2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)
July 6-11, 2014, Beijing, China

Fuzzy measures of pixel cluster compactness

Gleb Beliakov, Gang Li, Huy Quan Vu and Tim Wilkin
School of Information Technology, Deakin University,
221 Burwood Hwy, Victoria 3125, Australia
{gleb, gang.li, hqvu, tim.wilkin}@deakin.edu.au

***Abstract*— Pixel-scale fine details are often lost during image processing tasks such as image reduction and filtering. Block or region based algorithms typically rely on averaging functions to implement the required operation and traditional function choices struggle to preserve small, spatially cohesive clusters of pixels which may be corrupted by noise. This article proposes the construction of fuzzy measures of cluster compactness to account for the spatial organisation of pixels. We present two construction methods (minimum spannning trees and fuzzy measure decomposition) to generate measures with specific properties: monotonicity with respect to cluster size; invariance with respect to translation, reflection and rotation; and, discrimination between pixel sets of fixed cardinality with different spatial arrangements. We apply these measures within a non-monotonic mode-like averaging function used for image reduction and we show that this new function preserves pixel-scale structures better than existing monotonic averages.**

I. INTRODUCTION

The ubiquitous nature of small display screens in smartphones, tablets and laptop computers has reinforced the need for robust image reduction techniques. In particular, where these screens are being used to display images captured at higher resolutions than the screen is capable of displaying, image reduction is essential. Even where large screens are involved, the resolutions at which images may now be captured far exceeds the capability of many screens to display them at full resolution.

Another common application of image reduction is, along with image filtering, as a pre-processing task in computer vision applications. In such contexts reducing the number of pixels processed also reduces the computational complexity of the vision task and thus the memory and time requirements. In both of these scenarios it is important that the preprocessing of the image does not cause the loss of fine image details, which may convey important information relevant to the analysis of the image content.

Block-based image reduction operators based on averaging aggregation functions have been proposed for this purpose [7]. Recent work has focused on the problem of the preservation of fine, pixel-scale details in images, which are represented by small, spatially coherent clusters of pixels having similar intensities. Non-monotonic averaging functions have been shown to improve the robustness of image reduction compared to monotonic averages such as the arithmetic mean or median [9]. As with convolution-based image filters, in this previous work spatial organisation of pixels was accounted for using distance-based weights. This approach does not sufficiently describe the spatial structure of a set of pixels such that structured image details may be preserved during the reduction.

In this work we are interested in identifying geometrically compact clusters of pixels having similar intensity, which are tonally different from their local neighbourhood. Furthermore, we consider the possibility that such clusters may represent a minority of a local region, even when considering block sizes as small as 3×3 pixels. We also assume that pixel intensities may be corrupted by noise or undue variation due to sampling effects in discrete digital imaging arrays.

To extend and improve upon previous work in this area we propose that a measure of cluster compactness may be used to weight the contribution of specific pixels within a non-monotonic average, which is used to compute a representative pixel value for a given block of pixels. This article considers a novel approach of constructing cluster compactness measures based on fuzzy measure theory.

Given a subset $\mathcal{A}$ of pixels within an $m \times n$ block, the following requirements for a compactness measure SC would seem to be reasonable in the image reduction context: 1) the function $SC(\mathcal{A})$ must be non-decreasing in $|\mathcal{A}|$, as larger clusters are less likely to represent noise; 2) the function SC must be invariant with respect to translation, reflection and rotation (at least rotations by multiples of 90 degrees); and, 3) the function SC must discriminate between compact groups and disconnected subsets of fixed cardinality. For convenience we will require that SC is normalised.

We note that some simple measures of compactness exist, such as those based inter-set and intra-set distances, however these are not adequate for our purposes as they do not satisfy the first requirement. This requirement though is precisely the monotonicity condition used in the definition of fuzzy measures and therefore it makes sense to look for a solution within the class of fuzzy measure functions.

We present our work herein and our article is structured as follows. Section II formulates the image reduction task as an averaging problem over local image blocks and introduces the role of a cluster compactness measure within this context. Section III briefly presents fuzzy measures and Sections IV and V present our proposed measures based on a minimum spanning tree graph and a geometrical decomposition, respectively. We briefly compare measures produced by these methods in Section VI. In Section VII we present a validation of the proposed measures in the context of non-monotonic averaging for image reduction. Finally, Section VIII presents our conclusions and describes briefly some future work.

978-1-4799-2072-3/14/$31.00

II. PROBLEM FORMULATION

We formulate the compactness measure problem using an example of image reduction based on a local, block-based reduction operator, as shown in Figure 1. An image of size $M \times N$ is subdivided into non-overlapping blocks of size $m \times n$. Each block is aggregated to generate a single value representative of the original data, which becomes the intensity of a single pixel within the reduced image. Thus, the original image is reduced to the size $M' \times N' = \lfloor \frac{M}{m} \rfloor \times \lfloor \frac{N}{n} \rfloor$.

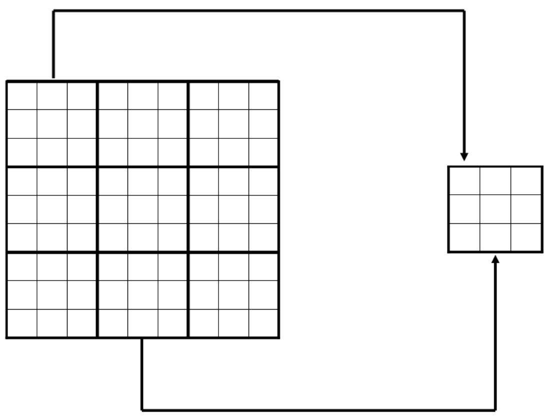


Fig. 1: A scheme for image reduction within 3×3 blocks.

The local operators based on aggregation functions have been shown to be both effective , efficient and easily admit parallel implementations [1], [8]. They require the properties of averaging functions such as means, so that the output is within the range of the intensities in the input pixels and is also idempotent [2]. This latter property ensures that if the block is of uniform intensity its representative value is exactly the same as the intensity of all input pixels. The simplest examples are the weighted arithmetic mean and the median, which play an important role in Gaussian and median filtering. All suitable averaging functions can be obtained by using a penalty-based approach [4], wherein a given penalty function for intensity deviations is minimised to obtain the aggregate value.

The most well-known averaging functions satisfy another condition, that of monotonicity [2]. Any increase in one or more input values does not cause a decrease in the output of the aggregation. This condition is useful when the data are noiseless, however it is not desirable when the data may be contaminated by noise, which appear as tonal outiers within small pixel blocks. Previous work in aggregation problems has focused on the design of non-monotonic averages using penalty-based methods [3], [4] and these approaches have been adapted for image reduction [9]. Many of the non-monotonic averages were recently shown to be weakly monotone averaging functions, which have desirable properties relevant to image processing tasks such as reduction and filtering [10] and our current work continues to pursue the design of such functions using penalty based methods.

We represent by the vector $\mathbf{x}$ the intensities of a set of p pixels with values in $\mathbb{I}^p$, where $\mathbb{I} = [a, b]$ is any closed, non-empty subset of the reals. The average intensity of these inputs is the solution to the minimisation problem

$$y = f(\mathbf{x}) = \arg\min_y \mathcal{P}(\mathbf{x}, y), \quad (1)$$

where $\mathcal{P} : \mathbb{I}^{p+1} \to \mathbb{R}$ is a penalty function satisfying the conditions:

1) $\mathcal{P}(\mathbf{x}, y) \geq c \quad \forall \mathbf{x} \in \mathbb{I}^p,\ y \in \mathbb{I}$;
2) $\mathcal{P}(\mathbf{x}, y) = d$ if all $x_i = y$,

for some constant $d \in \mathbb{R}$. In [9] a penalty function based on a weighted sum of intensity-based penalties was proposed, which was given by

$$\mathcal{P}(\mathbf{x}, y) = \sum_{i=1}^{p} w_i(y)\rho(x_i, y) \quad (2)$$

where

$$\rho(x_i, y) = \begin{cases} r_{(k)} & r_{(k)} < \tau, \\ \beta\tau & r_{(k)} \geq \tau. \end{cases} \quad (3)$$

$$\tau = \alpha \max(\epsilon, r_{(t)}) \text{ and } \alpha > 0, 0 \leq \beta \leq 1, 2 \leq t \leq p.$$

Given $r_i = \|x_i - y\|$ then $r_{(k)}$ denoted the kth smallest element of the set of ordered (ascending) values of r_i, given the aggregate value y. This function generates a mode-like non-monotonic average of the input vector $\mathbf{x}$ and was shown to outperform other monotonic and non-monotonic block-based reduction operators when applied to images corrupted by speckle or impulse noise.

Although the function ρ favours compact clusters of intensity values in $\mathbf{x}$ by assigning smaller penalties to inputs closer to the proposed output, it does not take into account spacial organisation of the pixels. This is achieved by using weights based on normalised distance between the pixels within the block:

$$w_i(y) = \frac{d(x_i, y)}{\sum_{i=1}^{p} d(x_i, y)}, \forall y = x_j \in \{x_1, ..., x_p\}. \quad (4)$$

This function arose from the additional constraint that the average must also be an internal function (i.e., the output should be one of the inputs), which is a reasonable requirement in image reduction tasks. In this current work we wish to replace the weights $w_i(y)$ with a function that incorporates spatial structure information and that appropriately orders candidate clusters of pixels according to their spatial arrangement. Consequently we desire a function over the power set 2^P, where P is the index set for the input vector $\mathbf{x}$ and $w : 2^P \to [0, 1]$.

As mentioned in the introduction we desire a monotone set function as larger subsets are favoured and have smaller penalties in the expression for $\mathcal{P}$. We also impose suitable boundary conditions so that the range of w is within the unit interval. Since these conditions are the same as the ones used in the definition of fuzzy measures we consider weighting functions of the type $w(\mathcal{A}) = w_0 - v(\mathcal{A})$, where $w_0 > 1$ ensures that w is strictly positive, and v is a fuzzy measure. Normalised weights are trivially obtained if required.

III. FUZZY MEASURES

A fuzzy measure is defined as follows [6]:

Definition 1: Let $\mathcal{N} = \{1, 2, \ldots, n\}$. A discrete fuzzy measure is a set function $v : 2^{\mathcal{N}} \to [0, 1]$ which is monotonic (i.e. $v(\mathcal{A}) \le v(\mathcal{B})$ whenever $\mathcal{A} \subset \mathcal{B}$) and satisfies $v(\emptyset) = 0$ and $v(\mathcal{N}) = 1$.

Definition 2: A fuzzy measure v is called submodular if for any $\mathcal{A}, \mathcal{B} \subseteq \mathcal{N}$

$$v(\mathcal{A} \cup \mathcal{B}) + v(\mathcal{A} \cap \mathcal{B}) \le v(\mathcal{A}) + v(\mathcal{B}). \quad (5)$$

It is called supermodular if

$$v(\mathcal{A} \cup \mathcal{B}) + v(\mathcal{A} \cap \mathcal{B}) \ge v(\mathcal{A}) + v(\mathcal{B}). \quad (6)$$

Definition 3: A fuzzy measure v is called subadditive if for any two nonintersecting subsets $\mathcal{A}, \mathcal{B} \subset \mathcal{N}$, $\mathcal{A} \cap \mathcal{B} = \emptyset$:

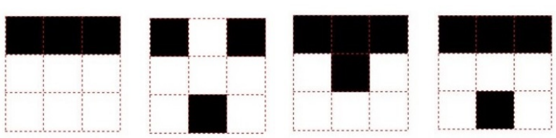
$$v(\mathcal{A} \cup \mathcal{B}) \le v(\mathcal{A}) + v(\mathcal{B}). \quad (7)$$

It is called superadditive if

$$v(\mathcal{A} \cup \mathcal{B}) \ge v(\mathcal{A}) + v(\mathcal{B}). \quad (8)$$

Definition 4: A fuzzy measure v is called symmetric if the value $v(\mathcal{A})$ depends only on the cardinality of the set $\mathcal{A}$, i.e., for any $\mathcal{A}, \mathcal{B} \subseteq \mathcal{N}$, if $|\mathcal{A}| = |\mathcal{B}|$ then $v(\mathcal{A}) = v(\mathcal{B})$.

We require a non-symmetric fuzzy measure as we wish to differentiate between compact and scattered groups of pixels of the same cardinality, such as those depicted in Figures 2a and 2b respectively. We also require symmetry among those subsets $\mathcal{A}$ that represent translation, rotation and reflection symmetry of the corresponding groups of pixels. In addition, we require a non-additive fuzzy measure. If we consider a compact cluster of pixels - such as the one shown in Figure 2a - and add another pixel to the cluster, the measure of this larger set should depend on how close (Figure 2c) or away (Figure 2d) this additional pixel is to the original cluster.



(a) Compact (b) Scattered (c) Close (d) Away

Fig. 2: Example of different clusters.

We shall consider two different approaches to constructing fuzzy measures that satisfy our requirements. In the first case, fuzzy measures reflecting cluster compactness are computed directly based on the Minimum Spanning Tree. In the second case we will use some extra information about the desired fuzzy measure in terms of reference points and constraints. A *reference point* is a specified value of v for a particular subset $\mathcal{A}$, which we believe is a reasonable choice. For example, we could specify $v(\mathcal{A}) = 1$ for all compact $\mathcal{A}$ of cardinality $|\mathcal{A}| \ge k$, for some $k < p$, and where compactness means that each pixel in $\mathcal{A}$ has a neighbour in $\mathcal{A}$ (that is, for $\forall a \in \mathcal{A}$: $d_H(a, \mathcal{A}) = 1$ and d_H is the Housdorff distance). Further, we will impose *constraints* such as $v(\mathcal{A}) \ge v(B) + \delta$ for a some $\delta > 0$ when we believe the measures of $\mathcal{A}$ and $\mathcal{B}$ should differ at least by δ. We assume there are r reference points and c inequality constraints.

IV. AN APPROACH BASED ON MINIMUM SPANNING TREES

Minimum Spanning Trees (MST) have been used for clustering for several decades [11]. Given a connected weighted graph the MST is a subgraph (a tree connecting all vertices) whose weight is the smallest. It is constructed from the adjacency matrix using Prim's or Kruskal's algorithms [5] and its complexity is quadratic in the number of vertices.

We shall use the weight of the MST (i.e., the sum of all edge weights in the MST), constructed from a complete graph connecting the elements of a cluster, whose weights are pairwise distances between the data. In cluster analysis, such MSTs are used to agglomerate the data and partition it into several clusters by removing the edges of maximum weight. Here we are interested in a measure of compactness of a single cluster, hence we shall only use the weight of the MST rather than its structure.

We are interested in devising a quantity that satisfies the three criteria set out in the Introduction. We use the formula

$$SC(\mathcal{A}) = C - \frac{\frac{W(MST(\mathcal{A}))}{TM} + 1}{|\mathcal{A}|}, \quad (9)$$

where $T = |\mathcal{U}|$ is the cardinality of the largest cluster, M is the largest distance between the elements of a cluster, and $C = 1 + \frac{T(M+1)-1}{T^2 M}$. For example, for a 3×3 block and the Euclidean distance, $T = 9$ and $M = 2\sqrt{2}$, and hence $C = 1 + \frac{1}{9} + \frac{4}{81\sqrt{2}}$. For brevity we will denote $w(\mathcal{A}) = w(MST(\mathcal{A}))$.

The constant C is a normalisation constant which ensures $SC(\mathcal{A}) \in (0, 1]$ (we implicitly assume that $SC(\emptyset) = 0$), and T and M are scaling parameters. This ensures the function SC is a fuzzy measure, invariant with respect to translation, refection and rotation and that it is discriminating with respect to cluster compactness.

Example 1: The numerical results illustrating the formula (9) are presented in Figure 3. We observe monotonicity with respect to set cardinality and differentiation between tightly compact and spread out sets of the same cardinality. One inconvenience of (9) is that the numerical values of $SC(\mathcal{A})$ are clogged at the higher end of the spectrum near one, so that the differences between more and less compact sets of cardinality more than three are in the second or third decimal place, see Figure 3(a). This can be rectified by raising $SC(\mathcal{A})$ to some power q, making the numerical values more evenly distributed and clearly differentiating numerically between the more compact and less compact sets. The CPU time required to compute these values was negligible.

V. DECOMPOSING FUZZY MEASURE

In this approach to computing a fuzzy measure we explicitly account for geometrical symmetries in the clusters by decomposing them into component blocks. Consider Figure 4 which shows examples of 4 building blocks of subsets, which we call the *basic components*, BC. Each subset $\mathcal{A}$ can be

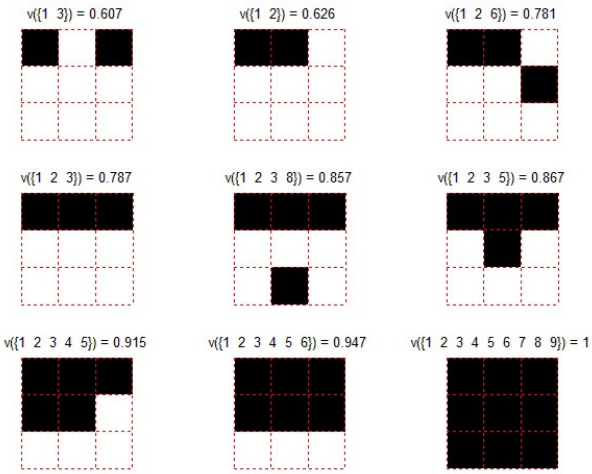


Fig. 3: Estimated fuzzy measure using Minimum Spanning Tree approach.

represented in a number of ways by the composition of a set of (possibly overlapping) basic components BC_i. The basic components account for all geometrical symmetries and are encoded using a hash function based on the average intra-pixel distance

$$h(\mathcal{A}) = \frac{1}{|\mathcal{A}|} \sum_{a \in \mathcal{A}} d_H(a, \mathcal{A} \setminus a).$$

The number of basic components and their shape can be specified by the user and allows for the customisation of the compactness measure (and thus penalty weights) to favour certain structural patterns over others.

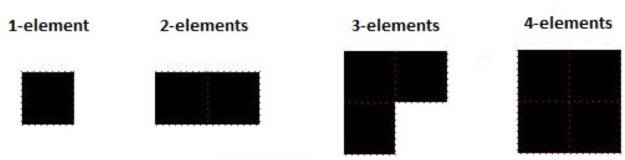


Fig. 4: Examples of basic components BC_1 – BC_4.

For a given subset $\mathcal{A}$ we identify its decompositions into one or more basic components of the same type and identify the number of possible ways to fit such a basic component into $\mathcal{A}$. Each of the basic components BC_i is assigned a value u_i. The values of $v(\mathcal{A})$ are computed as follows

$$SC(\mathcal{A}) = \min(1, n_1 u_1 + n_2 u_2 + \ldots + n_q u_q), \tag{10}$$

where q is the number of basic components, u_i is the value of the ith basic component, and n_i is the count of the ith basic component in the decomposition of $\mathcal{A}$.

Suppose a subset $\mathcal{A} = \{1, 2, 4\}$ is a cluster with 3 connected elements in a 3×3 mask. There are 3 possible ways to fit the 1-element component, 2 ways to fit 2-element component, and 1 way to fit 3-element component into the provided shape (note that overlapping is allowed). Assume that the weights of the basic components are provided as $\mathbf{u} = [0.1, 0.1, 0.05, 0.05]$. Thus, the value of the fuzzy measure would be:

$$\begin{aligned} v(\mathcal{A}) &= \min(1, 3u_1 + 2u_2 + 1u_3 + 0u_4) \\ &= \min(1, 3 * 0.1 + 2 * 0.1 + 01 * 0.05 + 0 * 0.05) \\ &= 0.55 \end{aligned}$$

Unlike the MST approach the fuzzy measure decomposition computes the fuzzy measure based on a model; the values $u_1, \ldots, u_q$. These values are estimated with respect to available reference points and constraints, which represent the desired numerical values and relations between the values of SC. Before we proceed with formulating a fitting problem, let us demonstrate that the function SC is a fuzzy measure. Let $\mathcal{U}$ denote the universal set (that is, the largest cluster, being the whole block of $m \times n$ pixels).

Proposition 1: The function SC in (10) is a fuzzy measure irrespective of the values $u_i \in [0, 1] | \sum_i n_i(\mathcal{U}) u_i \geq 1$. SC discriminates between more and less compact clusters.

Proof: Evidently $SC(\emptyset) = 0$ and $SC(\mathcal{U}) = 1$. We need to show monotonicity. For this we note that $\mathcal{A} \subset \mathcal{B}$ implies $n_i(\mathcal{A}) \leq n_i(\mathcal{B})$. Consequently $\sum_i n_i(\mathcal{A}) u_i \leq \sum_i n_i(\mathcal{B}) u_i$. Next, as more compact subsets $\mathcal{A}$ allow fitting more basic components that are larger in size, they will have larger values of n_i, and hence larger values of $SC(\mathcal{A})$. ■

We now need to fit the parameters u_i to the available reference points and constraints. Because the parameters enter our expressions linearly we can set up the following mathematical programming problem

$$\begin{aligned} \text{minimize} \quad & F(u_1, \ldots, u_q) = \sum_{i=1}^{r} |SC(\mathcal{A}_i) - v_i| \\ \text{subject to} \quad & SC(\mathcal{A}_l) - SC(\mathcal{B}_l) \geq \delta_l, \ l = 1, \ldots, c, \\ & SC(\mathcal{A}_k) = \min(1, \sum_{i=1}^{q} n_i(\mathcal{A}_k) u_i), \\ & \sum_{i=1}^{q} n_i(\mathcal{U}) u_i \geq 1, \\ & u_1, \ldots, u_q \geq 0. \end{aligned} \tag{11}$$

and convert this into a linear programming problem by introducing slack variables r_i^+, r_i^-, and setting $SC_i = SC(\mathcal{A}_i)$,

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{r} r_i^+ + r_i^- \\ \text{subject to} \quad & r_i^+ - r_i^- - SC_i = -v_i, \ i = 1, \ldots, r, \\ & - SC_i + SC_k \leq -\delta_{ik}, \\ & SC_k \leq \sum_{i=1}^{q} n_i(\mathcal{A}_k) u_i, \\ & SC_k \leq 1, \\ & \sum_{i=1}^{q} n_i(\mathcal{U}) u_i \geq 1, \\ & r_i^+, r_i^- \geq 0, SC_i \geq 0, \\ & u_1, \ldots, u_q \geq 0. \end{aligned} \tag{12}$$

The number of variables is $2r+c+q+t$ and the number of constraints is $r+c+2t+1$, where t is the total number of subsets engaged in all the constraints and reference points. Note that this number is smaller than the number of all possible subsets 2^n, because we care only about the values SC_i for those subsets. The values $\delta_{ik} = \delta_l$ correspond to the pairs of sets $(\mathcal{A}_l, \mathcal{B}_l) = (\mathcal{A}_i, \mathcal{A}_k)$ in the lth inequality constraint. The problem is solved by using the standard simplex method.

Example 2: We illustrate the output of this approach using an example of a 3×3 block. We provide a reference point $v(\{1,2,3,4,5,6\}) = 1$ and impose a constraint $v(\{1,2\}) \geq v(\{1,3\}) + 0.05$. We use the four basic components in Figure 4 to construct the fuzzy measure. The weights are estimated by solving the problem defined in eqn. (12), giving the values $\mathbf{u} = [0.0042, 0.0799, 0.0228, 0.1169]$. Figure 5 shows the estimated values of the fuzzy measure for several representative clusters.

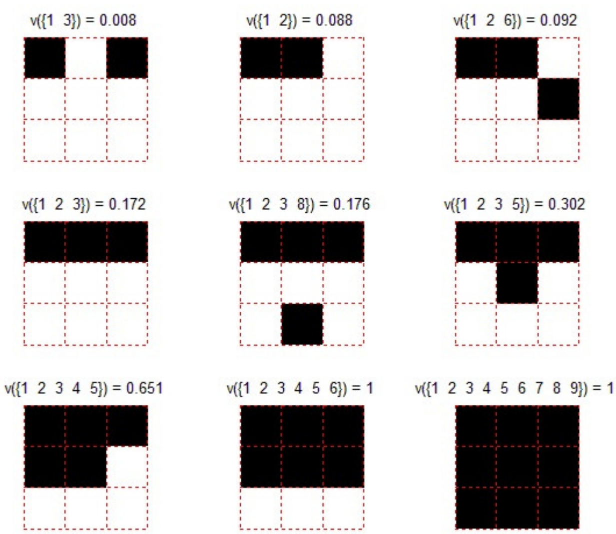
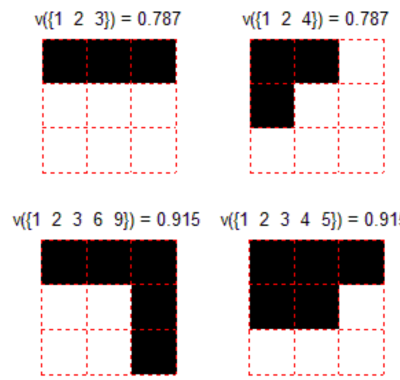


Fig. 5: Estimated fuzzy measure using decomposition approach.

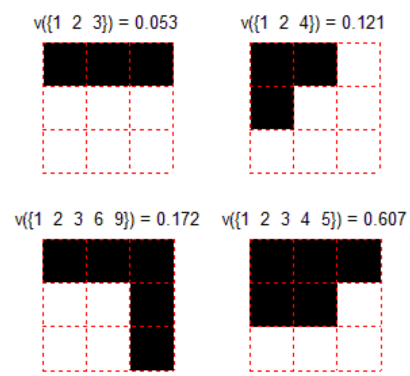
VI. Comparison of the Proposed Measures

A special feature of the MST construction method is that the computed fuzzy measure values are not only invariant with respect to translation, refection and rotation, but also invariant with respect to the shape of clusters with *connected* elements. Figure 6a shows a fuzzy measure computed using the Minimum Spanning Tree method. It can be observed that different cluster shapes with the same number of connected elements have the same fuzzy measure value, $v(\{1,2,4\}) = v(\{1,2,3\})$ and $v(\{1,2,3,4,5\}) = v(\{1,2,3,6,9\})$. This measure would be appropriate to apply in contexts where there is little or no need to discriminate between cluster shapes and only differentiate cluster cardinality.

On the other hand the Decomposing Fuzzy Measure provides shape discrimination due to its recognition of the basic components of a cluster. For example, if we desire that for clusters of connected elements with the same cardinality, a more dense cluster receives a higher value of the fuzzy measure, we may impose constraints such as $v(\{1,2,4\}) \geq v(\{1,2,3\}) + 0.01$ and $v(\{1,2,3,4,5\}) \geq v(\{1,2,3,6,9\}) + 0.01$. Using four basic components as in Figure 4 and one reference point, $v(\{1,2,3,4,5,6\}) = 1$, we compute the weights of the basic components by solving eqn. (12) and obtain the values $\mathbf{u} = [0.0106, 0.0321, 0.0621, 0.1075]$. Figure 6b displays the computed values for this example and it is clear that the constraints describing shape preferences are satisfied.



(a) Minimum Spanning Tree



(b) Decomposing Fuzzy Measure

Fig. 6: Fuzzy measure of different cluster shapes of the same cardinality.

VII. Application to Image Reduction

The image in figure 7 depicts a series of concentric circles as thin curves having a width of one pixel. This pattern contains a large variety of cluster patterns within small 3×3 blocks and in performing reduction on this image, any operator must cope with the problem that the image detail represents a minority of pixels at all scales. This image is also easily assessed visually for global continuity of the curves, making it useful for comparitive evaluation of various reduction operators.

The image $\mathcal{I}$ is constructed from the image

$$\mathcal{I} = max(\mathcal{C} \cdot \mathcal{F}, \mathcal{B})$$

where $\mathcal{C}$ is a binary image depicting the circles, $\mathcal{F}$ is a foreground noise field and $\mathcal{B}$ is a background noise field. These fields were generated as uniformly distributed 8 bit pixel intensities in the ranges $[0, 10]$ and $[245, 255]$ respectively.

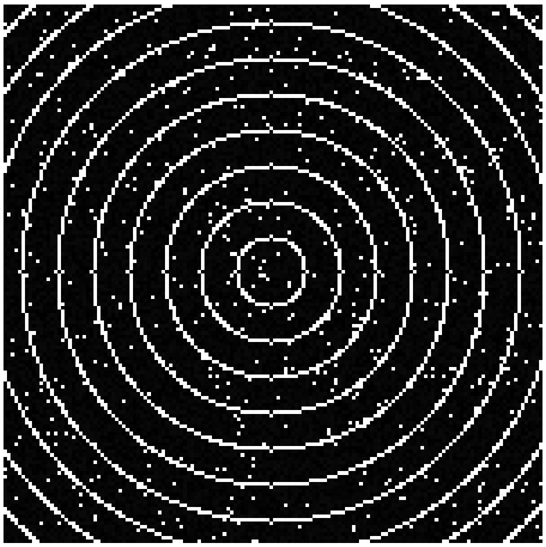


Fig. 7: Concentric circle pattern used to test image reduction operators.

We constructed a local block-based reduction operator based on eqn. 2, by replacing the individual pixel weights $w_i(y)$ with a cluster-based weight, such that

$$\mathcal{P}(\mathbf{x}, y) = w(\mathcal{A}) \sum_{i=1}^{p} \rho(x_i, y). \quad (13)$$

Here $w(\mathcal{A}) = 2 - v(\mathcal{A})^q \geq 1$, v is a fuzzy measure computed using either of the aforementioned approaches and $\mathcal{A}$ denotes the subset of pixels having intensities that satisfy $r_{(k)} < \tau$ as per eqn. (3). These pixels define a candidate cluster within the tonal space of the block for the given value y. This choice of penalty function means that for two candidate clusters of equal cardinality and equivalent pixel intensity differences within the cluster, the more spatially coherent cluster will have a lower penalty and thus be preferred as the significant cluster of the block. Conversely, if the spatial patterns are equivalent, the cluster with the more compact tonal range will be preferred. As with the method described in [9] the candidate average values y are taken from the set of input pixel values for that block, so that the output image is a proper subset of the input image.

We constructed versions of the penalty-based averaging function using the proposed fuzzy measures, given by equations (9) and (10) and denoted herein as MST and DFM respectively. For the MST measure we selected the exponent value $p = 200$, to ensure the measure covered the full range $[0, 1]$ and for the DFM measure we took $p = 1$, since this function already provides a well-distributed set of measure values in $[0, 1]$. Our cluster-based mode-like average was compared against the mode-like average using distance-based weights and the Shorth function, given in [9], as well as monotonic reduction operators using the arithmetic mean and median. The parameters for the mode-like averages were $\alpha = 12$ and $\beta = 0.3$.

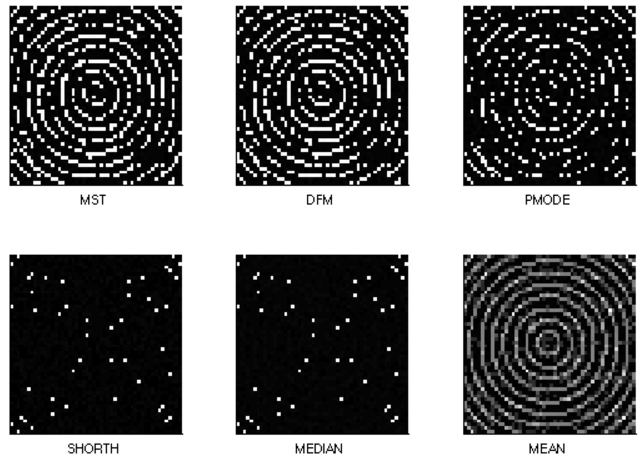


Fig. 8: Reduced circle test image using cluster-based mode-like averaging functions.

Image reduction was performed over disjoint blocks of size 3×3 (producing a $\frac{1}{9}$ scale image) and the resulting images are shown in Figure 8. It is apparent from these results that the median based reduction operator does not preserve the relevant curves of one pixel thickness. This is to be expected since within a 3×3 block the high valued pixels on the curve would appear as a minority of outliers against the more common background. In the case of the arithmetic mean, while the circles are preserved, their peak intensity is diminished and they are spatially broadened, which is to be expected. While the resulting image contains the desired visual continuity of the curves, the structural detail of these curves (specifically radial gradients) have been corrupted by the reduction.

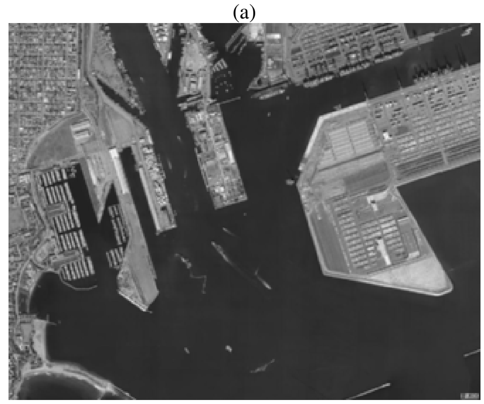
The mode-like average using distance-based weights outperforms both the mean and median operators and preserves sections of the curves. This indicates that certain cluster patterns will result in an aggregate selected from the cluster, whereas others are not significant enough to be preferred over the background pixels. In such cases the minimisation of the penalty favours reducing the number of outliers, even though they may be centrally located within the local block.

The cluster-based mode-like averaging functions, built using the various fuzzy measure construction methods, outperforms the other averaging functions on this test image, including the distance-based mode-like average, which is most similar to it. Interestingly, it is able to preserve nearly continuous curves. The missing sections appear to be associated with sections of the original curves that were locally corrupted by noise, or contained simple linear clusters of bright pixels (such as vertical lines). In such cases the background pixels would form a larger contiguous cluster and would be preferred in the penalty minimisation, since the image detail would represent a minor cluster of outliers.

To test the practicality of these reduction operators and their ability to preserve fine, pixel scale detail, they were applied to a real satellite image. Figure 9 depicts the Port of Los Angeles, showing many fine detail objects, include ships and boats with wakes, shipping containers, houses and other buildings, roads and parks.

Fig. 9: Port of Los Angeles, March 29, 2004. *Image courtesy of the U.S. Geological Survey* (used with permission).

(a)



(b)

Fig. 10: Reduction of Port of LA image using monotonic averages: (a) median; and (b) arithmetic mean.

Each of the averaging functions described above was tested using a variation on the disjoint block reduction algorithm. In this case averages were computed within overlapping 5×5 regions, where the inner 3×3 blocks of adjacent regions were disjoint (i.e., the overlap was a single row or column). The subsequent average was used to represent the inner block in the reduced image, which was thus $1/9$'th scale. This approach was chosen as it indicated better structure preservation than using averages of disjoint blocks.

Given the similarity of results for reducing circles using the cluster-based mode-like average, only the MST approach is tested herein (due to space constraints). Figure 11a shows the reduction using our proposed cluster-based weights, as well as reductions performed using the mode-like average with distance-based weight, and averages computed using the Shorth, median and arithmetic mean functions. It should be noted that all images (the original, as well as the reduced versions) have been scaled for publishing purposes, although this scaling has been performed using the same inbuilt algorithms within LaTeX. While there will be some affects on the resulting images, the differences between them are still apparent and thus we can assess the visual quality of the local block-based reduction algorithms based on these differences.

It is apparent that the mode-like averages produce images that retain fine detail, whereas the monotonic averaging functions operate as low pass filters and smooth the image, removing detail. For example, the small boats in the central channel are poorly preserved in Figures 10a and 10b. Detail such as the shipping containers (right side dock), the boats in the marina and the houses are better preserved by the mode-like averages (Figures 11a and 11b). The Shorth performs better than the monotonic averages, but not as well as the mode-like averages. Like the monotonic averages, the Shorth image is smoothed and fine detail has been lost. It is apparent from these tests that the mode-like operators not only preserve the fine detail present in each local block, but also retain information across neighbouring blocks to represent larger coherent structures in the final image. Both the cluster-based and distance-based weights produce very similar results on this image.

VIII. Conclusion

In this paper, we have tackled the problem of characterising the compactness of a single geometrical cluster of spatially organised values, using a fuzzy measure with specific reasonable properties. This problem arises in the context of image processing, where we need to distinguish between compact groups of pixels (of similar intensity) representing an object, from scattered groups likely to be noise. We proposed two novel approaches to defining a fuzzy measure, based on the geometry of the cluster. One

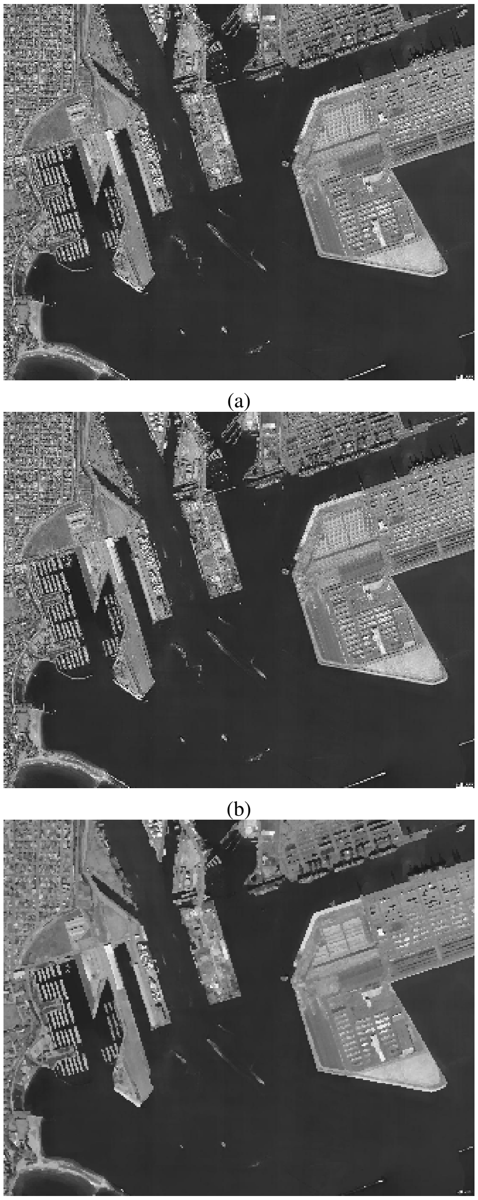


Fig. 11: Reduction of Port of LA image using non-monotonic averages: (a) mode-like average with cluster-based weights; (b) mode-like average with distance-based weights; and (c) Shorth.

approach involves a suitably scaled weight of the minimum spanning tree graph. It is simple to construct numerically and requires negligible computational capacity, however it is less flexible than the alternatives that involve fitting the measure to user specified reference points and constraints. The second approach is based on decomposing clusters into simple geometrical (basic) components. Fitting of the fuzzy measure to the power set is achieved by solving a small scale linear programming problem.

We have demonstrated that the fuzzy measure was monotone in set cardinality and discriminated between more and less compact subsets. By construction, the resulting values were invariant with respect to basic geometrical manipulations of the clusters, such as translation, rotation and reflection. The application of our construction is within image filtering and image reduction, where mode-like non-monotonic averages have been applied to remove noise while preserving fine details of the images, such as small objects. We presented initial results validating the use of a fuzzy measure based penalty weight and showed that it is capable of preserving pixel-scale details in images corrupted by impulse noise and having reasonable variation in intensity values within both the background and foreground classes. We compared this new approach to previously published methods involving both monotonic and non-monotonic averaging functions and found that our new non-monotonic averaging function outperforms these previous methods on our test images. We also tested these operators on a real satellite image and showed that, as expected, the monotonic averages performed as low pass filters and obliterated fine detail which would be relevant to analysis of these images. The mode-like averages were able to preserve this fine detail and clearly depict fine, pixel-scale structures, such as individual boats, houses, cranes and shipping containers. These initial results are promising for pursuing mode-like non-monotonic averages as image reduction operators and for using penalty weights based on fuzzy measures of pixel cluster compactness for preferencing spatially coherent pixels during reduction operations. Further results from this research program will appear in an upcoming publication and thesis.

References

[1] G. Beliakov, H. Bustince, and D. Paternain. Image reduction using means on discrete product lattices. *IEEE Transactions on Image Processing*, 21(3):1070–1083, 2012.

[2] G. Beliakov, A. Pradera, and T. Calvo. *Aggregation Functions: A Guide for Practitioners*. Springer, Heidelberg, Berlin, New York, 2007.

[3] G. Beliakov, S. Yu, and D. Paternain. Non-monotone averaging aggregation. In *IEEE International Conference on Fuzzy Systems*, volume DOI: 10.1109/FUZZY.2011.6007643, pages 2905 – 2908. IEEE, 2011.

[4] T. Calvo and G. Beliakov. Aggregation functions based on penalties. *Fuzzy Sets and Systems*, 161:1420–1436, 2010.

[5] T.H. Cormen, C.E. Leiserson, R.L. Rivest, and C. Stein. *Introduction to Algorithms*. MIP Press, 3rd edition, 2009.

[6] M. Grabisch, T. Murofushi, and M. Sugeno, editors. *Fuzzy Measures and Integrals. Theory and Applications*. Physica-Verlag, Heidelberg, 2000.

[7] D. Paternain, H. Bustince, J. Fernandez, G. Beliakov, and R. Mesiar. Image reduction with local reduction operators. In *IEEE International Conference on Fuzzy Systems*, pages 1–8, Barcelona, Spain, 2010. IEEE.

[8] D. Paternain, C. Lopez-Molina, H. Bustince, R. Mesiar, and G. Beliakov. Image reduction using fuzzy quantifiers. In P. Melo-Pinto, P. Couto, C. Serodio, J. Fodor, and B. De Baets, editors, *Advances in Intelligent and Soft Computing*, volume 107 of *Advances in Intelligent and Soft Computing*, pages 351–362. Springer, Berlin / Heidelberg, 2012.

[9] T. Wilkin. Image reduction operators based on non-monotonic averaging functions. In *Proc. of the 10th IEEE Int. Conf. on Fuzzy Systems*, pages DOI: 10.1109/FUZZ–IEEE.2013.6622458, Hyderabad, India, 2013.

[10] T. Wilkin and G. Beliakov. Weakly monotone averaging functions. *Submitted*, 2014.

[11] C.T. Zahn. Graph-theoretical methods for detecting and describing gestalt clusters. *IEEE Transaction on Computers*, C20:68–86, 1971.