{\mu}_{\tilde{A}}: X \to L[0,1]$ and $\tilde{v}_{\tilde{A}}: X \to L[0,1]$ satisfy the condition $0 \le \sup(\tilde{\mu}_{\tilde{A}}(x))+\sup(\tilde{v}_{\tilde{A}}(x)) \le 1$, for each $x\in X$, the intervals $\tilde{\mu}_{\tilde{A}}(x)$ and $\tilde{v}_{\tilde{A}}(x)$ denote, respectively, the degree of membership and non-membership of x to A. Similar to IFSs, for each element $x\in X$ we can compute its hesitation interval relative to $\tilde{A}$ as $\tilde{\pi}_{\tilde{A}}(x)=1_L-\tilde{\mu}_{\tilde{A}}(x)-\tilde{v}_{\tilde{A}}(x)$. Especially, if each of the intervals $\tilde{\mu}_{\tilde{A}}(x)$ and $\tilde{v}_{\tilde{A}}(x)$ contains only one real value, i.e., $\inf(\tilde{\mu}_{\tilde{A}}(x)) = \sup(\tilde{\mu}_{\tilde{A}}(x)) = \mu(x)$, $\inf(\tilde{v}_{\tilde{A}}(x)) = \sup(\tilde{v}_{\tilde{A}}(x)) = v(x)$, for each $x\in X$, then the given IVIFS $\tilde{A}$ is reduced to an ordinary IFS $\hat{A}=\{<x,\mu(x),\ v(x)>|\, x\in X\}$.

For any given x, the pair $(\tilde{\mu}_{\tilde{A}}(x),\tilde{v}_{\tilde{A}}(x))$ is called an interval-valued intuitionistic fuzzy number (IVIFN), and it must satisfy the condition: $\tilde{\mu}_{\tilde{A}}(x)\in L[0,1]$, $\tilde{v}_{\tilde{A}}(x)\in L[0,1]$ and $0 \le \sup(\tilde{\mu}_{\tilde{A}}(x))+\sup(\tilde{v}_{\tilde{A}}(x)) \le 1$, we can describe the same truth values by two interval numbers $\bar{\bar{r}}_{\tilde{A}}^{L}=\tilde{\mu}_{\tilde{A}}(x)$ and $\bar{\bar{r}}_{\tilde{A}}^{U}=1_L-\tilde{v}_{\tilde{A}}(x)$ which satisfy the condition $\bar{\bar{r}}_{\tilde{A}}^{L} \le_L \bar{\bar{r}}_{\tilde{A}}^{U}$, this condition is the same as the condition under which two interval numbers form an interval-valued interval $[\bar{\bar{r}}_{\tilde{A}}^{L},\bar{\bar{r}}_{\tilde{A}}^{U}]$, that is to say, an interval-valued intuitionistic fuzzy number $(\tilde{\mu}_{\tilde{A}}(x),\tilde{v}_{\tilde{A}}(x))$ is equivalent to an interval-valued interval fuzzy number $[\bar{\bar{r}}_{\tilde{A}}^{L},\bar{\bar{r}}_{\tilde{A}}^{U}]$.

In what follows, we consider the situations where the preference values given by the decision maker are interval-valued intuitionistic fuzzy numbers.

Suppose a multiple criteria decision making problem with a finite set of n criteria, and let $\mathbf{X}=\{x_1,x_2,\cdots,x_n\}$ be the set of criteria, a decision maker compares each pair of criteria in X, and provides his/her interval-valued intuitionistic preference degree $\tilde{r}_{ij}=(\tilde{\mu}_{ij},\tilde{v}_{ij})$ of the criterion x_i over x_j, where $\tilde{r}_{ij}$ is an interval-valued intuitionistic fuzzy number, consisting of the interval degree $\tilde{\mu}_{ij}$ to which x_i is preferred to x_j and the interval degree $\tilde{v}_{ij}$ to which x_i is not preferred to x_j, and $\tilde{\pi}_{ij}(x)=1_L-\tilde{\mu}_{ij}-\tilde{v}_{ij}$ is interpreted as the hesitation interval degree to which x_i is preferred to x_j. All these interval-valued intuitionistic preference degree $\tilde{r}_{ij}, i,j=1,2,\cdots,n$ compose an interval- valued intuitionistic preference relation [16]:

$$\tilde{R}=(\tilde{r}_{ij})_{n\times n}=\left((\tilde{\mu}_{ij},\tilde{v}_{ij})\right)_{n\times n} \quad (10)$$

with the conditions:

$$\begin{aligned}&\tilde{\mu}_{ij}(x)\in L[0,1],\tilde{v}_{ij}(x)\in L[0,1],\\ &0\le \sup(\tilde{\mu}_{ij}(x))+\sup(\tilde{v}_{ij}(x))\le 1,\\ &\tilde{\mu}_{ji}=\tilde{v}_{ij},\tilde{v}_{ji}=\tilde{\mu}_{ij},\tilde{\mu}_{ii}=\tilde{v}_{ii}=[0.5,0.5].\end{aligned} \quad (11)$$

Since each element $\tilde{r}_{ij}$ in the interval-valued intuitionistic preference relation $\tilde{R}$ consists of an IVIFN $(\tilde{\mu}_{ij},\tilde{v}_{ij})$, which is equivalent to an interval-valued interval fuzzy number $[\bar{\bar{r}}_{ij}^{L},\bar{\bar{r}}_{ij}^{U}]=[\tilde{\mu}_{ij}(x),1_L-\tilde{v}_{ij}(x)]$, then we have

Theorem 1. The interval-valued intuitionistic preference relation $\tilde{R}=(\tilde{r}_{ij})_{n\times n}=\left((\tilde{\mu}_{ij},\tilde{v}_{ij})\right)_{n\times n}$ is equivalent to the corresponding interval-valued interval preference relation

$$\bar{\bar{R}}=(\bar{\bar{r}}_{ij})_{n\times n}=\left([\bar{\bar{r}}_{ij}^{L},\bar{\bar{r}}_{ij}^{U}]\right)_{n\times n} \quad (12)$$

where $\bar{\bar{r}}_{ij}^{L}=\tilde{\mu}_{ij}(x)$ and $\bar{\bar{r}}_{ij}^{U}=1_L-\tilde{v}_{ij}(x)$ are the lower and upper limits of the interval-valued interval fuzzy number $\bar{\bar{r}}_{ij}$, for all $i,j=1,2,\cdots,n$, respectively, furthermore, $\bar{\bar{r}}_{ij}^{L}$ and $\bar{\bar{r}}_{ij}^{U}$ satisfy the following characteristics:

$$\begin{aligned}&\bar{\bar{r}}_{ij}^{L},\bar{\bar{r}}_{ij}^{U}\in L[0,1],\bar{\bar{r}}_{ii}^{L}=\bar{\bar{r}}_{ii}^{U}=[0.5,0.5],\\ &\bar{\bar{r}}_{ij}^{L}\le_L\bar{\bar{r}}_{ij}^{U},\bar{\bar{r}}_{ji}^{L}=1_L-\bar{\bar{r}}_{ij}^{U},\bar{\bar{r}}_{ji}^{U}=1_L-\bar{\bar{r}}_{ij}^{L},\end{aligned} \quad (13)$$

In this paper, assume that $\bar{\omega}_i^{L}$ and $\bar{\omega}_i^{U}$ be the lower and upper degrees of membership of the criteria $x_i\in X$ on the fuzzy concept "importance", respectively, where $0\le\bar{\omega}_i^{L}\le\bar{\omega}_i^{U}\le 1$. In other words, weights of the criteria $x_i\in X$ given by the decision maker are IVFSs $\bar{\omega}_i=\{[x_i,\bar{\omega}_i^{L},\bar{\omega}_i^{U}]$

$\mid x_i \in X\}$, denoted by $\bar{\omega}_i = [\bar{\omega}_i^L,\ \bar{\omega}_i^U]$ for short. Then a weight vector of all criteria can be expressed concisely in the interval-valued format as follows:

$$\bar{\boldsymbol{\omega}} = ([\bar{\omega}_1^L, \bar{\omega}_1^U], [\bar{\omega}_2^L, \bar{\omega}_2^U], \ldots, [\bar{\omega}_n^L, \bar{\omega}_n^U])^T$$

Let $\bar{\omega}_i = [\bar{\omega}_i^L, \bar{\omega}_i^U]$ be a set of interval weights with $0 \le \bar{\omega}_i^L \le \bar{\omega}_i^U \le 1$, $i = 1, 2, \cdots, n$, and $N = \{(\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T) \mid \bar{\omega}_i^L \le \omega_i \le \bar{\omega}_i^U, i = 1, 2, \cdots, \mathrm{n}, \sum_{i=1}^n \omega_i = 1\}$ be a set of normalized weight vectors, based on which we introduce the following definition of normalization for interval weights [17, 24-30].

Definition 7. If an interval weight vector $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n)^T$ with $\bar{\omega}_i = [\bar{\omega}_i^L, \bar{\omega}_i^U]$, $0 \le \bar{\omega}_i^L \le \bar{\omega}_i^U \le 1$, for $i = 1, 2, \cdots, n$ satisfies

$$\sum_{i \ne j}^{n} \bar{\omega}_i^L + \bar{\omega}_j^U \le 1, j = 1, 2, \cdots, n, \tag{14}$$

$$\bar{\omega}_j^L + \sum_{i \ne j}^{n} \bar{\omega}_i^U \ge 1, j = 1, 2, \cdots, n, \tag{15}$$

then it is normalized; otherwise it is not normalized.

Theorem 2. Let an interval weight vector $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n)^T$ with $\bar{\omega}_i = [\bar{\omega}_i^L, \bar{\omega}_i^U]$, $0 \le \bar{\omega}_i^L \le \bar{\omega}_i^U \le 1$, for $i = 1, 2, \cdots, n$ and $N = \{(\omega = (\omega_1, \omega_2, \cdots, \omega_n)^T) \mid \bar{\omega}_i^L \le \omega_i \le \bar{\omega}_i^U, i = 1, 2, \cdots, n, \sum_{i=1}^n \omega_i = 1\}$ be a set of normalized weight vectors, $c_{ij} = 0.5(\omega_i - \omega_j + 1), \bar{c}_{ij} = 0.5(\bar{\omega}_i - \bar{\omega}_j + 1_L) = [0.5(\bar{\omega}_i^L - \bar{\omega}_j^U + 1), 0.5(\bar{\omega}_i^U - \bar{\omega}_j^L + 1)]$ for $i, j = 1, 2, \cdots, n$, then for $i, j = 1, 2, \cdots, n$,

(1) $c_{ij} \in \bar{c}_{ij} \in L([0,1])$, that is

$$0.5(\omega_i - \omega_j + 1) \in [0.5(\bar{\omega}_i^L - \bar{\omega}_j^U + 1), 0.5(\bar{\omega}_i^U - \bar{\omega}_j^L + 1)] \in L([0,1])$$

(2) $\bar{c}_{ji} = 1_L - \bar{c}_{ij}$.

Motivated by the models for deriving priority weights based on interval fuzzy preference relations in [10], we define consistent interval-valued intuitionistic fuzzy preference relations as follows:

Definition 8. Let an interval-valued intuitionistic preference relation $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = \left((\tilde{\mu}_{ij}, \tilde{\nu}_{ij})\right)_{n\times n}$ ($\bar{\bar{R}} = (\bar{\bar{r}}_{ij})_{n\times n} = \left([\bar{\bar{r}}_{ij}^L, \bar{\bar{r}}_{ij}^U]\right)_{n\times n}$ with $\bar{\bar{r}}_{ij}^L = \tilde{\mu}_{ij}(x)$, $\bar{\bar{r}}_{ij}^U = 1_L - \tilde{\nu}_{ij}(x)$ and $\bar{\bar{r}}_{ij}^L \le_L \bar{\bar{r}}_{ij}^U$), if there exists a normalized interval priority vector $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n)^T$ with $\bar{\omega}_i = [\bar{\omega}_i^L,\ \bar{\omega}_i^U]$ such that for all $i, j = 1, 2, \cdots, n$.

$$\bar{\bar{r}}_{ij}^L \le_L 0.5(\bar{\omega}_i - \bar{\omega}_j + [1,1]) \le_L \bar{\bar{r}}_{ij}^U, \tag{16}$$

where $\bar{\omega}$ satisfies the condition (14-15), then $\tilde{R}$ is a consistent interval-valued intuitionistic preference relation, otherwise $\tilde{R}$ is not a consistent interval-valued intuitionistic preference relation.

By the characteristics (13) of interval-valued interval preference relation and Theorem 1, it is easy to prove that Definition 8 is equivalent to Definition 9.

Definition 9. Let an interval-valued intuitionistic preference relation $\tilde{R} = (\tilde{r}_{ij})_{n\times n} = \left((\tilde{\mu}_{ij}, \tilde{\nu}_{ij})\right)_{n\times n}$ ($\bar{\bar{R}} = (\bar{\bar{r}}_{ij})_{n\times n} = \left([\bar{\bar{r}}_{ij}^L, \bar{\bar{r}}_{ij}^U]\right)_{n\times n}$ with $\bar{\bar{r}}_{ij}^L = \tilde{\mu}_{ij}(x)$, $\bar{\bar{r}}_{ij}^U = 1_L - \tilde{\nu}_{ij}(x)$ and $\bar{\bar{r}}_{ij}^L \le_L \bar{\bar{r}}_{ij}^U$), if there exists a normalized interval priority vector $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n)^T$ with $\bar{\omega}_i = [\bar{\omega}_i^L,\ \bar{\omega}_i^U]$ such that for all $i = 1, 2, \cdots, n-1; j = i+1, \cdots, n$

$$\bar{\bar{r}}_{ij}^L \le_L 0.5(\bar{\omega}_i - \bar{\omega}_j + [1,1]) \le_L \bar{\bar{r}}_{ij}^U, \tag{17}$$

where $\bar{\omega}$ satisfies the condition (14-15), then $\tilde{R}$ is a consistent interval-valued intuitionistic preference relation, otherwise $\tilde{R}$ is not a consistent interval-valued intuitionistic preference relation.

In particular, let the lower and upper boundaries of the intervals $\tilde{\mu}_{ij}$ and $\tilde{\nu}_{ij}$ are denoted by $\tilde{\mu}_{\tilde{A}}^L(x), \tilde{\mu}_{\tilde{A}}^U(x), \tilde{\nu}_{\tilde{A}}^L(x)$ and $\tilde{\nu}_{\tilde{A}}^U(x)$, respectively. Therefore, an equivalent way to express an IVIFS $\tilde{r}_{ij}$ is $\tilde{r}_{ij} = ([\tilde{\mu}_{ij}^L(x), \tilde{\mu}_{ij}^U(x)], [\tilde{\nu}_{ij}^L(x), \tilde{\nu}_{ij}^U(x)])$ where $0 \le \tilde{\mu}_{ij}^L(x) \le \tilde{\mu}_{ij}^U(x) \le 1$, $0 \le \tilde{\nu}_{ij}^L(x) \le \tilde{\nu}_{ij}^U(x) \le 1$ and $0 \le \tilde{\mu}_{ij}^U(x) + \tilde{\nu}_{ij}^U(x) \le 1$. Correspondingly among the $\bar{\bar{R}} = (\bar{\bar{r}}_{ij})_{n\times n} = \left([\bar{\bar{r}}_{ij}^L, \bar{\bar{r}}_{ij}^U]\right)_{n\times n}$, the interval numbers $\bar{\bar{r}}_{ij}^L = \tilde{\mu}_{ij}(x) = [\tilde{\mu}_{ij}^L, \tilde{\mu}_{ij}^U]$ and $\bar{\bar{r}}_{ij}^U = [1 - \tilde{\nu}_{ij}^U, 1 - \tilde{\nu}_{ij}^L]$. Thus the Definition 9 can be transformed to the following form.

Definition 10. Let an interval-valued intuitionistic preference relation $\tilde{R} = (\tilde{r}_{ij})_{n\times n}$ with $\tilde{r}_{ij} = (\tilde{\mu}_{ij}, \tilde{\nu}_{ij}) = ([\tilde{\mu}_{ij}^L, \tilde{\mu}_{ij}^U], [\tilde{\nu}_{ij}^L, \tilde{\nu}_{ij}^U])$, if there exists a normalized interval priority vector $\bar{\omega} = (\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_n)^T$ with $\bar{\omega}_i = [\bar{\omega}_i^L, \bar{\omega}_i^U]$ such that for all $i = 1, 2, \cdots, n-1; j = i+1, \cdots, n$

$$[\tilde{\mu}_{ij}^L, \tilde{\mu}_{ij}^U] \le_L 0.5(\bar{\omega}_i - \bar{\omega}_j + [1,1]) \le_L [1 - \tilde{\nu}_{ij}^U, 1 - \tilde{\nu}_{ij}^L] \tag{18}$$

i.e.,

$$\begin{cases} \tilde{\mu}_{ij}^L \le 0.5(\bar{\omega}_i^L - \bar{\omega}_j^U + 1) \le 1 - \tilde{\nu}_{ij}^U \\ \tilde{\mu}_{ij}^U \le 0.5(\bar{\omega}_i^U - \bar{\omega}_j^L + 1) \le 1 - \tilde{\nu}_{ij}^L \end{cases} \tag{19}$$

where $\hat{\boldsymbol{\omega}}$ satisfies the condition (14-15), then $\tilde{\mathbf{R}}$ is called a consistent interval-valued intuitionistic preference relation, otherwise $\tilde{\mathbf{R}}$ is not a consistent interval-valued intuitionistic preference relation.

IV. A METHOD FOR ESTIMATING CRITERIA WEIGHTS

Priority weights of criteria obviously play an important role in a multiple criteria decision making process. Next, we shall develop some simple and practical goal programming models for deriving priority weights based on both consistent and inconsistent interval-valued intuitionistic preference relations:

(1) If $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$ is a consistent interval-valued intuitionistic fuzzy preference relation, then the interval weight vector $\bar{\omega}=(\bar{\omega}_1,\bar{\omega}_2,\cdots,\bar{\omega}_n)^T$ derived from $\tilde{R}$ should satisfy (14) and (15). In general, the interval weight vector satisfying these two conditions is not unique, but each interval weight $\bar{\omega}_i, i=1,2,...,n$ should belong to an interval-valued range. As such, based on conditions (14) and (15), we establish the following four linear programming models:

$$
\begin{aligned}
(M-1)\ \ & \bar{\omega}_i^{L-}=\min \bar{\omega}_i^L \\
s.t\ \ & 0.5(\bar{\omega}_i^L-\bar{\omega}_j^U+1)\ge \tilde{\mu}_{ij}^L, \\
& 0.5(\bar{\omega}_i^L-\bar{\omega}_j^U+1)\le 1-\tilde{v}^U, \\
& 0.5(\bar{\omega}_i^U-\bar{\omega}_j^L+1)\ge \tilde{\mu}^U, \\
& 0.5(\bar{\omega}_i^U-\bar{\omega}_j^L+1)\le 1-\tilde{v}_{ij}^L, \\
& i=1,2,\cdots,n-1;\ j=i+1,\cdots,n; \\
& 0\le \bar{\omega}_i^L\le \bar{\omega}_i^U\le 1, i=1,2,\cdots n \\
& \sum_{i=1,i\ne j}^{n}\bar{\omega}_i^L+\bar{\omega}_j^U\le 1,\ j=1,2,\cdots n \\
& \bar{\omega}_j^L+\sum_{i=1,i\ne j}^{n}\bar{\omega}_i^U\ge 1, j=1,2,\cdots n.
\end{aligned}
$$

$$
\begin{aligned}
(M-2)\ \ & \bar{\omega}_i^{L+}=\max \bar{\omega}_i^L \\
s.t\ \ & 0.5(\bar{\omega}_i^L-\bar{\omega}_j^U+1)\ge \tilde{\mu}_{ij}^L, \\
& 0.5(\bar{\omega}_i^L-\bar{\omega}_j^U+1)\le 1-\tilde{v}^U, \\
& 0.5(\bar{\omega}_i^U-\bar{\omega}_j^L+1)\ge \tilde{\mu}^U, \\
& 0.5(\bar{\omega}_i^U-\bar{\omega}_j^L+1)\le 1-\tilde{v}_{ij}^L, \\
& i=1,2,\cdots,n-1;\ j=i+1,\cdots,n; \\
& 0\le \bar{\omega}_i^L\le \bar{\omega}_i^U\le 1, i=1,2,\cdots n; \\
& \sum_{i=1,i\ne j}^{n}\bar{\omega}_i^L+\bar{\omega}_j^U\le 1,\ \bar{\omega}_j^L+\sum_{i=1,i\ne j}^{n}\bar{\omega}_i^U\ge 1, \\
& j=1,2,\cdots n.
\end{aligned}
$$

$$
\begin{aligned}
(M-3)\ \ & \bar{\omega}_i^{U-}=\min \bar{\omega}_i^U \\
s.t\ \ & 0.5(\bar{\omega}_i^L-\bar{\omega}_j^U+1)\ge \tilde{\mu}_{ij}^L, \\
& 0.5(\bar{\omega}_i^L-\bar{\omega}_j^U+1)\le 1-\tilde{v}^U, \\
& 0.5(\bar{\omega}_i^U-\bar{\omega}_j^L+1)\ge \tilde{\mu}^U, \\
& 0.5(\bar{\omega}_i^U-\bar{\omega}_j^L+1)\le 1-\tilde{v}_{ij}^L, \\
& i=1,2,\cdots,n-1;\ j=i+1,\cdots,n; \\
& 0\le \bar{\omega}_i^L\le \bar{\omega}_i^U\le 1, i=1,2,\cdots n; \\
& \sum_{i=1,i\ne j}^{n}\bar{\omega}_i^L+\bar{\omega}_j^U\le 1, \bar{\omega}_j^L+\sum_{i=1,i\ne j}^{n}\bar{\omega}_i^U\ge 1, \\
& j=1,2,\cdots n.
\end{aligned}
$$

and

$$
\begin{aligned}
(M-4)\ \ & \bar{\omega}_i^{U+}=\max \bar{\omega}_i^U \\
s.t\ \ & 0.5(\bar{\omega}_i^L-\bar{\omega}_j^U+1)\ge \tilde{\mu}_{ij}^L, \\
& 0.5(\bar{\omega}_i^L-\bar{\omega}_j^U+1)\le 1-\tilde{v}^U, \\
& 0.5(\bar{\omega}_i^U-\bar{\omega}_j^L+1)\ge \tilde{\mu}^U, \\
& 0.5(\bar{\omega}_i^U-\bar{\omega}_j^L+1)\le 1-\tilde{v}_{ij}^L, \\
& i=1,2,\cdots,n-1;\ j=i+1,\cdots,n; \\
& 0\le \bar{\omega}_i^L\le \bar{\omega}_i^U\le 1, i=1,2,\cdots n; \\
& \sum_{i=1,i\ne j}^{n}\bar{\omega}_i^L+\bar{\omega}_j^U\le 1, \bar{\omega}_j^L+\sum_{i=1,i\ne j}^{n}\bar{\omega}_i^U\ge 1, \\
& j=1,2,\cdots n.
\end{aligned}
$$

Solving models (M-1)~(M-4), we can obtain the bounds of the interval-valued weights as follows:

$$
\begin{aligned}
\theta_1=\{&\bar{\omega}=(\bar{\omega}_1,\bar{\omega}_2,\cdots,\bar{\omega}_n)^T \mid \bar{\omega}_i=[\bar{\omega}_i^L,\bar{\omega}_i^U], \bar{\omega}_i^{L-}\le \\
&\bar{\omega}_i^L\le \bar{\omega}_i^{L+}, \bar{\omega}_i^{U-}\le \bar{\omega}_i^U\le \bar{\omega}_i^{U+}, i=1,2,\cdots,n\}
\end{aligned}
\tag{20}
$$

(2) If $\tilde{R}=(\tilde{r}_{ij})_{n\times n}$ is not a consistent interval-valued intuitionistic fuzzy preference relation, then (19) does not always hold. In this case, we relax (19) by introducing the deviation variables $d_{ij}^-, d_{ij}^+, e_{ij}^-$ and e_{ij}^+, for all $i=1,2,\cdots,n-1; j=i+1,\cdots,n$:

$$
\begin{cases}
\tilde{\mu}_{ij}^L-d_{ij}^-\le 0.5(\bar{\omega}_i^L-\bar{\omega}_j^U+1)\le 1-\tilde{v}_{ij}^U+d_{ij}^+ \\
\tilde{\mu}_{ij}^U-e_{ij}^-\le 0.5(\bar{\omega}_i^U-\bar{\omega}_j^L+1)\le 1-\tilde{v}_{ij}^L+e_{ij}^+
\end{cases}
\tag{21}
$$

where $d_{ij}^-, d_{ij}^+, e_{ij}^-$ and e_{ij}^+ are nonnegative real numbers. Obviously, the smaller the deviation variables $d_{ij}^-, d_{ij}^+, e_{ij}^-$ and e_{ij}^+, the closer $\tilde{R}$ is to a consistent interval-valued intuitionistic preference relation. Hence, we establish the following optimization model:

$$(M-5)\ \ J = min\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(d_{ij}^{-}+d_{ij}^{+}+e_{ij}^{-}+e_{ij}^{+})$$
$$\begin{aligned} s.t\ \ & 0.5(\varpi_i^L-\varpi_j^U+1)+d_{ij}^{-}\ge \tilde{\mu}_{ij}^{L}, \\ & 0.5(\varpi_i^L-\varpi_j^U+1)-d_{ij}^{+}\le 1-\tilde{\nu}^{U}, \\ & 0.5(\varpi_i^U-\varpi_j^L+1)+e_{ij}^{-}\ge \tilde{\mu}^{U}, \\ & 0.5(\varpi_i^U-\varpi_j^L+1)-e_{ij}^{+}\le 1-\tilde{\nu}_{ij}^{L}, \\ & i=1,2,\cdots,n-1;\ j=i+1,\cdots,n; \\ & 0\le\varpi_i^L\le\varpi_i^U\le 1, i=1,2,\cdots n; \\ & \sum_{i=1,i\ne j}^{n}\varpi_i^L+\varpi_j^U\le 1, \varpi_j^L+\sum_{i=1,i\ne j}^{n}\varpi_i^U\ge 1, \\ & j=1,2,\cdots n; \\ & d_{ij}^{-}\ge 0, d_{ij}^{+}\ge 0, e_{ij}^{-}\ge 0, e_{ij}^{+}\ge 0, \\ & i=1,2,\cdots,n-1;\ j=i+1,\cdots,n. \end{aligned}$$

Solving the model (M-5), we can get the optimal deviation values $\dot{d}_{ij}^{-}$, $\dot{d}_{ij}^{+}$, $\dot{e}_{ij}^{-}$ and $\dot{e}_{ij}^{+}$, for $i=1,2,\cdots,n-1;$. $j=i+1,\cdots,n$.

Based on the optimal deviation values $\dot{d}_{ij}^{-},\dot{d}_{ij}^{+},\dot{e}_{ij}^{-}$ and $\dot{e}_{ij}^{+}$, we further establish the following optimization models:

$$(M-6)\ \ \varpi_i^{L-}=\min\varpi_i^L$$
$$\begin{aligned} s.t\ \ & 0.5(\varpi_i^L-\varpi_j^U+1)+\dot{d}_{ij}^{-}\ge \tilde{\mu}_{ij}^{L}, \\ & 0.5(\varpi_i^L-\varpi_j^U+1)-\dot{d}_{ij}^{+}\le 1-\tilde{\nu}^{U}, \\ & 0.5(\varpi_i^U-\varpi_j^L+1)+\dot{e}_{ij}^{-}\ge \tilde{\mu}^{U}, \\ & 0.5(\varpi_i^U-\varpi_j^L+1)-\dot{e}_{ij}^{+}\le 1-\tilde{\nu}_{ij}^{L}, \\ & i=1,2,\cdots,n-1;\ j=i+1,\cdots,n; \\ & 0\le\varpi_i^L\le\varpi_i^U\le 1,\ i=1,2,\cdots n; \\ & \sum_{i=1,i\ne j}^{n}\varpi_i^L+\varpi_j^U\le 1, \varpi_j^L+\sum_{i=1,i\ne j}^{n}\varpi_i^U\ge 1, \\ & j=1,2,\cdots n; \end{aligned}$$

$$(M-7)\ \ \varpi_i^{L+}=\max\varpi_i^L$$
$$\begin{aligned} s.t\ \ & 0.5(\varpi_i^L-\varpi_j^U+1)+\dot{d}_{ij}^{-}\ge \tilde{\mu}_{ij}^{L}, \\ & 0.5(\varpi_i^L-\varpi_j^U+1)-\dot{d}_{ij}^{+}\le 1-\tilde{\nu}^{U}, \\ & 0.5(\varpi_i^U-\varpi_j^L+1)+\dot{e}_{ij}^{-}\ge \tilde{\mu}^{U}, \\ & 0.5(\varpi_i^U-\varpi_j^L+1)-\dot{e}_{ij}^{+}\le 1-\tilde{\nu}_{ij}^{L}, \\ & i=1,2,\cdots,n-1;\ j=i+1,\cdots,n; \\ & 0\le\varpi_i^L\le\varpi_i^U\le 1,\ i=1,2,\cdots n; \\ & \sum_{i=1,i\ne j}^{n}\varpi_i^L+\varpi_j^U\le 1, \varpi_j^L+\sum_{i=1,i\ne j}^{n}\varpi_i^U\ge 1, \\ & j=1,2,\cdots n. \end{aligned}$$

$$(M-8)\ \ \varpi_i^{U-}=\min\varpi_i^U$$
$$\begin{aligned} s.t\ \ & 0.5(\varpi_i^L-\varpi_j^U+1)+\dot{d}_{ij}^{-}\ge \tilde{\mu}_{ij}^{L}, \\ & 0.5(\varpi_i^L-\varpi_j^U+1)-\dot{d}_{ij}^{+}\le 1-\tilde{\nu}^{U}, \\ & 0.5(\varpi_i^U-\varpi_j^L+1)+\dot{e}_{ij}^{-}\ge \tilde{\mu}^{U}, \\ & 0.5(\varpi_i^U-\varpi_j^L+1)-\dot{e}_{ij}^{+}\le 1-\tilde{\nu}_{ij}^{L}, \\ & i=1,2,\cdots,n-1;\ j=i+1,\cdots,n; \\ & 0\le\varpi_i^L\le\varpi_i^U\le 1, i=1,2,\cdots n; \\ & \sum_{i=1,i\ne j}^{n}\varpi_i^L+\varpi_j^U\le 1, \varpi_j^L+\sum_{i=1,i\ne j}^{n}\varpi_i^U\ge 1, \\ & j=1,2,\cdots n. \end{aligned}$$

and

$$(M-9)\ \ \varpi_i^{U+}=\max\varpi_i^U$$
$$\begin{aligned} s.t\ \ & 0.5(\varpi_i^L-\varpi_j^U+1)+\dot{d}_{ij}^{-}\ge \tilde{\mu}_{ij}^{L}, \\ & 0.5(\varpi_i^L-\varpi_j^U+1)-\dot{d}_{ij}^{+}\le 1-\tilde{\nu}^{U}, \\ & 0.5(\varpi_i^U-\varpi_j^L+1)+\dot{e}_{ij}^{-}\ge \tilde{\mu}^{U}, \\ & 0.5(\varpi_i^U-\varpi_j^L+1)-\dot{e}_{ij}^{+}\le 1-\tilde{\nu}_{ij}^{L}, \\ & i=1,2,\cdots,n-1;\ j=i+1,\cdots,n; \\ & 0\le\varpi_i^L\le\varpi_i^U\le 1,\ i=1,2,\cdots n; \\ & \sum_{i=1,i\ne j}^{n}\varpi_i^L+\varpi_j^U\le 1, \varpi_j^L+\sum_{i=1,i\ne j}^{n}\varpi_i^U\ge 1, \\ & j=1,2,\cdots n. \end{aligned}$$

Solving the models (M-6)~(M-9), we can similarly obtain the bounds of interval-valued weights for an inconsistent interval-valued intuitionistic preference relation as follows:

$$\theta_2=\{\varpi=(\varpi_1,\varpi_2,\cdots,\varpi_n)^T \mid \varpi_i=[\varpi_i^L,\varpi_i^U],\varpi_i^{L-}\le \varpi_i^L\le\varpi_i^{L+},\varpi_i^{U-}\le\varpi_i^U\le\varpi_i^{U+}, i=1,2,\cdots,n\} \quad (22)$$

From the model (M-5), we can get the following result:

Theorem 3. $\tilde{R}$ is a consistent interval-valued intuitionistic preference relation if and only if $J=0$.

These analyses reveal that the weights derived from an interval-valued intuitionistic preference relation are in the form of bounded interval numbers, which can be expressed as the following form of interval-valued interval numbers:

$$\mathbf{\theta}=\{\bar{\bar{\boldsymbol{\omega}}}=(\bar{\bar{\varpi}}_1,\bar{\bar{\varpi}}_2,\cdots,\bar{\bar{\varpi}}_n)^T \mid \bar{\bar{\varpi}}_i=[[\varpi_i^{L-},\varpi_i^{L+}],[\varpi_i^{U-},\varpi_i^{U+}]], i=1,2,\cdots n\} \quad (23)$$

After priority weights are derived as interval-valued interval numbers, we may use a simple defuzzification approach such as the midpoint method [17] to rank the interval $\bar{\bar{\varpi}}_i (i=1,2,...,n)$ .First, we compute the midpoint $m(\bar{\bar{\varpi}}_i)$ of interval-valued interval numbers $\bar{\bar{\varpi}}_i$, $i=1,2,\cdots,n$:

$$m(\bar{\bar{\omega}}_i) = 0.5\left([\bar{\omega}_i^{L-}, \bar{\omega}_i^{L+}] + [\bar{\omega}_i^{U-}, \bar{\omega}_i^{U+}]\right) = \left[0.5(\bar{\omega}_i^{L-} + \bar{\omega}_i^{U-}), 0.5(\bar{\omega}_i^{L+} + \bar{\omega}_i^{U+})\right] \quad (24)$$

for which are still interval numbers, and then further compute by the midpoints of $m(\bar{\bar{\omega}}_i)$, $(i = 1, 2, \cdots n)$.

$$m\left[m(\bar{\bar{\omega}}_i)\right] = 0.25(\bar{\omega}_i^{L-} + \bar{\omega}_i^{L+} + \bar{\omega}_i^{U-} + \bar{\omega}_i^{U+}) \quad (0.25)$$

Therefore, based on the values $m\left[m(\bar{\bar{\omega}}_i)\right]$ $(i = 1, 2, \cdots, n)$, we can compare the interval-valued interval numbers $\bar{\bar{\omega}}_i$ $(i = 1, 2, \cdots n)$ simply.

V. ILLUSTRATIVE EXAMPLES

Example 1. Assume that a multiple criteria decision making problem consists of four criteria x_i $(i = 1, 2, 3, 4)$. A DM provides its pairwise comparison of two criteria weights as interval-valued intuitionistic fuzzy preference value $\tilde{r}_{ij}$ = $(\tilde{\mu}_{ij}, \tilde{v}_{ij}) = ([\tilde{\mu}_{ij}^L, \tilde{\mu}_{ij}^U], [\tilde{v}_{ij}^L, \tilde{v}_{ij}^U]), (i, j = 1, 2, 3, 4)$, where $\tilde{\mu}_{ij} = [\tilde{\mu}_{ij}^L, \tilde{\mu}_{ij}^U]$ provides an interval degreeto which x_i is preferred to x_j and $\tilde{v}_{ij} = [\tilde{v}_{ij}^L, \tilde{v}_{ij}^U]$ gives an interval degree to which x_i is not preferred to x_j. Assume further that the DM's pairwise compassions is given in the following interval-valued intuitionistic preference relation $\tilde{R}$.

We first transform the interval-valued intuitionistic preference relation $\tilde{R}$ into its equivalent the interval-valued interval fuzzy preference relation $\bar{\bar{R}} = (\bar{\bar{r}}_{ij})_{n\times n}$.

Solving model (*M*-5), we have $J = 0$, indicating that there exists an optimal solution such that all deviation variables, $\dot{d}_{ij}^-, \dot{d}_{ij}^+, \dot{e}_{ij}^-$ and $\dot{e}_{ij}^+ (i = 1, 2, 3; j = i + 1, \cdots, 4)$ are equal to zero. Therefore, we can tell that $\tilde{R}$ is a consistent interval-valued intuitionistic preference relation, then by the models (*M*-1) ~ (M-4), we get the corresponding interval- valued interval weights $\bar{\bar{\omega}}_i$ $(i = 1, 2, 3, 4)$.

Based on Definition 8, we compute the midpoint values $m\left(m(\bar{\bar{\omega}}_i)\right)$ of $\bar{\bar{\omega}}_i$, $(i = 1, 2, 3, 4)$, then we have $m\left(m(\bar{\bar{\omega}}_2)\right) > m\left(m(\bar{\bar{\omega}}_4)\right) > m\left(m(\bar{\bar{\omega}}_1)\right) > m\left(m(\bar{\bar{\omega}}_3)\right)$, it is clear that the ranking of $\bar{\bar{\omega}}_i$ $(i = 1, 2, 3, 4)$ is as follows:

$$\bar{\bar{\omega}}_2 \succ \bar{\bar{\omega}}_4 \succ \bar{\bar{\omega}}_1 \succ \bar{\bar{\omega}}_3.$$

Example 2. If we replace the elements $\tilde{r}_{12} = ([0.25, 0.35], [0.55, 0.65])$ and $\tilde{r}_{21} = ([0.55, 0.65], [0.25, 0.35])$ of $\tilde{R}$ in Example 1 with a pair of new elements $\tilde{r}'_{12} = ([0.05, 0.15], [0.75, 0.85])$ and $\tilde{r}'_{21} = ([0.75, 0.85], [0.05, 0.15])$, respectively, i.e., the interval-valued intuitionistic preference relation $\tilde{R}'$, then the corresponding interval-valued interval fuzzy preference relation $\bar{\bar{R}}'$.

Solving the model (M-5), we have $J = 0.2$. Therefore, there does not exist any optimal solution such that Eq. (19) is satisfied without any positive deviation. By Definition 12, $\tilde{R}'$ is not a consistent interval-valued intuitionistic preference relation, and the corresponding optimal deviation values are computed.

Plugging these optimal deviation values into models (M-6)~(M-9), based on Definition 8, we compute the midpoint values $m\left(m(\bar{\bar{\omega}}_i)\right)$ of the corresponding interval-valued interval weights $\bar{\bar{\omega}}_i$, $(i = 1, 2, 3, 4)$, and

$$m\left(m(\bar{\bar{\omega}}_2)\right) > m\left(m(\bar{\bar{\omega}}_4)\right) > m\left(m(\bar{\bar{\omega}}_1)\right) = m\left(m(\bar{\bar{\omega}}_3)\right),$$

it is clear that the ranking of $\bar{\bar{\omega}}_i$ $(i = 1, 2, 3, 4)$ is as follows:

$$\bar{\bar{\omega}}_2 \succ \bar{\bar{\omega}}_4 \succ \bar{\bar{\omega}}_1 \sim \bar{\bar{\omega}}_3.$$

Obviously, the ranking of $\bar{\bar{\omega}}_1$ and $\bar{\bar{\omega}}_3$ is changed due to that the element $\tilde{r}_{12}$ becomes smaller from $([0.25, 0.35], [0.55, 0.65])$ to $([0.05, 0.15], [0.75, 0.85])$.

VI. CONCLUSION

We have introduced the notion of consistent interval-valued intuitionistic preference relation and established some simple linear programming models to develop a method for estimating criteria weights from interval-valued intuitionistic preference relations. The method can be applicable to multi-criteria decision making problems in many fields, such as the high technology project investment of venture capital firms, supply chain management, medical diagnosis, etc. In the future, we shall study the approach to improving the consistency of inconsistent interval-valued intuitionistic preference relations.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China (NSFC) (71171048, 71371049), the Ph.D. Program Foundation of Chinese Ministry of Education (20120092110038), the Scientific Research and Innovation Project for College Graduates of Jiangsu Province (CXZZ11_0185), the Scientific Research Foundation of Graduate School of Southeast University (YBJJ1135) and the State Key Laboratory of Rail Traffic Control and Safety (RCS2011K002) of Beijing Jiaotong University.

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$$\tilde{R}=\begin{bmatrix} ([0.50,0.50],[0.50,0.50]) & ([0.25,0.35],[0.55,0.65]) & ([0.45,0.55],[0.25,0.35]) & ([0.35,0.45],[0.45,0.55]) \\ ([0.55,0.65],[0.25,0.35]) & ([0.50,0.50],[0.50,0.50]) & ([0.55,0.65],[0.15,0.25]) & ([0.15,0.25],[0.35,0.45]) \\ ([0.25,0.35],[0.45,0.55]) & ([0.15,0.25],[0.55,0.65]) & ([0.50,0.50],[0.50,0.50]) & ([0.35,0.45],[0.15,0.25]) \\ ([0.45,0.55],[0.35,0.45]) & ([0.35,0.45],[0.15,0.25]) & ([0.15,0.25],[0.35,0.45]) & ([0.50,0.50],[0.50,0.50]) \end{bmatrix}$$

$$\bar{\bar{R}}=\begin{bmatrix} [[0.50,0.50],[0.50,0.50]] & [[0.25,0.35],[0.35,0.45]] & [[0.45,0.55],[0.65,0.75]] & [[0.35,0.45],[0.45,0.55]] \\ [[0.55,0.65],[0.65,0.75]] & [[0.50,0.50],[0.50,0.50]] & [[0.55,0.65],[0.75,0.85]] & [[0.15,0.25],[0.55,0.65]] \\ [[0.25,0.35],[0.45,0.55]] & [[0.15,0.25],[0.35,0.45]] & [[0.50,0.50],[0.50,0.50]] & [[0.35,0.45],[0.75,0.85]] \\ [[0.45,0.55],[0.55,0.65]] & [[0.35,0.45],[0.75,0.85]] & [[0.15,0.25],[0.55,0.65]] & [[0.50,0.50],[0.50,0.50]] \end{bmatrix}$$

$$\tilde{R}'=\begin{bmatrix} ([0.50,0.50],[0.50,0.50]) & ([0.05,0.15],[0.75,0.85]) & ([0.45,0.55],[0.25,0.35]) & ([0.35,0.45],[0.45,0.55]) \\ ([0.75,0.85],[0.05,0.15]) & ([0.50,0.50],[0.50,0.50]) & ([0.55,0.65],[0.15,0.25]) & ([0.15,0.25],[0.35,0.45]) \\ ([0.25,0.35],[0.45,0.55]) & ([0.15,0.25],[0.55,0.65]) & ([0.50,0.50],[0.50,0.50]) & ([0.35,0.45],[0.15,0.25]) \\ ([0.45,0.55],[0.35,0.45]) & ([0.35,0.45],[0.15,0.25]) & ([0.15,0.25],[0.35,0.45]) & ([0.50,0.50],[0.50,0.50]) \end{bmatrix}$$

$$\bar{\bar{R}}'=\begin{bmatrix} [[0.50,0.50],[0.50,0.50]] & [[0.05,0.15],[0.15,0.25]] & [[0.45,0.55],[0.65,0.75]] & [[0.35,0.45],[0.45,0.55]] \\ [[0.75,0.85],[0.85,0.95]] & [[0.50,0.50],[0.50,0.50]] & [[0.55,0.65],[0.75,0.85]] & [[0.15,0.25],[0.55,0.65]] \\ [[0.25,0.35],[0.45,0.55]] & [[0.15,0.25],[0.35,0.45]] & [[0.50,0.50],[0.50,0.50]] & [[0.35,0.45],[0.75,0.85]] \\ [[0.45,0.55],[0.55,0.65]] & [[0.35,0.45],[0.75,0.85]] & [[0.15,0.25],[0.55,0.65]] & [[0.50,0.50],[0.50,0.50]] \end{bmatrix}$$