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Rotation of Triangular Fuzzy Numbers via Quaternion

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***Abstract*—In this paper we introduced the concept of three-dimensional triangular fuzzy number and their properties are investigated. It is shown that this set has important metrical properties, e.g *convexity*. The paper also provides a rotation method for such numbers based on quaternion and aggregation operator.**

I. Introduction

The quaternions was discovered by William Rowan Hamilton in 1837. The History says that Hamilton and his wife Helen were walking at the Royal Irish Academy when he thought how to add and multiply four dimensional elements. Excited with his discovery, while the couple was going over the Broome Bridge of the Royal Canal, he caverd in the stone wall of the bridge the famous equations:

$$i^2 = j^2 = k^2 = ijk = -1$$

which, implicitly, contain the equations.

$$ij = k = -ji, \ jk = i = -kj, \ ki = j = -ik$$

He was also aware of the greatest problem of his time, coming from Physics: constructing a language which would be appropriate to develop the field of Dynamics in a similar way that Newton created Calculus. In order to achieve it, it was necessary to create an algebra to manipulate the vectors. He noticed that it would not be possible to construct such structure based on geometrical considerations, but on operators acting on vectors, more precisely with a four-dimensional algebra.

He considered elements of the form $\alpha = a + bi + cj + dk$, which he called *quaternions*, where the coefficients a, b, c, d are real numbers and i, j, k are formal symbols called basic units. It was obvious to him that two elements should be added componentwise by the formula:

$(a + bi + cj + dk) + (a' + b'i + c'j + d'k) = (a + a') + (b + b')i + (c + c')j + (d + d')k.$

The main difficulty was to define the product of two elements. Since this product should have the usual properties of a multiplication, such as the distributive law, it would actually be enough to decide how to multiply the symbols i, j, k among themselves. This demanded a considerable effort from the young Hamilton. He also implicitly assumed that the product should be commutative. It was perfectly possible, since he was about to find the first non-commutative algebra in the entire history of Mathematics. Afterwards, he presented an extensive memoir on quaternions to the Royal Irish Academy. His discovery came as a shock to the mathematicians of the time, because it opened the possibilities for new extensions of the field of complex numbers.

There are many applications of quaternions. In Physics, we highlight applications in quantum mechanics [1] and theory of relativity [2]. Moreover, we can find applications in aerospace projects [3] and flight simulators [4]. In computer graphics, it is relatively easy to visualize a translation and express it mathematically. However, the same does not happen to rotations. Nowadays, Mathematicians offer a wide variety of rotation techniques such as quaternion rotation which has a more compact representation than a rotation matrix. Also, the quaternion algebra allows us to compose rotations easily. Thus, game programmers discovered the *high* potential of quaternions and started using it as a powerful tool to describing rotation about an arbitrary axis [5].

According to the standard literature, a *fuzzy number* is a convex and a normalized fuzzy subset of real numbers. The Zadeh's extension principle [6] allows us to define the arithmetical operations among fuzzy numbers by extending the classical ones. Dubois and Prade [7], [8], [9] drew attention to their arithmetic properties and Buckley [10] gave the first steps towards the extension from fuzzy real numbers to complex fuzzy numbers. This paper shows that the fuzzy complex numbers is closed under arithmetic operations and they may be performed in terms of α-cuts. In 1992, Zhang [11] introduced a new definition for fuzzy complex numbers and obtained some results which are analogous to those in Mathematical Analysis. In 2011, Tamir [12] introduced fuzzy complex numbers with an axiomatic approach. Finally, in 2013, *the authors in [13] proposed to extend the real fuzzy numbers to quaternions fuzzy numbers and to investigate their properties*.

In this paper, we investigate a special subset of fuzzy numbers called triangular fuzzy numbers as well as their properties. Also, the concept of *three-dimensional triangular fuzzy numbers* (T^3) is built. At the end of the paper we give an example of the rotation algorithm in T^3 based on an

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aggregation operator; we drew attention to the fact that each aggregation operator produces different results.

This paper has the following structure: section 2 gives an overview of fuzzy numbers; section 3 and section 4 develop some properties about triangular fuzzy numbers, specially that the set of triangular fuzzy numbers is convex; section 5 introduces an overview about the rotation in $\mathbb{R}^3$ via quaternions; section 6 provides a method to rotate two vectors in T^3 based on an aggregation operator; section 7 provides the final remarks.

II. Preliminaries

We consider $\mathbb{R}$ as the set of real numbers and $\mathbb{H}$ as the set of quaternion numbers.

Definition 1: A fuzzy real set is a function $\bar{A} : \mathbb{R} \longrightarrow [0,1]$.

Definition 2: A fuzzy subset $\bar{A}$ of X is convex if $\bar{A}(\lambda x_1 + (1-\lambda x_2)) \geq \min(\bar{A}(x_1), \bar{A}(x_2))$, $x_1, x_2 \in X$ and $\lambda \in [0,1]$. Alternatively, a fuzzy subset is convex if all of its α-cut sets are convex.

Definition 3: A fuzzy subset $\bar{A}$ of X is normalized if there exists $x \in X$ such that $\bar{A}(x) = 1$.

Definition 4: **A fuzzy number is a convex and normalized fuzzy subset of** $\mathbb{R}$.

In this paper the set of all fuzzy real numbers is denoted by $\mathbb{R}_F$.

Definition 5: Given $\bar{A}, \bar{B} \in \mathbb{R}_F$ we define:

$\bar{A} * \bar{B}(z) = sup\{\bar{A}(x) \wedge \bar{B}(y) \ : \ z = x * y\}$, where $*$ represents the usual arithmetical operations: $+, -, \times, \div$.

According to Dubois and Prade [7] the structures $(\mathbb{R}_F, +)$ and $(\mathbb{R}_F, \times)$ are semi-groups.

We can see that $\mathbb{R} \subset \mathbb{R}_F$, since every $a \in \mathbb{R}$ can be written as $a : \mathbb{R} \longrightarrow [0,1]$, where $a(x) = 1$ if $x = a$ and $a(x) = 0$ if $x \neq a$.

III. Triangular Fuzzy Numbers

In this section we prove some properties of triangular fuzzy numbers.

A **triangular fuzzy number** can be represented by three points as follows: $\bar{A} = (a_1, a_2, a_3)$, $a_1, a_2, a_3 \in \mathbb{R}$. This representation is interpreted as the following membership function:

$$\bar{A}(x) = \begin{cases} \frac{x-a_1}{a_2-a_1}, & x \in [a_1, a_2] \\ \frac{a_3-x}{a_3-a_2}, & x \in [a_2, a_3] \\ 0 & otherwise \end{cases}$$

We get a crisp interval by an α-cut operation as follows: $\bar{A}[\alpha] = [a_1 + (a_2 - a_1)\alpha, a_3 - (a_3 - a_2)\alpha]$ where $\alpha \in [0,1]$.

A triangular number is called **positive**, whenever $a_i > 0$, for all $i = 1,2,3$, and **negative**, whenever $a_i < 0$, for all $i = 1,2,3$.

Two triangular fuzzy numbers $\bar{A} = (a_1, a_2, a_3)$,$\bar{B} = (b_1, b_2, b_3)$ are equal if $a_i = b_i$ for all $i = 1,2,3$.

Consider $\bar{A} = (a_1, a_2, a_3), \bar{B} = (b_1, b_2, b_3)$ triangular fuzzy numbers and $\lambda \in \mathbb{R}$, then:

(i) **Addition**: $\bar{A} + \bar{B} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$.

(ii) **Subtraction**: $\bar{A} - \bar{B} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$.

(iii) **Multiplication**:
$\bar{A} \times \bar{B} = (\min(a_1b_1, a_1b_3, a_3b_1, a_3b_3), a_2b_2,$
$\max(a_1b_1, a_1b_3, a_3b_1, a_3b_3))$.

(iv) **Scalar Multiplication**: $\lambda\bar{A} = (\lambda a_1, \lambda a_2, \lambda a_3)$;

(v) **Division**: $\frac{\bar{A}}{\bar{B}} =$
$(\min(a_1/b_1, a_1/b_3, a_3/b_1, a_3/b_3), a_2/b_2,$
$\max(a_1/b_1, a_1/b_3, a_3/b_1, a_3/b_3))$.

Note that it is possible $\bar{A} - \bar{A} \neq (0,0,0)$ and $\frac{\bar{A}}{\bar{B}} \neq (1,1,1)$. The solution of the fuzzy linear equation $\bar{A} + \bar{B} = \bar{C}$ is not as we would always expect: $\bar{B} = \bar{C} - \bar{A}$.

In this paper the set of all triangular fuzzy numbers will be denoted by T.

Definition 6: Let $\bar{A} = (a_1, a_2, a_3)$ and $\bar{B} = (b_1, b_2, b_3)$ in T. We say $\bar{A} \leq \bar{B}$ iff $a_2 \leq b_2$. Note that $\leq$ is a total order.

Definition 7: Let $\bar{A} = (a_1, a_2, a_3)$ and $\bar{B} = (b_1, b_2, b_3)$ in T. We define the distance of $\bar{A}, \bar{B}$ as $d(\bar{A}, \bar{B}) = |a_1 - b_1| + |a_2 - b_2| + |a_3 - b_3|$.

It is easy to see that d is a metric in T. Thus, (T, d) is a metric space.

The next two definitions are necessary to T become a topological space.

Definition 8: The open ball of radius $r > 0$ centered at a point $\bar{A} \in T$ is the set $B(\bar{A}, r) = \{\bar{B} \ : \ d(\bar{A}, \bar{B}) < r\}$. The closed ball is $B[\bar{A}, r] = \{\bar{B} \ : \ d(\bar{A}, \bar{B}) \leq r\}$.

Definition 9: Let $A \subset T$. We say that A is an open set if for each $\bar{A} \in A$ there exists an open ball $B(\bar{A}, r) \subset A$. The collection of all open sets of T is denoted by Γ_T.

Proposition 1: **(T, Γ_T) is a topological space.**

Proof: Straightforward. ■

Proposition 2: Let $\Delta_K = \{(a_1, a_2, a_3) \in T \ : \ a_1, a_2, a_3 \in K \subset \mathbb{R}\}$. If K is a compact set of $\mathbb{R}$, then Δ_K is a compact set of T.

Proof: Straightforward. ■

Note that the boundary $B_K = \{\bar{A} \in T \ : \ B(\bar{A}, r) \cap K \neq \emptyset \ and \ B(\bar{A}, r) \cap K \neq \emptyset \ for \ all \ r > 0.\} = \{(a, 0, b) \in T \ : \ a, b \in \mathbb{R}\} \cup \{(a, 1, b) \in T \ : \ a, b \in \mathbb{R}\}$.

Again, it is not hard to see that T is a vector space over $\mathbb{R}$ and $\bar{V}_1 = (1, 1, 1), \bar{V}_2 = (0, 1, 1), \bar{V}_3 = (0, 0, 1)$ is a basis for T, e.g given $\bar{A} = (a_1, a_2, a_3) \in T$ let $\lambda_1 = a_1, \lambda_2 = a_2 - a_1, \lambda_3 = a_3 - a_2$. Thus, $\bar{A} = \lambda_1 \bar{V}_1 + \lambda_2 \bar{V}_2 + \lambda_3 \bar{V}_3$.

Proposition 3: The function $\| \cdot \|: T \longrightarrow \mathbb{R}$, where $\| \bar{A} \| = d(\bar{0}, \bar{A}) = |a_1| + |a_2| + |a_3|$ is a norm on T.

Proof: i) If $\| \bar{A} \| = 0$, then $a_1 = a_2 = a_3 = 0$. ii) $\| \lambda \bar{A} \| = |\lambda|(|a_1| + |a_2| + |a_3|) = |\lambda| \ \| \bar{A} \|$. iii) $\| \bar{A} + \bar{B} \| = |a_1 + b_1| + |a_2 + b_2| + |a_3 + b_3| \leq |a_1| + |a_2| + |a_3| + |b_1| + |b_2| + |b_3| = \| \bar{A} \| + \| \bar{B} \|$. ■

Proposition 4: Let $f_x : T \longrightarrow T$, $x \in \mathbb{R}$ such that $f_x(\bar{A}) = f((a_1 + x, a_2 + x, a_3 + x))$. Then f_x preserves the distance d.

Proof: $d(f_x(\bar{A}, f_x(\bar{B}))) = d((a_1 + x, a_2 + x, a_3 + x), (b_1 + x, b_2 + x, b_3 + x)) = |(a_1 + x) - (b_1 + x)| + |(a_2 + x) - (b_2 + x)| + |(a_3 + x) - (b_3 + x)| = d(\bar{A}, \bar{B})$. ■

Proposition 5: T **is convex.**

Proof: Given $\bar{A}, \bar{B} \in T$. Let $f : [0, 1] \longrightarrow T$, where $f(t) = (1 - t)\bar{A} + t\bar{B}$. Thus, $f(0) = \bar{A}$ and $f(1) = \bar{B}$. ■

IV. Three-dimensional triangular fuzzy numbers

In this section we deal only with the concept of tree-dimensional triangular fuzzy numbers.

Definition 10: $T^3 = T \times T \times T \subset \mathbb{R}^3_F$ is called **three-dimensional triangular fuzzy numbers**; i.e. if $v \in T^3$, then $v = (\bar{v}_1, \bar{v}_2, \bar{v}_3)$, where $\bar{v}_i \in T$ for $i = 1, 2, 3$.

Definition 11: Let $v = (\bar{v}_1, \bar{v}_2, \bar{v}_3)$,$u = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ in T^3 and $\lambda \in \mathbb{R}$. We can define:

i) **Addition**: $v + u = (\bar{v}_1 + \bar{u}_1, \bar{v}_2 + \bar{u}_2, \bar{v}_3 + \bar{u}_3)$;

ii) **Subtraction**: $v - u = (\bar{v}_1 - \bar{u}_1, \bar{v}_2 - \bar{u}_2, \bar{v}_3 - \bar{u}_3)$;

iii) **Scalar Multiplication**: $\lambda v = (\lambda \bar{v}_1, \lambda \bar{v}_2, \lambda \bar{v}_3)$

Clearly T^3 is a vector space over $\mathbb{R}$, where $0 = (\bar{0}, \bar{0}, \bar{0})$.

Definition 12: Let be $v = (\bar{v}_1, \bar{v}_2, \bar{v}_3)$, $u = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$ in T^3. We define the distance of v to u as: $D(v, u) = \min\{d(\bar{v}_1, \bar{u}_1), d(\bar{v}_2, \bar{u}_2), d(\bar{v}_3, \bar{u}_3)\}$.

Proposition 6: Let $v = (\bar{v}_1, \bar{v}_2, \bar{v}_3) \in T^3$, where $\bar{v}_i = (v_i^1, v_i^2, v_i^3) \in T$ for $i = 1, 2, 3$. Then the function $\| \cdot \|: T^3 \longrightarrow \mathbb{R}$, where $\| v \| = D(0, v) = \min\{|v_1^2|, |v_2^2|, |v_3^2|\}$ is a norm on T.

Proof: Straightforward. ■

It is easy to see that D is a metric in T^3.

Definition 13: The open ball of radius $r > 0$ centered at a point $v \in T^3$ is the set $B(v, r) = \{u \ : \ D(v, u) < r\}$. The closed ball is $B[v, r] = \{u \ : \ D(v, u) \leq r\}$.

Definition 14: Let $A \subset T^3$. We say that A is an open set if for each $v \in A$ there exists an open ball $B(v, r) \subset A$. The collection of all open sets of T^3 is denoted by Γ^3_T.

V. Quaternion Rotation Operator

In this section we introduce the quaternion rotation operator (see [14]).

We can see a quaternion number $q = q_0 + q_1 i + q_2 j + q_3 k$ as a sum of $q_0 = (q_0, 0, 0)$ and vector $\mathbf{q} \in \mathbb{R}^3$, where $i = (1, 0, 0), j = (0, 1, 0)$, and $k = (0, 0, 1)$; i.e.:

$$q = q_0 + \mathbf{q},$$

and $\mathbf{q} = q_1 i + q_2 j + q_3 k$

The **addition** of quaternions is componentwise. Let $p = p_0 + p_1 i + p_2 j + p_3 k$ and $q = q_0 + q_1 i + q_2 j + q_3 k$. Then

$$p + q = (p_0 + q_0) + (p_1 + q_1)i + (p_2 + q_2)j + (p_3 + q_3)k$$

The **multiplication** of quaternions satisfies the fundamental rules defined by Hamilton.

$pq = (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3) + (p_0q_1 + q_0p_1 + p_2q_3 - p_3q_2)i + (p_0q_2 + q_0p_2 + p_3q_1 - p_1q_3)j + (p_0q_3 + q_0p_3 + p_1q_2 - p_2q_1)$

or $pq = p_0q_0 - \mathbf{p}\cdot\mathbf{q} + p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q}$, where $\mathbf{p} = (p_1, p_2, p_3), \mathbf{q} = (q_1, q_2, q_3)$, $\mathbf{p}\times\mathbf{q}$ is an cross product in $\mathbb{R}^3$ and $\cdot$ is a inner product.

Let $q = q_0 + q_1 i + q_2 j + q_3 k$ be a quaternion. The conjugate of q is denoted by q^* and defined as $q^* = q_0 - \mathbf{q} = q_0 - q_1 i - q_2 j - q_3 k$. It is not hard to see that $(q^*)^* = qq^*$ and $(pq)^* = q^*p^*$.

The norm of quaternion q is denoted by $|q| = \sqrt{q^*q}$ and a quaternion is called a unit if its norm is 1. It is not hard to see that $|pq|^2 = |p|^2|q|^2$. The inverse of q is $q^{-1} = \dfrac{q^*}{|q|^2}$. If q is a unit, i.e. $|q| = 1$, then $q^{-1} = q^*$.

Let $q = q_0 + \mathbf{q}$, if $q_0 = 0$, then q is called **pure quaternion**.

Let $\|\cdot\|$ be the euclidian norm in $\mathbb{R}^3$ and a unit quaternion $q = q_0 + \mathbf{q}$. Then $q_0^2 + \|\mathbf{q}\|^2 = 1$ implies that there exists θ such that $cos^2\theta = q_0^2$ and $sin^2\theta = \|\mathbf{q}\|^2$. In fact, there exists $\theta \in [0, \pi]$, where $cos\ \theta = q_0$ and $sin\ \theta = \|\mathbf{q}\|$. Thus, we can write $q = cos\ \theta + \mathbf{u} sin\ \theta$, where $\mathbf{u} = \mathbf{q}/\|\mathbf{q}\|$.

Theorem 1: For any unit quaternion $q = q_0 + \mathbf{q} = cos\frac{\theta}{2} + \mathbf{u} sin\frac{\theta}{2}$ and any vector $\mathbf{v} \in \mathbb{R}^3$ the action of the operator $L_q(\mathbf{v}) = q\mathbf{v}q^* = (q_0^2 - \|\mathbf{q}\|)\mathbf{v} + 2(\mathbf{q}\cdot\mathbf{v})q + 2q_0(\mathbf{q}\times\mathbf{v})$

on $\mathbf{v}$ may be *interpreted geometrically as the **rotation** of the vector **v** through an angle θ about **u** as the axis of rotation.*

VI. Application: Rotation of Three-dimensional Triangular Fuzzy Numbers

Consider T^3 (three-dimensional triangular fuzzy numbers). We will show in this section a method to rotate a vector $\mathbf{v}$ through an angle θ about $\mathbf{u}$ as the axis of rotation.

Remark: Aggregation Operator has the purpose to summarize simultaneous pieces of information. Aggregations have been applied in many fields; for example: Neural networks, possibility theory and fuzzy sets theory [15]. A well-known aggregation is the Arithmetic Mean. Formally:

Definition 15: An aggregation operator is a function $A : \bigcup_{n\in\mathbb{N}} [0,1]^n \longrightarrow [0,1]$ such that:

i) $A(x) = x$ for all $x \in [0,1]$ (*Identity when unary*) .

ii) $A(x_1, \ldots, x_n) \leq A(y_1, \ldots, y_n)$ if $(x_i \leq y_i)$ for $1 \leq i \leq n$ (Non decreasing).

iii) $A(0, \ldots, 0) = 0$ and $A(1, \ldots, 1) = 1$ (boundary conditions).

Each aggregation operator A can be represented by a family $(A_n)_{n\in\mathbb{N}}$ of n-ary operation functions $A_n : [0,1]^n \longrightarrow [0,1]$.

The most common aggregations are shown in Table I.

Arithmetic Mean	$\mathbf{M}(x_1,\ldots,x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i$
Geometric Mean	$\mathbf{G}(x_1,\ldots,x_n) = \left(\prod_{i=1}^{n} x_i\right)^{\frac{1}{n}}$
Harmonic Mean	$\mathbf{H}(x_1,\ldots,x_n) = \dfrac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}$
Quadratic Mean	$\mathbf{Q}(x_1,\ldots,x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}}$

TABLE I. COMMON AGGREGATION OPERATORS.

Application

Given $\mathbf{v}, \mathbf{u} \in T^3$. Let's rotate a vector $\mathbf{v}$ through an angle θ about $\mathbf{u}$ as the axis of rotation. For this end, consider $\mathbf{v} = (\bar{v}_1, \bar{v}_2, \bar{v}_3)$ and $\mathbf{u} = (\bar{u}_1, \bar{u}_2, \bar{u}_3)$, where $\bar{v}_i = (v_i^1, v_i^2, v_i^3)$ and $\bar{u}_i = (u_i^1, u_i^2, u_i^3)$ for $i = 1, 2, 3$.

The first step is to rotate the vector $m(v) = (v_1^2, v_2^2, v_3^2)$ (where $m : T^3 \longrightarrow T$ is a projection on the second coordinate, i.e $m(v) = m(\bar{v}_1, \bar{v}_2, \bar{v}_3) = \bar{v}_2$) through an angle θ about $m(u)$ via quaternion method for $\mathbb{R}^3$, in which we call $(w_1, w_2, w_3) \in T$. Now consider the function $D : T \longrightarrow \mathbb{R}$, where $D(\bar{v}_i) = v_i^3 - v_i^1$ and $\bar{v}_i = (v_i^1, v_i^2, v_i^3)$.

The second step is to calculate the dispersion $dis_i = A(D(\bar{u}_i), D(\bar{v}_i)$, where A is an **aggregation operator**. The rotation of v around the axis u with angle θ is given by:

$w = (\bar{w}_1, \bar{w}_2, \bar{w}_3) \in T^3$, where

$\bar{w}_i = (w_i - dis_i, w_i, w_i + dis_i)$ for $i = 1, 2, 3$.

Example 1: Now consider the following numerical example, let

$$\mathbf{v} = \begin{pmatrix} (-0.5,0,\frac{1}{3}) \\ (0.5,1,2) \\ (-\frac{1}{3},0,0.5) \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} (-0.25,0,\frac{1}{3}) \\ (-1,0,-1) \\ (-\frac{1}{3},1,1.5) \end{pmatrix} \in T^3$$

and $\theta = \pi$.

The rotation of $\mathbf{v}$ around the axis $\mathbf{u}$, with a rotation angle π is:

Algorithm 1: Algorithm to build triangular fuzzy number

Input: $\mathbf{v}$ and $\mathbf{u} \in T^3$ - Vector for rotation; θ - Angle for rotation
Output: $\mathbf{w} \in T^3$ - Vector obtained by rotation
$vmid \leftarrow m(v)$
$umid \leftarrow m(u)$
/* unitary quaternion form */
$q \leftarrow \cos\theta + umid \times \sin\theta$
/* Multiplication quaternion explained in Theorem 1. */
$wmid \leftarrow q \times vmid \times q^{-1}$
for (i =0 to 3) **do**
  $dispersion \leftarrow A(D(u_i), D(v_i))$ /* A is aggregation operator */
  $\mathbf{w}_i = (wmid_i - dispersion, wmid_i, wmid_i + dispersion)$

1 - Using A as arithmetic mean:

$$\mathbf{w} = \begin{pmatrix} (-1.354, -1, -0.6458) \\ (-0.875, 0, 0.875) \\ (-0.667, 0, 0.667) \end{pmatrix}$$

2 - Using A as geometric mean:

$$\mathbf{w} = \begin{pmatrix} (-1.517, -1, -.483) \\ (-1.118, 0, 1.118) \\ (-0.917, 0, 0.917) \end{pmatrix}$$

3 - Using A as quadratic mean:

$$\mathbf{w} = \begin{pmatrix} (-1.68, -1, -0.319) \\ (-1.132, 0, 1.132) \\ (-0.917, 0, 0.917) \end{pmatrix}$$

Each triangle in the figure 1 shows the components of $\mathbf{v}$ and $\mathbf{u}$. Figure 2 shows the rotation using different aggregation operator.

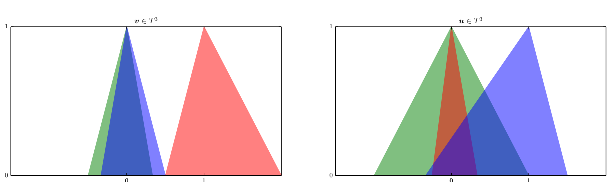

Fig. 1. Components of $\mathbf{v}$ and $\mathbf{u}$.

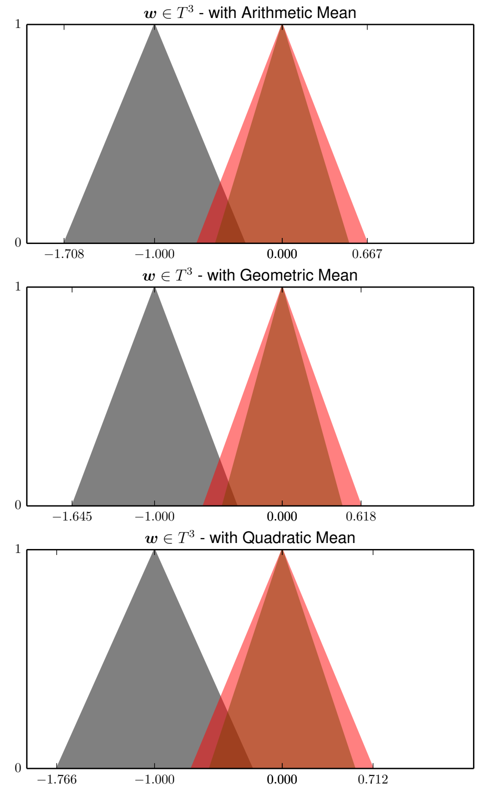

Fig. 2. The rotation of $\mathbf{v}$ around the axis $\mathbf{u}$, with a rotation angle π using different aggregation operator

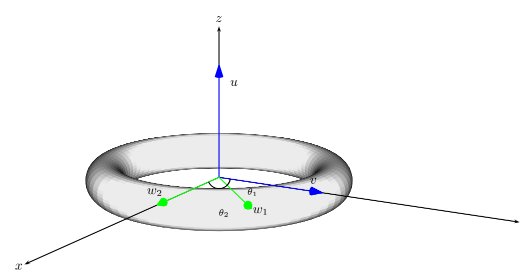

Fig. 3. The rotation of $\mathbf{v}$ around the axis $\mathbf{u}$, with a rotation angle Θ_1 and Θ_2.

VII. Final Remarks

The rotation in T^3 proposed here is different from rotation in $\mathbb{R}^3$. This fact was observed when we evaluated the distance from the axis vector. In other words, let $\mathbf{v}, \mathbf{u}$ be in T^3 and $\mathbf{w}$ the rotation of $\mathbf{v}$ around $\mathbf{u}$ by θ. The distance $\| \mathbf{v} - \mathbf{u} \| \neq \| \mathbf{w} - \mathbf{u} \|$, but when we rotate $\mathbf{v}$ around $\mathbf{u}$ we have an oscillation correspondent to the dispersion, then $\mathbf{v}$ lies in the torus. The figure 3 give us an idea about this oscillation, **W1** and **W2** is $\mathbf{v}$ rotate aroud $\mathbf{u}$ by Θ_1 and Θ_2 respectively.

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