2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)
July 6-11, 2014, Beijing, China

Ranking Fuzzy Numbers by Their Expansion Center

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***Abstract*-Based on the area between the curve of the membership function and the horizontal real axes, a new index, called the expansion center for fuzzy numbers is proposed. An intuitive and reasonable ranking method for fuzzy numbers based on their expansion center is also established. This new ranking method is useful in fuzzy decision making and fuzzy data mining.**

Keywords-fuzzy numbers; ranking fuzzy numbers; total ordering; fuzzy decision making; soft computation

I. INTRODUCTION

Fuzzy number [5, 9, 17] is one of the most important mathematical concepts concerning fuzziness. Ranking fuzzy numbers is a necessary link of analyzing fuzzy information in optimization, data mining, decision making, and related areas. Since the concept of fuzzy sets and fuzzy numbers were introduced in the sixties of the last century, many significant contributions have been made in ranking fuzzy numbers [1-4, 6-8, 10-13, 15, 16, 18]. They have respect intuition based on some geometric characteristics (e.g., area, distance, or centroid), and can be used for various purposes. Mostly, these methods can only rank fuzzy numbers but allow different fuzzy numbers to have the same rank, or order some special types of fuzzy numbers, such as the triangular (or, more generally, trapezoidal) fuzzy numbers.

Generally, based on one or more characteristics of fuzzy numbers, an equivalence relation and an opposite but transitive relation on the set of all fuzzy numbers can be defined. Using these two relations, it is easy to define a total ordering on its quotient space with respect to the equivalence relation, but not on the set of all fuzzy numbers themselves.

In this paper, a new concept of expansion center is proposed. Based on this concept, an alternative ranking method for fuzzy numbers is established. It is intuitive and reasonable. For some common types of fuzzy numbers, the calculation for ranking is not complex and, therefore, it is convenient to be used in fuzzy data mining and fuzzy decision making.

After the introduction, this paper is arranged as follows. In Section 2, the concept of fuzzy number and some common types of fuzzy numbers are recalled. The new concept of expansion center of fuzzy numbers is presented in Section 3. Its existence and uniqueness is proved in this section. The formula for calculating the expansion center is also presented for those common types of fuzzy numbers. For other fuzzy numbers, a soft computing method may be adopted in the calculation. Then Section 4 shows the relevant ranking method. By using the expansion center, an equivalence relation on the set of all fuzzy numbers is defined in Section 5 and a total ordering on the quotient space is also shown there. Some conclusions are summarized in Section 6.

II. FUZZY NUMBERS

Let $\mathbb{R} = (-\infty, \infty)$. A fuzzy subset of $\mathbb{R}$, denoted by $\tilde{e}$, is called a fuzzy number if its membership function $m_e: \mathbb{R} \to [0, 1]$ satisfies the following conditions.

(FN1) Set $\{x | m_e(x) \geq \alpha\}$, the α-cut of $\tilde{e}$ (denoted by e_α), is a closed interval, denoted as $[e_\alpha^l, e_\alpha^r]$, for every $\alpha \in (0, 1]$.

(FN2) Set $\{x | m_e(x) > 0\}$, the support set of $\tilde{e}$ (denoted by supp(e)), is bounded.

Condition (FN1) implies that $\tilde{e}$ is a convex fuzzy subset of $\mathbb{R}$ and is equivalent to the following three conditions.

(FN1.1) There exists at least one real number a_0 such that $m_e(a_0) = 1$.

(FN1.2) Function m_e is nondecreasing on $(-\infty, a_0]$ and nonincreasing on $[a_0, \infty)$.

(FN1.3) Function m_e is upper semi-continuous, or say, m_e is right-continuous (i.e., $\lim_{x \to x_0^+} m_e(x) = m_e(x_0)$ when $x_0 < a_0$) and is left-continuous (i.e., $\lim_{x \to x_0^-} m_e(x) = m_e(x_0)$ when $x_0 > a_0$).

The set of all fuzzy numbers is denoted by $\mathcal{N}_F$. Now, let us recall some special but common types of fuzzy numbers in $\mathcal{N}_F$.

Each rectangular fuzzy number can be regarded as a closed interval. So, we can still use $\mathcal{N}_I$, which is a proper subset of $\mathcal{N}_F$, to denote the set of all rectangular fuzzy numbers. The membership function of a rectangular number $\tilde{e}$ has a form as

$$m_e(x) = \begin{cases} 1 & \text{if } x \in [a_l, a_r] \\ 0 & \text{otherwise} \end{cases}, \quad (1)$$

and can be simply denoted by vector $[a_l \; a_r]$, where real-valued parameters a_l and a_r satisfy $a_l \le a_r$. Its core is closed interval $[a_l, a_r]$.

Another common special type of fuzzy numbers is the triangular numbers whose membership function has a form

$$m_e(x) = \begin{cases} \dfrac{x - a_l}{a_0 - a_l} & \text{if } x \in [a_l, a_0) \\ 1 & \text{if } x = a_0 \\ \dfrac{x - a_r}{a_0 - a_r} & \text{if } x \in (a_0, a_r] \\ 0 & \text{otherwise} \end{cases}, \quad (2)$$

where real-valued parameters a_l, a_0, and a_r satisfy $a_l \le a_0 \le a_r$. The set of all triangular fuzzy numbers is denoted by, which is also a proper subset of $\mathcal{N}_F$. Such a fuzzy number can be simply denoted by a vector $[a_l \; a_0 \; a_r]$.

As a generalization of both rectangular fuzzy numbers and triangular fuzzy numbers, trapezoidal fuzzy numbers are also an important common type of fuzzy numbers, whose membership function has a form

$$m_e(x) = \begin{cases} \dfrac{x - a_l}{a_b - a_l} & \text{if } x \in [a_l, a_b) \\ 1 & \text{if } x \in [a_b, a_d] \\ \dfrac{x - a_r}{a_d - a_r} & \text{if } x \in (a_d, a_r] \\ 0 & \text{otherwise} \end{cases}, \quad (3)$$

Where a_l, a_b, a_d and a_r are real-valued parameters with $a_l \le a_b \le a_d \le a_r$. Such a trapezoidal fuzzy number can be simply denoted by a vector $[a_l \; a_b \; a_d \; a_r]$. Its core is a closed interval $[a_b, a_d]$.

Definition 1. Fuzzy number $\tilde{e}$ is said to be symmetric iff there exists real number t_0 such that $m_e(t_0 - t) = m_e(t_0 + t)$ for any $t \in \mathbb{R}$. Real number t_0 is called the symmetry center of fuzzy number $\tilde{e}$.

Any rectangular fuzzy number with membership function (1) is symmetric. Any triangular fuzzy number with membership function (2) is symmetric iff $a_0 - a_l = a_r - a_0$. Any trapezoidal fuzzy number with membership function (3) is symmetric iff $a_b - a_l = a_r - a_d$.

III. THE EXPANSION CENTER OF FUZZY NUMBERS

Let $\tilde{e}$ be a fuzzy number and m_e be its membership function.

Lemma 1. For any real number *x*, definite integrals $\int_{-\infty}^{x} m_e(t)dt$ and $\int_{x}^{\infty} m_e(t)dt$, as well as $\int_{-\infty}^{\infty} m_e(t)dt$ exist and are finite.

Proof. This is a direct result of the fact that membership function m_e is piecewise monotone. The finiteness comes from the fact that the support set of any fuzzy number is bounded.

Lemma 2. $\int_{-\infty}^{\infty} m_e(t)dt = 0$ if and only if $\tilde{e}$ is a crisp real number.

Proof. This is just an exercise problem in Chapter 3 of [17]. We omit the proof here.

Lemma 3. If $\int_{-\infty}^{b} m_e(t)dt > 0$ and $\int_{d}^{\infty} m_e(t)dt > 0$ for some real numbers *b* and *d* with $b < d$, then $\int_{b}^{d} m_e(t)dt > 0$.

Proof. From $\int_{-\infty}^{b} m_e(t)dt > 0$, we may find point $t_1 < b$ such that $m_e(t_1) > 0$. Similarly, we may find point $t_2 > d$ such that $m_e(t_2) > 0$. Denote min $[m_e(t_1), m_e(t_2)]$ by s. Then $s > 0$. Since the convexity of $\tilde{e}$, we have $m_e(t) > s$ for any t between t_1 and t_2. Thus, $\int_{b}^{d} m_e(t)dt > s(d - b) > 0$. ■

Definition 2. Let $\tilde{e}$ be a fuzzy number but not a crisp real number. A real number c_e is called the expansion center of $\tilde{e}$ iff $\int_{-\infty}^{c_e} m_e(t)dt = \int_{c_e}^{\infty} m_e(t)dt$. For any real number, its expansion center is just itself.

The role of expansion center for a given fuzzy number in fuzzy mathematics is similar to the one of medium (50 percentile) in statistics. The former is one of the numerical indexes of fuzziness possessed by a fuzzy number, while the latter is used for describing the randomness of a random variable. For any crisp real number, its expansion center is defined as itself. For any symmetric fuzzy number, its expansion center coincides with the symmetry center.

Theorem 1. For any given fuzzy number, its expansion center exists and is unique.

Proof. We only need to prove the conclusion for fuzzy numbers that are not crisp real numbers. Denote a given non crisp fuzzy number by c_e. From Lemma 1, we know that definite integrals $\int_{-\infty}^{x} m_e(t)dt$ and $\int_{x}^{\infty} m_e(t)dt$ exist and are finite for any real number x and can be regarded as functions of x. Let $\int_{-\infty}^{\infty} m_e(t)dt = r$ and $f(x) = \int_{-\infty}^{x} m_e(t)dt - \int_{x}^{\infty} m_e(t)dt$, where $x \in \mathbb{R}$. Function f is continuous. Since c_e is not a crisp real number and supp(e) is bounded, we have $0 < r < \infty$ and $\lim_{x\to-\infty} f(x) = -r$ as well as $\lim_{x\to\infty} f(x) = r$. By the well-known Intermediate Value Theorem of Continuous Function, we know that there exists a real number c_e such that $f(c_e) = 0$ and, therefore, $\int_{-\infty}^{c_e} m_e(t)dt = \int_{c_e}^{\infty} m_e(t)dt$, that is, c_e is an expansion center of $\tilde{e}$. Furthermore, we have $\int_{-\infty}^{c_e} m_e(t)dt = \int_{c_e}^{\infty} m_e(t)dt = r/2$. Now, let's prove the uniqueness of the expansion center. Assume that, for given fuzzy number c_e, there exists another expansion center c_e'. Without any loss of generality, we may assume that $c_e' < c_e$. From both $\int_{-\infty}^{c_e} m_e(t)dt = \int_{c_e}^{\infty} m_e(t)dt = r/2$ and $\int_{-\infty}^{c_e'} m_e(t)dt = \int_{c_e'}^{\infty} m_e(t)dt = r/2$, we obtain $\int_{c_e'}^{c_e} m_e(t)dt = \int_{c_e}^{c_e'} m_e(t)dt$. So, $\int_{c_e'}^{c_e} m_e(t)dt = 0$. By using Lemma 3, we get $c_e' = c_e$. The proof is now complete. ■

Example 1. Let $\tilde{e}$ be a trapezoidal fuzzy number with membership function (3) shown in Section 2. Its expansion center c_e can be expressed as

$$c_e = \begin{cases} a_l + \sqrt{\frac{1}{2}(a_b - a_l)(a_d + a_r - a_l - a_b)} & \text{if } a_l + a_d + a_r \le 3a_b \\ a_r - \sqrt{\frac{1}{2}(a_r - a_d)(a_d + a_r - a_l - a_b)} & \text{if } a_l + a_b + a_r \ge 3a_d \\ \frac{(a_l + a_b + a_d + a_r)}{4} & \text{otherwise} \end{cases} . \quad (4)$$

As a special case, when $\tilde{e}$ be a triangular fuzzy number with membership function (2), its expansion center is

$$c_e = \begin{cases} a_l + \sqrt{\frac{1}{2}(a_0 - a_l)(a_r - a_l)} & \text{if } a_l + a_r \le 2a_0 \\ a_r - \sqrt{\frac{1}{2}(a_r - a_0)(a_r - a_l)} & \text{otherwise} \end{cases} . \quad (5)$$

As another special case, when $\tilde{e}$ be a rectangular fuzzy number with membership function (1), i.e., when $a_b - a_l = a_r - a_d = 0$, expression (4) is reduced to $c_e = \frac{1}{2}(a_l + a_r)$. This coincides with the intuition.

For a given fuzzy number, in general cases, a numerical method may be used to calculate the values of the involved definite integrals and a soft computing technique, such as the genetic algorithm, can be adopted to optimize the location of its expansion center.

IV. RANKING FUZZY NUMBER BY THEIR EXPANSION CENTER

By using the expansion center of fuzzy numbers, we may propose a ranking method as follows.

Definition 3. Let $\tilde{e}$ and $\tilde{f}$ be two fuzzy numbers. We say that fuzzy number $\tilde{e}$ precedes fuzzy number $\tilde{f}$, denoted by $\tilde{e} \preccurlyeq \tilde{f}$, iff their expansion centers satisfy $c_e \le c_f$.

Relation $\preccurlyeq$ is well defined on the set of all fuzzy numbers, $\mathcal{N}_F$. It does not satisfy the antisymmetry, i.e., $\tilde{e}$ and $\tilde{f}$ may be different even both $\tilde{e} \preccurlyeq \tilde{f}$ and $\tilde{f} \preccurlyeq \tilde{e}$ hold. So, it is not a partial ordering on $\mathcal{N}_F$. However, this relation can be used to rank fuzzy numbers.

Example 2. Let fuzzy numbers $\tilde{e}$ and $\tilde{f}$ have membership functions

$$m_e(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ \frac{5-x}{4} & \text{if } x \in (1,5] \\ 0 & \text{otherwise} \end{cases}$$

and

$$m_f(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 2-x & \text{if } x \in (1,1.5] \\ 0.5 & \text{if } x \in (1.5,3.5] \\ 0 & \text{otherwise} \end{cases}$$

shown in Figures 1 and 2 respectively. $\tilde{e}$ is a trapezoidal fuzzy number but $\tilde{f}$ is not. Their expansion centers are $c_e \approx 1.5359$ and $c_f = 1.5625$ respectively. So, $\tilde{e} \preccurlyeq \tilde{f}$.

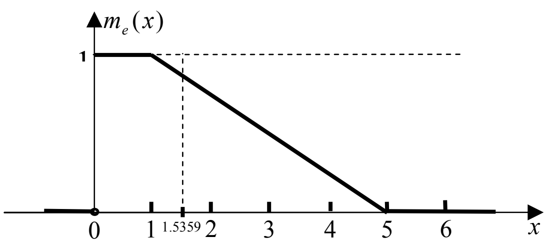

Figure 1. The membership function of fuzzy number $\tilde{e}$ and its expansion center $c_e \approx 1.536$ in Example 2

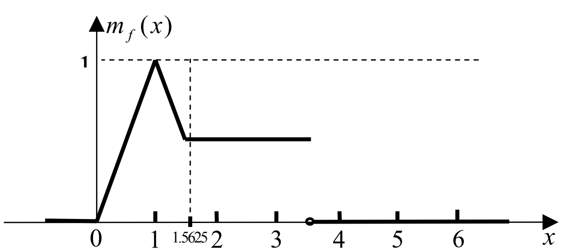

Figure 2. The membership function of fuzzy number $\tilde{f}$ and its expansion center $c_f = 1.5625$ in Example 2

V. GEOMETRIC INTUITIVITY OF RANKINGS

The geometric intuitivity can be defined by the geometric location of the non-zero parts of membership functions corresponding to the given fuzzy numbers.

Definition 4. For two fuzzy numbers $\tilde{e}$ and $\tilde{f}$, if the α-cuts $e_\alpha \le f_\alpha$, i.e., $l_\alpha(e) \le l_\alpha(f)$ and $r_\alpha(e) \le r_\alpha(f)$ for every $\alpha \in (0,1]$, then $\tilde{e} \preccurlyeq \tilde{f}$.

Intuitively, if the curve of the membership function of $\tilde{e}$ is totally on the left of the curve of the membership function of $\tilde{f}$, then the ranking method should issue $\tilde{e} \preccurlyeq \tilde{f}$.

In our case, the proposed ranking method obviously follows the above-mentioned definition. So, our ranking method adheres to the geometric intuitivity of rankings.

VI. EQUIVALENCE CLASSES AND A TOTAL ORDERING ON THE QUOTIENT SPACE

We may define a relation $\sim$ on the set of all fuzzy number, $\mathcal{N}_F$, as follows.

Definition 5. We say that two fuzzy numbers $\tilde{e}$ and $\tilde{f}$ are equivalent, denoted as $\tilde{e} \sim \tilde{f}$, iff they have the same expansion center, i.e., $c_e = c_f$.

Example 3. Let fuzzy numbers $\tilde{g}$ and $\tilde{h}$ have membership functions

$$m_g(x) = \begin{cases} \frac{2x}{3} & \text{if } x \in [0,1.5) \\ 1 & \text{if } x = 1.5 \\ \frac{6-2x}{3} & \text{if } x \in (1.5,3] \\ 0 & \text{otherwise} \end{cases}$$

and

$$m_h(x) = \begin{cases} 1 & \text{if } x \in [0,2) \\ \frac{4-x}{2} & \text{if } x \in (2,4] \\ 0 & \text{otherwise} \end{cases}.$$

Figure 3 shows the membership functions of fuzzy numbers $\tilde{g}$ and $\tilde{h}$ with their expansion center. Their expansion centers c_g and c_h have the same value 1.5.

So, $\widetilde{g} \sim \widetilde{h}$, that is, these two fuzzy numbers have the same rank when the ranking method shown in Section 4 is used.

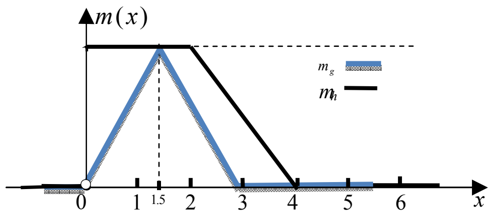

Figure 3. The membership functions of fuzzy numbers $\widetilde{g}$ and $\widetilde{h}$ with their expansion center $c_g = c_h = 1.5$ in Example 3

Relation $\sim$ is reflective, symmetric, and transitive. Hence, it is an equivalent relation [14] on $\mathcal{N}_F$. The collection of all equivalent classes with respect to $\sim$ forms the quotient space, denoted by $Q_{\sim}$. Regarding $\preccurlyeq$ as a relation on $Q_{\sim}$, $(Q_{\sim}, \preccurlyeq)$ is a total ordered set.

VII. CONLUSIONS

The introduced new concept of expansion center for fuzzy numbers is effective in ranking fuzzy numbers. This new way of ranking fuzzy numbers is intuitive and practicable. It provides an alternative choice in decision making and data mining within fuzzy environment.

ACKNOWLEDGEMENT

This work is partially supported by National Science Foundation of China #70921061 and #71110107026, and by the CAS/SAFEA International Partnership Program for Creative Research Teams.

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