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A New Monotonic Type-Reducer for Interval Type-2 Fuzzy Sets

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***Abstract*— This paper presents a new defuzzification algorithm for interval type-2 fuzzy sets. The algorithm exploits the fact that we can treat an interval type-2 fuzzy set as two type-2 fuzzy sets. We suggest in this paper that monotonicity is an important property for defuzzifiers and so we provide a definition of monotonicity for type-2 defuzzifiers based on previous work by Runkler. The research reported here shows that our new operator is monotonic and provides a defuzzified value that lies within the interval computed by the popular Karnik-Mendel algorithm.**

I. Introduction

It is well known that defuzzification in interval type-2 fuzzy logic systems is problematic [1], [2]. In particular, the result of inferencing in a type-2 fuzzy logic system is a (usually) non-normal, non-convex type-2 fuzzy sets that by its very nature has a complex structure. To reduce this type-2 set (in either the interval or general case) to a single number is computationally and mathematically demanding. There are a number of algorithms reported in the literature including the exhaustive approach [3], KM [4], sampling [5], geometric[1], [6], the collapsing method [7], alpha-cuts [8], Nie-Tan [9] and Wu-Mendel [10]. They all take different approaches, are of differing complexities and produce different solutions (e.g. [11], [12]). It is our assertion that the complexities of most algorithms mean that to produce a new, simple algorithm is a significant contribution. The results of most of these algorithms is both a final defuzzified value and an interval associated with that value.

There seems to be a view that we need to have this interval associated with final defuzzified value and some authors attribute this to the uncertainty associated with the final defuzzified value. We take the view that the jury is still out on whether this interval actually in any way represents the uncertainty around the defuzzified value. Depending on the algorithm used to arrive at the defuzzified value it may, or may not represent some uncertainty about the defuzzified value. However, given that the most widely adopted algorithm for defuzzification is the Karnik-Mendel algorithm (KM) [4] we thought it would be interesting to see how the result of our algorithm related to the interval provided by the KM algorithm.

The rest of the paper is structured as follows: in Section II we provide the related material necessary to explain our algorithm including the notation and discussion about monotonicity; Section III describes our new algorithm including its properties, and Section IV provides a conclusion and discussion.

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II. Related Material

To enable us to explain our algorithm we provide some definitions. First, we define a type-1 fuzzy set.

Definition 1: A type-1 fuzzy set, A we define as a mapping between the universal set X and an interval $[0, 1]$:

$$\tilde{A} : X \rightarrow [0, 1] \quad (1)$$

The defuzzification algorithm in this paper is for interval type-2 fuzzy sets only. There are a number of definitions in the literature using different notations but we extend the type-1 definition above.

Definition 2: An interval type-2 fuzzy set, $\tilde{A}$ we define as a mapping between the universal set X and an interval set within $[0, 1]$:

$$\tilde{A} : X \rightarrow [0, 1] \rightarrow \{0, 1\} \quad (2)$$

Along the lines of [13] we define the general case of a defuzzifier as:

Definition 3: For a given type-1 fuzzy set, A, we define a defuzzifier as follows where, as usual, X is the universal set:

$$A^{-1} : A \rightarrow X \quad (3)$$

For a type-2 fuzzy set (general or interval) we have the following definition for a defuzzifier for $\tilde{A}$:

$$\tilde{A}^{-1} : \tilde{A} \rightarrow X \times X \quad (4)$$

To explain monotonicity for interval type-2 fuzzy sets we need to provide a definition that shows whether one interval is smaller than another:

Definition 4: Given two intervals A, B and $a, a' \in A$, $b, b' \in B$ then we define

$$A < B \;\equiv\; \forall a \; \exists b | a < b \wedge \forall b' \exists a' | a' \leq b' \quad (5)$$

There are a number of properties that a defuzzifier should have (see [13] for a discussion for the type-1 fuzzy case). In this paper we explore the relationship between the Karnik-Mendel algorithm [4] and ours and what we believe to be the most important property of any defuzzifier - monotonicity[1].

Monotonicity [13] has the following properties:

1) If you decrease the membership grade on the right hand side of the centre then the centre will move to the left
2) If you decrease the membership grade on the left hand side of the centre then the centre will move to the right
3) If you increase the membership grade on the right hand side of the centre then the centre will move to the right

[1]A journal article will explore in detail the type-2 defuzzifiers and the properties as discussed in [13]

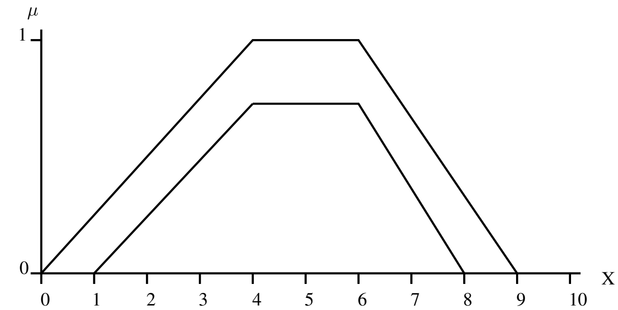

Fig. 1. The Interval Type-2 Fuzzy Set $\tilde{A}$.

4) If you increase the membership grade on the left hand side of the centre then the centre will move to the left

We explain this more formally using the notation above:

$$\tilde{A}'^{-1} < \tilde{A}^{-1} \text{where}$$
$$\tilde{A}' = \{u | u = \tilde{A}(x)\ \forall\ x < \tilde{A}^{-1}, u \leq \tilde{A}(x)\ \forall x > \tilde{A}^{-1}\} \quad (6)$$

$$\tilde{A}'^{-1} > \tilde{A}^{-1} \text{where}$$
$$\tilde{A}' = \{u | u = \tilde{A}(x)\ \forall\ x > \tilde{A}^{-1}, u \leq \tilde{A}(x)\ \forall x < \tilde{A}^{-1}\} \quad (7)$$

$$\tilde{A}'^{-1} > \tilde{A}^{-1} \text{where}$$
$$\tilde{A}' = \{u | u = \tilde{A}(x)\ \forall\ x < \tilde{A}^{-1}, u \geq \tilde{A}(x)\ \forall x > \tilde{A}^{-1}\} \quad (8)$$

$$\tilde{A}'^{-1} < \tilde{A}^{-1} \text{where}$$
$$\tilde{A}' = \{u | u = \tilde{A}(x)\ \forall\ x > \tilde{A}^{-1}, u \geq \tilde{A}(x)\ \forall x < \tilde{A}^{-1}\} \quad (9)$$

So, in this Section we have provided the necessary definitions and notation to enable the description of our algorithm. We also have discussed monotonicity of a type-2 defuzzifier and provided the properties needed for a type-2 defuzzifier to be monotonic.

III. A New Simple Defuzzification Algorithm

It is a widely accepted fact that the inferencing process, prior to defuzzification, of an interval type-2 fuzzy system is identical to the the inferencing process of two independent type-1 fuzzy system operating in parallel [14]. Implementing a type-2 interval system in this way greatly reduces the computational complexity of the system and introduces a simple opportunity to introduce parallel execution if should wish to do so.

In this paper we propose to carry this notion of parallel type-1 systems one step further into the type-reduction phase of the inference process. Here, we propose a new, simple type-reduction algorithm for interval type-2 fuzzy sets. Essentially we treat the upper and lower membership functions as type-1 fuzzy sets and calculate the centroid of each of these to give us an interval. So, given an interval type-2 fuzzy set $\tilde{A}$ we have $\overline{\tilde{A}}$ as the upper membership function and $\underline{\tilde{A}}$.

$$C_L = \overline{\tilde{A}} \wedge \underline{\tilde{A}} \quad (10)$$

$$C_R = \overline{\tilde{A}} \vee \underline{\tilde{A}} \quad (11)$$

$$\tilde{A}'^{-1} = [C_L, C_R] \quad (12)$$

The KM algorithm has been shown to be essentially a Newton-Rapheson root finding approach [15] which finds the maximum (minimum) of a particular function. The functions that contain the left endpoint of the centroid and the right end have values from the upper and lower membership functions. This means when the KM algorithm calculates the left end point by finding the minimum value of this function the value obtained must be less than or equal to the minimum of the centroid of the upper and lower membership functions. Also, the right endpoint obtained by KM is greater than the maximum of the the centroid of the upper and lower membership functions.

An essential property of a type-reducer we believe is that of monotonicity as described above. Runkler [13] stated that the most common type-1 defuzzifiers (centre of area and centre of gravity) satisfy monotonicity. Since our type-reducer relies on type-1 defuzzifiers then our new approach is also monotonic. We also believe the KM algorithm to be monotonic but this is much more complex to demonstrate and is for future work. The left and right endpoint calculations (type-1 centroid) are monotonic and this property allows us to state that the interval obtained by our new method lays within the interval obtained by KM. We now demonstrate this point with an example.

Consider the interval type-2 fuzzy set $\tilde{A}$ depicted in figure 1. We now perform type-reduction on this set which has been discretised in to 11 points starting at 0 with an interval of 1. If we consider the KM algorithm it operates by identifying the minimum of the function $l(x)$ to give the left end point and identifies the maximum of the function $r(x)$ to give the right end point. The functions $l(x)$ and $r(x)$ are depicted in figures 2 and 3 respectively. The domain of these functions is the switch-point used in the KM algorithm, the point at which use of the upper or lower membership function is used in the centroid calculation. Most importantly from the point of view of the current discussion the follow is clear:

- $l(0)$ is the centroid of the upper membership function.
- $l(10)$ is the centroid of the lower membership function.
- $r(0)$ is the centroid of the lower membership function.
- $r(10)$ is the centroid of the upper membership function.

Since the centroids of both the upper and lower membership functions lay on both functions $l(x)$ and $r(x)$ the interval of our new type reducer must lay inside the interval produced by KM. Proof of this is simple to obtain. First consider the left end point of the new type-reducer which lays inside the KM interval.

$$C_l^{KM} \leq \overline{\tilde{A}} \wedge \underline{\tilde{A}} \leq C_r^{KM} \quad (13)$$

We know that C_l^{KM} cannot be greater than $\overline{\tilde{A}} \wedge \underline{\tilde{A}}$ since

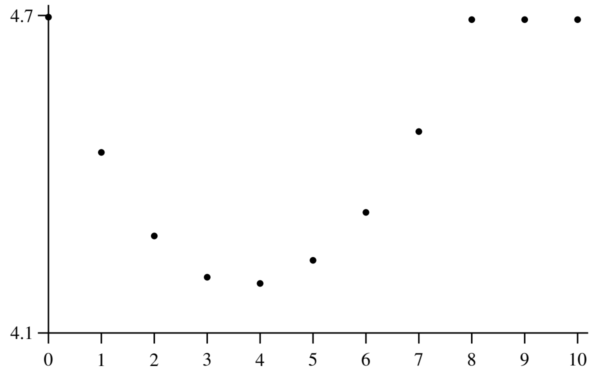

Fig. 2. The Function $l(x)$.

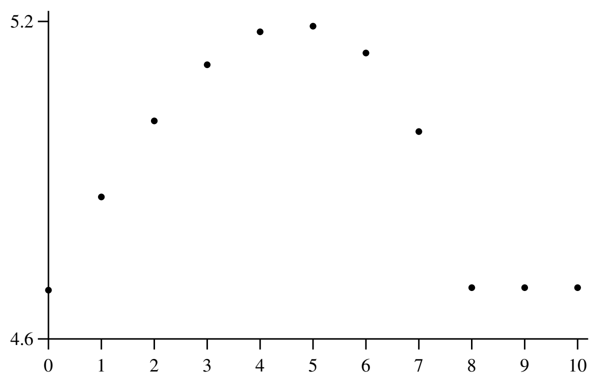

Fig. 3. The Function $r(x)$.

C_l^{KM} is the minimum of a function on which both $\overline{\tilde{A}}$ and $\underline{\tilde{A}}$ lay. We know that C_r^{KM} cannot be less than $\overline{\tilde{A}} \wedge \underline{\tilde{A}}$ since C_r^{KM} is the maximum of a function on which both $\overline{\tilde{A}}$ and $\underline{\tilde{A}}$ lay. Therefore the end points given in equations 10 and 11 must lay inside the KM endpoints.

IV. Results

In order to characterise the properties of our new type-reduction operator we compare the defuzzification with that of range of other defuzzifiers using the experiment carried out in [12]. This experiment requires an output surface to be created for a particular problem. We use the same rule fuzzy inference system used by Coupland and John, given below:

- Rule 1: *IF* x_1 *is* $\tilde{F}_1$ *THEN* y *is* $\tilde{G}_1$
- Rule 2: *IF* x_2 *is* $\tilde{F}_2$ *THEN* y *is* $\tilde{G}_2$

where y is some output variable, $\tilde{G}_1$ and $\tilde{G}_2$ are two interval type-2 fuzzy sets as depicted in Figure 4. We are not concerned with the rule inputs or antecedents. This is because to produce an output surface only the rule consequents and a range rule firing strengths are required. To make the results in this paper comparable to those in [12] we shall use exactly the same parameters to create the output surface, namely

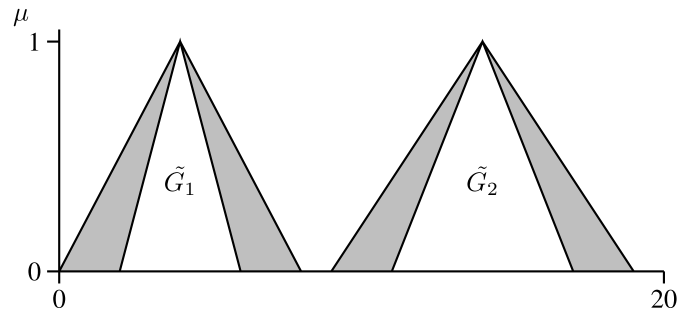

Fig. 4. The Interval Type-2 Fuzzy Sets $\tilde{G}_1$ and $\tilde{G}_2$.

rule firing strengths of 0 to 1 at intervals 0.01 plotted as a contoured surface where the contours have intervals of 0.25 from 3 to 14.

We now give surface plots using our new type-reducer where the rule implication has been performed by the minimum and product t-norms. Figure 5 gives the surface plot from our new type-reducer under minimum whilst figure 6 and figure 7 are the surface plots produced using Karnik-Mendel type-reduction [16] and geometric defuzzification [17], [1] respectively. Figure 8 gives the surface plot from our new type-reducer under minimum whilst figure 9 and figure 10 are the surface plots produced using Karnik-Mendel type-reduction and geometric defuzzification respectively.

We now state our observations from these surface plots. For both minimum and product we find that the general properties of the new defuzzifier follows that of Karnik-Mendel type-reduction. The overall topography of the new approach and Karnik-Mendel appear to be more similar with each other than they are with the geometric defuzzifier. Under product Karnik-Mendel and the new approach appear to be very similar indeed. However, there are subtle differences under minimum. The new approach obtains the extreme values (3 and 14) considerably less often than Karnik-Mendel. This make sense when the operational differences are considered. Karnik-Mendel is using sets (non-triangular) which are more extreme i.e. to the left or right than our new approach. There is another difference which can be observed in the plateau in the top right hand corner of all the surface plots obtained with minimum. The value at this plateau as obtained by our new type-reducer is slightly higher than both the Karnik-Mendel type-reducer and the geometric defuzzifier. We are unable to offer a suggestion on why this should be so. More work will need on our new approach to fully understand it's properties, however it has strong and clear similarities to the properties of Karnik-Mendel type-reduction.

V. Conclusions

Defuzzification algorithms for an interval type-2 fuzzy set have been defined by a number of researchers - all with their strengths and weaknesses. In this paper we have presented a new, simple, algorithm for defuzzifying an interval type-2 fuzzy set that works with the upper and lower membership

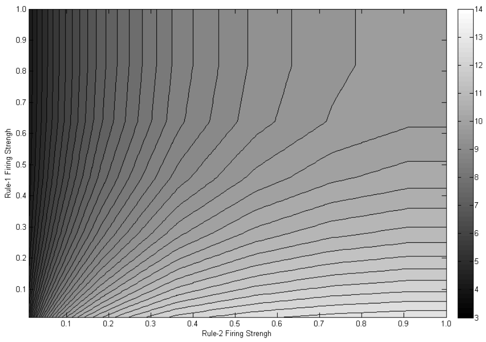

Fig. 5. Output Surface of the New Type-Reduction System Using The Minimum T-norm.

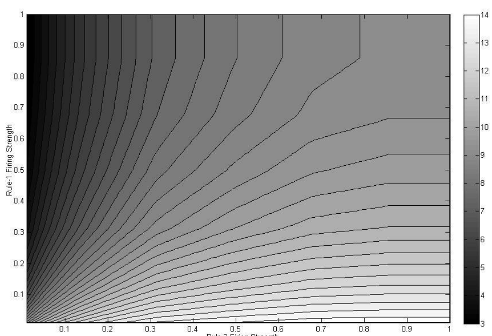

Fig. 6. Output Surface of the Type-Reduction System Using The Minimum T-norm. Reproduced from [17].

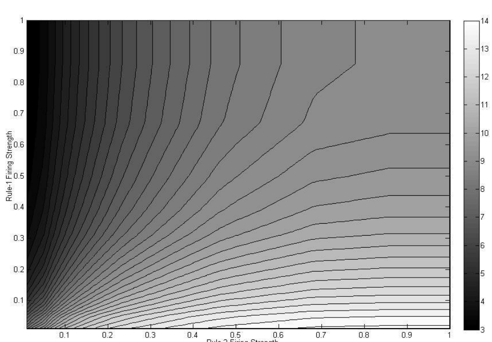

Fig. 7. Output Surface of the Geometric Defuzzfier System Using The Minimum T-norm. Reproduced from [17].

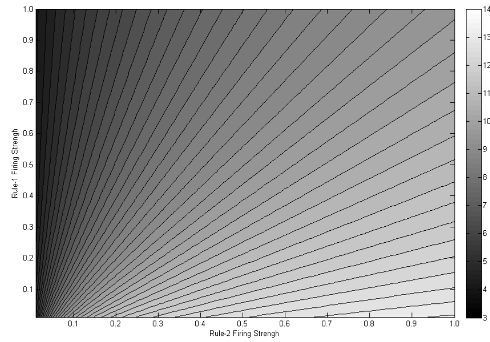

Fig. 8. Output Surface of the New Type-Reduction System Using The Product T-norm.

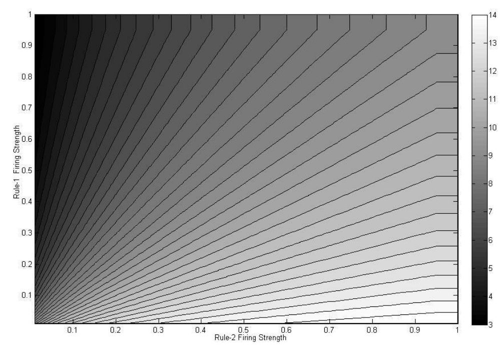

Fig. 9. Output Surface of the Type-Reduction System Using The Product T-norm. Reproduced from [17].

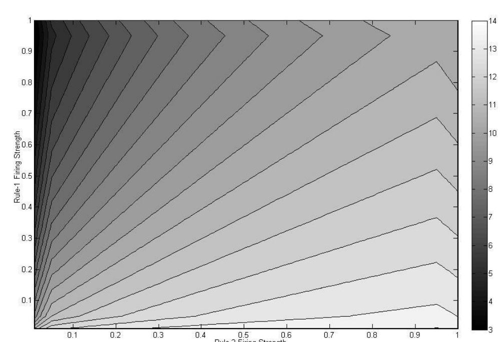

Fig. 10. Output Surface of the Geometric Defuzzfier System Using The Product T-norm. Reproduced from [17].

functions. We show that the algorithm satisfies a monotonicity property and produces an interval that lies within the interval of the defuzzified solutions using the well regarded KM algorithm. Future work will investigate our and others algorithm against the properties one would expect of any defuzzifier.

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