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A New Approach to Improve the Consistency of Linguistic Pair-wise Comparison Matrix and Derive Interval Weight Vector

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***Abstract*—H. Zhang, Q. Zheng, and T. Liu et al. proposed a discrete region based approach to improve the consistency of the pair-wise comparison matrix. The approach is able to significantly improve the consistency of pair-wise comparison matrix without to revise the decision maker's opinion. In the approach, a discrete region matrix is transformed into a set-matrix in which the elements are the real number set. In this paper, a discrete region matrix is transformed into a reciprocal interval matrix. A new iterative searching algorithm (NISA) is proposed to find the pair-wise comparison matrix with approximate optimum consistency from the reciprocal interval matrix. Based on the similarly principle, a new algorithm is proposed to derive the interval weight vector for the reciprocal interval matrix. The key character of this algorithm is that the derived interval weight vector includes the weight vector got by NISA for the same reciprocal interval matrix. In the experiment, five experimental strategies are designed, and the experimental results show that H. Zhang et al. proposed approach and NISA can get approximately similar weight vector according to the same pair-wise comparison matrix used the discrete region.**

Keywords—consistency, interval matrix, region, pair-wise comparison matrix, linguistic term.

I. INTRODUCTION

In multi-criteria decision making (MCDM) problems, decision makers usually use preference relations to express their preferences over each pair of alternatives (or criteria). Because of the uncertainty of real problems and intuitiveness of human judgments, it often happens that the given comparisons by a decision maker are inconsistent each other. The evaluated priority is plausible, while the pair-wise comparison metric does not pass the consistency test. In this case, the transitivity and reciprocity rules are not respected within the pair-wise comparison process (Ishizaka and Lusti [1]). As a result, the interval evaluations are more suitable for representing uncertain information. The consistency of MCDM could be increased by use the interval evaluation pair-wise comparison matrix, and an interval weight vector could be produced [2].

The consistency ratio (CR) is proposed by Saaty and et al to measure the inconsistency of pair-wise comparison matrices. And they introduce the concept of acceptable consistency for CR <0.1[3]. Other researchers also introduce some index to measure the inconsistency [4-11], such as GCI [10], which sums the difference between the ratio of the calculated priorities and the given comparisons. Some methods are given a different consistency definition. For example, G. Zhang, Y. Dong and Y. Xu [11] propose linguistic index measures the consistency degree of linguistic preference relations via a linguistic way. Furthermore, Jaroslav and Petr [8] propose a new consistency index based on the idea of distance of the matrix to special ratio matrix measured by a particular metric. This consistency index is introduced for reciprocal matrix with fuzzy elements. Because of the CR is well known for its widespread use. In this paper, the CR is applied to measure the consistency.

The lack of consistency in decision making with preference relations can lead to inconsistent conclusions. The consistency improving approaches have been widely studied [12]. One can convincingly note that reaching consensus requires flexibility and willingness on a part of each member of the group to adjust his/her original position [13]. In this manner, the decision makers are dispirited to revise their judgments. Such approach does not provide any automated aid, and the decision makers don't know how to improve the consistency. Another method is to develop a heuristic algorithm to improve ordinal consistency by identifying and eliminating intransitivity in multiplicative preference relation matrices [1, 14-16]. Such as Ishizaka and Lusti [1] suggest an expert module to improve the consistency of pair-wise comparison matrices, which detects rule transgressions, explains them, suggests alternatives and gives hints on how to continue the comparison process. It can help the expert to build a consistent matrix or limited the inconsistency within controlled area. Wang and Chin et al [16] propose an approach which looks into decision makers' over all judgments which can be obtained through the aggregation of their direct and indirect judgments. This method can guide the decision maker to make more consistent judgment. But the deducted value may contradict with the decision maker's intention. Many methods are proposed to modify pair-wise comparison matrices so that the revised matrices are of acceptable consistency [4, 11, 12, 17]. Such as M. Xia, Z. Xu

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and J. Chen [12] have proposed a method for improving consistency or consensus of reciprocal [0, 1]-valued preference relations. G. Zhang, Y. Dong and Y. Xu [11] propose an algorithm to improving consistency degree in linguistic preference relations. These approaches attempts to modify the values in the matrix automatically so that the inconsistency is below predefined threshold. Obviously, the adjusted matrix may no longer respect evaluator's intention.

In [18], H. Zhang, Q. Zheng, and T. Liu et al. propose a discrete region-based approach to improve the consistency of the pair-wise comparison. The approach is that it enables the decision makers express their fuzzy cognition with discrete regions rather than specific values. The advantage of this approach is that it not only can improve the consistency of the pair-wise comparison, but also can faithful represent the evaluators' opinions without modifying them. However, this approach does not consider the case in which the region is continued. In this paper, we proposed a new approach which can improve the consistency of pair-wise comparison matrix, while the region is considered as a continue interval. At same time, another approach with the same principle is proposed to derive an interval weight vector according to the reciprocal interval comparison matrix. The interval preference relations include two principal ways, and they are the interval multiplicative preference relations [19-23] and the interval fuzzy preference relations [24-30]. For interval multiplicative preference relations where pairwise comparison matrices consist of interval values, a large body of literature has been developed over the years [22, 25]. Such as Z. Wang and K. W. Li [30] put forward a goal programming approach to deriving interval weights based on a consistent or inconsistent interval comparison matrix. However, what is the relation of the interval weight vector and the exact weight vector derived from the same interval matrix. This paper proposed approach can handle this issue.

As same as in [18], in this paper proposed approach, the linguistic term is used to compare the two criteria, and a linguistic term discrete region is introduced to express the judgment of the decision makers. If the decision maker can give a clear and definite judgment, the discrete region would contain only one linguistic term. Otherwise, they can express their fuzzy cognition with the discrete region containing multiple linguistic terms. The comparing result is stored as a linguistic term set matrix, and this set matrix is transformed into a set-matrix in which the elements are a discrete region of real number [18]. But in this paper, the linguistic term set matrix is transformed into a reciprocal interval matrix. Then, a new iterative searching algorithm (NISA) is proposed to find the pair-wise comparison matrix with approximate optimal consistency from the reciprocal interval matrix, which is considered as the final pair-wise comparison matrix to evaluate the preference of criteria. At same time, a method is proposed to derive the interval weight vector based on the reciprocal interval matrix. The derived interval weight vector incudes the weight vector which is computed based on the searched matrix by NISA. The preference orders of them are same.

In the experiment, ten volunteers are invited to evaluate the heights of 7 people who are a few personages in the world and some teachers in one laboratory using the linguistic term discrete regions. All participants get a comparison matrix with much better consistency than the stochastic choose matrix in reciprocal interval matrix. To all evaluation results, the weight vectors are included in the interval weight vector derived based on same reciprocal interval matrix in this paper proposed approach. The experimental results also show that the final derived weight vector based on the discrete region of real number in [18] is approximately similar with the one based on reciprocal interval matrix in this paper.

The main contribution of this paper is that: first, proposed a new iterative searching algorithm to search a matrix with approximate optimum consistency in reciprocal interval matrix. Second, based on similar theory, a method derived interval weight vector according to reciprocal interval matrix is proposed. Third, 5 experimental strategies are designed in the experiment to verify the efficiency and key character of the proposed methods in this paper. The experimental results show that the proposed methods in this paper and the method proposed in [18] can get the approximately [31-33] similar weight vector according to the same discrete region matrix.

The rest of this paper is organized as follows. The preliminary concepts are briefly reviewed in Section II. In section III, we present the two approaches. The experiment and analysis are shown in Section IV. Section V is the conclusion and future work.

II. Preliminary Knowledge

In this section, we review the main concepts of linguistic, the 2-tuple linguistic model, and the interval pair-wise comparison [23] of the analytic hierarchy process (AHP) [34], which is the basis of our approach.

A. Linguistic Scale

The scale used in analytic hierarchy process AHP [34] can be decomposed into two parts [35]: linguistic scale and numerical scale. Let $S=\{s_\alpha \mid \alpha=0,1,2,\ldots,g\}$ be a linguistic term set with odd cardinality. The term s_α represents a possible value for a linguistic variable. We also call this linguistic term set S as the linguistic scale [35]. The AHP linguistic scale, provided in Saaty, is nine gradations [35]. Miller [36] demonstrated that an individual cannot simultaneously compare more than 7 ± 2 objects without confusion.

The numerical scale of AHP can be described as follows[35]:

$\left\{f_i, f_1, \frac{1}{f_i}\right\}, i=2,3,\cdots,g$, where $f_1=1$, $f_{i+1}>f_i>1$, $(5 \le g \le 9)$.

The value of $f_i\,(i=1, 2, \ldots, 9)$ corresponds to the ith gradation of the AHP linguistic scale.

B. 2-tuple fuzzy linguistic representation model

Herrera and Mart´ınez[31-33] contributed a 2-tuple fuzzy linguistic representation model.

Let $S=\{s_k \mid k=0,1,2,\ldots,g\}$ be a linguistic term set with odd cardinality, and the linguistic term set satisfies the following characteristics:

a) The set is ordered: $s_i > s_j$ if and only if $i > j$.

b) There is a negation operator: $Neg(s_i) = s_j$ such that $j = g - i$.

Definition 1 [32]: Let $\beta \in [0, g]$ be a number in the granularity interval of the linguistic term set S, and let $i = round(\beta)$ and $\alpha = \beta - i$ be two values such that $i \in [0, g]$ and $\alpha \in [-0.5,\ 0.5)$. Then, *α* is called a symbolic translation, with round being the usual rounding operation.

The 2-tuple linguistic model [31-33] represents the linguistic information by means of 2-tuples (s_i, α_i), where $s_i \in S$ and $\alpha_i \in [-0.5,\ 0.5)$ [35]. This linguistic representation model defines a function with the purpose of making transformations between linguistic 2-tuples and numerical values.

Definition 2 [32]: Let S is a linguistic term set and $\beta \in [0, g]$ be a value representing the result of a symbolic aggregation operation; then, the 2-tuple that expresses the equivalent information to β is obtained with the following function:

$\Delta:\ [0, g]\ \rightarrow S \times\ [-0.5,\ 0.5)$

$\Delta(\beta) = (s_i, \alpha)$ with $s_i,\ i = round(\beta)$

$\alpha = \beta - i, \alpha \in [-0.5,\ 0.5)\cdot$

Clearly, Δ is one to one. For convenience, its range is denoted as S. Then, Δ has an inverse function with $\Delta^{-1}: \mathrm{S} \rightarrow [0, \mathrm{g}],\ \Delta^{-1}(\mathrm{s}_i, x) = i + x$.

More details about 2-tuple model can be seen in [31-33].

C. Interval pair-wise comparison matrix

An interval pair-wise comparison matrix can be represented as $\overline{A} = ([a_{ij}^l, a_{ij}^h])_{n \times n}$, where $0 < a_{ij}^l \le a_{ij}^h$, $a_{ji}^l = 1/a_{ij}^h$, $a_{ji}^h = 1/a_{ij}^l$. About the above interval comparison matrix, we have the following definitions:

Definition 3 [23]. Given an interval comparison matrix $\overline{A} = ([a_{ij}^l, a_{ij}^h])_{n \times n}$ with $a_{ij}^l \le a_{ij} \le a_{ij}^h$ and $a_{ii}^l = a_{ii} = a_{ii}^h = 1$, for $i, j = 1, \cdots, n$. If the following convex feasible region $S_w = \{w = (w_1, \cdots, w_n) \mid a_{ij}^l \le w_i / w_j \le a_{ij}^h, \sum_{i=1}^n w_i = 1, w_i\ > 0\}$ is nonempty, then $\overline{A}$ is said to be a consistent interval comparison matrix, otherwise, $\overline{A}$ is said to be inconsistent.

Let $w = (w_1, \cdots, w_n)$ be a weight vector, on which two different types of constraints may be imposed. One is the additive constraint, namely $\sum_{i=1}^n w_i = 1$. The other is the multiplicative constraint, i.e. $\prod_{i=1}^n w_i = 1$, which is equivalent to $\sum_{i=1}^n \ln w_i = 0$. Such a multiplicative constraint is used in this paper.

In order to compare or rank global interval weights, a preference ranking approach [23] is proposed to compare the weights of criteria or rank alternatives in a multiplicative aggregation process.

TABLE 1 LINGUISTIC SCALE

Linguistic term	Linguistic scale	Linguistic term	Linguistic scale
S_0	Absolutely less important (AbL)	S_5	Weakly important (Wk)
S_1	Strongly less important (StL)	S_6	Essentially important (Es)
S_2	Essentially less important (EsL)	S_7	Strongly important (St)
S_3	Weakly less important (WkL)	S_8	Absolutely important (Ab)
S_4	Equally important (Eq)		

Let $a = [a_1,\ a_2]$, and $b = [b_1,\ b_2]$ be two interval weights. The degree of an interval weight being greater than another one is defined as the degree of preference [23].

Definition 4[23]. The degree of preference of *a* over *b* (or *a* > *b*) is defined as

$$P(a > b) = \frac{\max(0,\ a_2 - b_1) - \max(0,\ a_1 - b_2)}{(a_2 - a_1)\ +\ (b_2 - b_1)}.$$

In [23], some useful properties about the degree of preference of *a* over *b* is summarized.

III. PROPOSED ALGORITHMS

First, we simple describe the discrete region-based evaluation approach proposed in [18].

A. Discrete Region-Based Approach Review

For illustrating this approach, we introduce a predefined linguistic term set (TABLE 1)[35].

Definition 5[18]: Let *S* be a linguistic term set, $S = \{s_k \mid k = 0, 1, 2, \cdots, g\}$. The discrete region of linguistic term is an ordered finite subset of *S*. Let Ψ be a discrete region of *S*, which has follow properties:

1). $\Psi \subseteq S$;

2) If Ψ is a discrete region, then $\Psi = \{s_i, s_{i+1}, \cdots, s_j\}$ $(i < j)$, and $s_i < s_{i+1} < \ldots < s_j$.

3) The discrete region Ψ express as $[s_i, s_j]$ $(i \le j,\ i \in [0, g], j \in [0, g])$.

We describe the details of this approach used a simple example.

Example 1: Three quality features of software system are "efficiency" (C1), "reliability" (C2) and "functionality" (C3).

First, using above concepts, the decision maker can give a pair-wise comparison matrix, as shown in TABLE 2.

The value at [C2, C3] is s_2, which indicates reliability is "essentially less important" than functionality. The discrete region [s_5, s_7] at [C1, C2] states that efficiency is "weakly, essentially" or "strongly more important" than reliability. Similarly, discrete region [s_2, s_3] at [C1, C3] states that efficiency is "strongly", "essentially" or "weakly less

TABLE 2 PAIR-WISE COMPARISON MATRIX

	C1	C2	C3
C1	[s_4]	[s_5,s_7]	[s_2,s_3]
C2	[s_1,s_3]	[s_4]	[s_2]
C3	[s_5,s_6]	[s_6]	[s_4]

important" than functionality. Finally, the lower triangular matrix is inferred based on the up triangular.

Second, the comparison matrix which elements are represented as the discrete regions is transformed into a set-matrix [18] by use the 2-tuple model. In [18], the used scale function is the geometrical scale [37] : $f(s)=(\sqrt{c})^{\Delta^{-1}(s)-4}$, where c=2. As a result, the TABLE 2 is transformed into the follow set-matrix.

$$\begin{bmatrix} 1:[1.000] & 3:[1.414,2.828] & 2:[0.500,0.707] \\ 3:[0.354,0.707] & 1:[1.000] & 1:[0.500] \\ 2:[1.414,2.000] & 1:[2.000] & 1:[1.000] \end{bmatrix}$$

Third, an iterative searching algorithm (ISA) is proposed to search a matrix with the approximate consistency, and the weight vector is derived based on this matrix by use the Logarithmic least-squares method (LLM) [38]. The procedure could be illustrated follow.

An initial matrix is stochastic selected from the set-matrix, such as the follow matrix A_0.

$$A_0=\begin{bmatrix} 1.000 & 2.000 & 0.500 \\ 0.707 & 1.000 & 0.500 \\ 2.000 & 2.000 & 1.000 \end{bmatrix},\ A_0^*=\begin{bmatrix} 1.000 & 1.414 & 0.630 \\ 0.794 & 1.000 & 0.445 \\ 1.782 & 2.245 & 1.000 \end{bmatrix}$$

The weight follow matrix $A_0^*=(\omega_i^{A_0}/\omega_j^{A_0})_{n\times n}$ of matrix A_0 is computed, and CR=0.060346. The next matrix $A_1=(a_{ij}^1)_{n\times n}$ is searched from the set-matrix, and its elements make that the distance $\left|\ln a_{ij}^1-\ln(\omega_i/\omega_j)\right|$ ($1\le i\le n, i\le j\le n$) is the minimum. Such as, $|\ln 0.500-\ln 0.630|>|\ln 0.707-\ln 0.630|$, then $a_{13}^1=0.707$.

$$A_1=\begin{bmatrix} 1.000 & 1.414 & 0.707 \\ 0.707 & 1.000 & 0.500 \\ 1.414 & 2.000 & 1.000 \end{bmatrix}$$

Iterating this process until $A_{k+1}=A_k$, then A_k is the matrix searched by ISA. In this example, the optimum matrix is A_1, and it is a consistency matrix, CR=0.

B. New Iterative Searching Algorithm

In the discrete region-based approach [18], the linguistic term discrete region matrix is transformed into a set-matrix. In this paper, we consider the case, in which the discrete region matrix is transformed into a reciprocal interval matrix. For example, the TABLE 2 may be transformed into a follow interval matrix.

$$\begin{bmatrix} [1.000,1.000] & [1.414,2.828] & [0.500,0.707] \\ [0.354,0.707] & [1.000,1.000] & [0.500,0.500] \\ [1.414,2.000] & [2.000,2.000] & [1.000,1.000] \end{bmatrix}.$$

The algorithm to search a matrix with approximate optimum consistency from the reciprocal interval matrix is proposed, and named as New Iterative Searching Algorithm (NISA).

Firstly, the useful notations are introduced.

Let an interval matrix is presented as $\overline{A}=([a_{ij}^l,a_{ij}^h])_{n\times n}$, the matrix sequence which is generated by NISA is denoted as $A^{(k)}(k=0,1,\cdots m)$. The prioritization method is the Logarithmic least-squares method (LLM) [37]. The weight vector of matrix $A^{(k)}$ is denoted as $\{\omega_i(A^k)\,|\,i=1,2,\cdots,n\}$. Let $a_{ij}^{(k)*}=\omega_i(A^k)/\omega_j(A^k)$, the weight follow matrix of matrix $A^{(k)}$ is denoted as $A^{(k)*}=(a_{ij}^{(k)*})_{n\times n}$. The consistency index of matrix $A^{(k)}$ is denoted as $CR(A^k)$.

The steps of NISA are presented as follow.

Step1. Chose a matrix from the given reciprocal interval matrix $\overline{A}=([a_{ij}^l,a_{ij}^h])_{n\times n}$, and it is presented as $A^{(0)}=(a_{ij}^{(0)})_{n\times n}$, where $a_{ij}^{(0)}=\dfrac{a_{ij}^l+a_{ij}^h}{2}$ ($i\le j,\ i,j=1,2,\cdots,n$), and $a_{ji}^{(0)}=1/a_{ij}^{(0)}$.

Step2. For matrix $A^{(k)}$ $(k=0,1,2,\cdots)$, calculating the following parameters:

$$\omega_i(A^k)=\prod_{j=1}^{n}a_{ij}^k \qquad (1)$$

$$A^{(k)*}=(a_{ij}^{(k)*})_{n\times n} \qquad (2)$$

$$CR(A^{(k)})=\frac{\lambda_{\max}^{A^{(k)}}-n}{(n-1)\times RI} \qquad (3)$$

Step3. For $1\le i\le j\le n$, if $a_{ij}^l\le a_{ij}^{(k)*}\le a_{ij}^h$, let $a_{ij}^{(k+1)}=a_{ij}^{(k)*}$. Otherwise, if $a_{ij}^{(k)*}<a_{ij}^l$, let $a_{ij}^{(k+1)}=a_{ij}^l$, and if $a_{ij}^{(k)*}>a_{ij}^h$, and let $a_{ij}^{(k+1)}=a_{ij}^h$. Finally, if $(i>j)$, then assignment $a_{ji}^{(k+1)}=1/a_{ij}^{(k+1)}$.

Step4. Let $A^{(k+1)}=(a_{ij}^{(k+1)})_{n\times n}$, and calculated the consistency ratio $CR(A^{(k+1)})$. If $CR(A^{(k+1)})\ne CR(A^{(k)})$, then $A^{(k)}=A^{(k+1)}$. The algorithm is gone to step 2. Otherwise, if $CR(A^{(k+1)})=CR(A^{(k)})$, the algorithm is finished.

The final matrix $A^{(k)}$ in step4 is the one with the approximate optimal consistency $CR(A^{(k)})$.

If $\{A^{(k)}\}(k=0,1,\cdots,m)$ is the matrix sequence which is generated by NISA algorithm, then we can get $CR(A^{(k+1)})<CR(A^{(k)})$. This conclusion has been proofed based on matrix theory.

C. Derived Interval Weight Vector Algorithm

After the linguistic term discrete region matrix is transformed into a reciprocal interval matrix, the interval weight vector is needed to be derived. The algorithm derived interval weight vector is presented in the following, and named as Iterate Search Optimum Interval Weight vector Algorithm (ISOIWA).

Step1 Initializing $k=0$, let $\overline{A}_k=\overline{A}$, and $\overline{A}_k$ is represented as

TABLE 3 THE RESULTS FOR ALL EXPERIMENTAL STRATEGIES ABOUT A LINGUISTIC TERM REGION MATRIX

		Weight 0	Weight 1	Weight 2	Weight 3	Weight 4	Weight 5	Weight 6
Interval weight vector	Initialize interval weight vector	[0.609507, 0.905724]	[0.580065, 0.905724]	[1.160129, 1.485994]	[0.951695, 1.561418]	[1.640671, 2.208179]	[0.475848, 0.609507]	[0.905724, 1.414214]
	Optimum interval weight vector	[0.710688, 0.749785]	[0.749027, 0.876021]	[1.219014, 1.277010]	[1.094095, 1.200677]	[1.791425, 1.891886]	[0.517476, 0.580065]	[1.076549, 1.277010]
Weight vector	S1	0.742997	0.820335	1.219014	1.10409	1.811447	0.580065	1.160129
	S2	0.736705	0.79043	1.259863	1.12589	1.877776	0.573692	1.123832
	S3	0.733545	0.817347	1.230027	1.120668	1.841471	0.572566	1.147584
	S4	0.737761	0.814554	1.227666	1.111926	1.837254	0.580065	1.143834
	S5	0.737761	0.814554	1.227666	1.111926	1.837254	0.580065	1.143834

$\overline{A}_k = ([a_{ij}^l(k), a_{ij}^h(k)])_{n\times n}$. Let $A_k^L = (a_{ij}^l(k))_{n\times n}$, $A_k^H = (a_{ij}^h(k))_{n\times n}$.

Step2 Compute the weight vectors for two matrixes $A_k^L = (a_{ij}^l(k))_{n\times n}$ and $A_k^H = (a_{ij}^H(k))_{n\times n}$ (Based on LLM [37]), and they are represented as $\varpi^L(k) = \{\omega_1^l(k), \omega_2^l(k), \cdots, \omega_n^l(k)\}$, $\varpi^H(k) = \{\omega_1^h(k), \omega_2^h(k), \cdots, \omega_n^h(k)\}$ respectively.

Step3 Let $v_{ij}^l(k) = \omega_i^l(k) \big/ \omega_j^h(k)$ and $v_{ij}^h(k) = \omega_i^h(k) \big/ \omega_j^l(k)$, $1 \le i \le n$, $1 \le j \le n$. A new reciprocal interval matrix is get, and it is presented as $\overline{V}(k) = ([v_{ij}^l(k), v_{ij}^h(k)])_{n\times n}$.

Step4 If $(a_{ij}^l \ge v_{ij}^h(k))$, then $a_{ij}^l(k+1) = a_{ij}^l$, and $a_{ij}^h(k+1) = a_{ij}^h(k)$. If $(a_{ij}^h \le v_{ij}^l(k))$, then $a_{ij}^l(k+1) = a_{ij}^l(k)$, and $a_{ij}^h(k+1) = a_{ij}^h$, else $a_{ij}^l(k+1) = \max\{a_{ij}^l, v_{ij}^l(k)\}$, $a_{ij}^h(k+1) = \min\{a_{ij}^h, v_{ij}^h(k)\}$. Let $\overline{A}_{k+1} = ([a_{ij}^l(k+1), a_{ij}^h(k+1)])_{n\times n}$, and computing $\delta_1 = \max_{1\le i,j\le n}\{|a_{ij}^l(k+1) - a_{ij}^l(k)|\}$, $\delta_2 = \max_{1\le i,j\le n}\{|a_{ij}^h(k+1) - a_{ij}^h(k)|\}$. If $\delta_1 > 0$ or $\delta_2 > 0$, then let $\overline{A}_k = \overline{A}_{k+1}$, and the algorithm is gone to step2. Else, the algorithm is stop. The interval weight vector derived by this algorithm is follow:

$\varpi = \{[\omega_1^l(k), \omega_1^h(k)], [\omega_2^l(k), \omega_2^h(k)], \ldots, [\omega_n^l(k), \omega_n^h(k)]\}$.

IV. EXPERIMENT

A. Experiment 1

a) Definition

In the experiment, ten volunteers are invited to evaluate the heights of 7 people who are a few personages in the world and some teachers in one laboratory. The volunteers known well these people and can simply estimate their heights. All researchers are asked to give a linguistic term discrete region pair-wise comparison matrix for the heights of 7 people. In the experiment, the got linguistic term discrete region matrices are transformed into the reciprocal interval matrix and the set-matrix [18] respectively.

After the set-matrix and reciprocal interval matrix are got, some experimental strategies are designed to verify the efficiency and character of the algorithms proposed in this paper. First, ISA is used to search the matrix with the approximate optimum consistency from the set-matrix, and deduce the weight vector. This experimental strategy is called S1. Second, NISA is used to search the matrix with the approximate optimum consistency in the reciprocal interval matrix and compute the weight vector. This strategy is called S2. Third, ISOIWA is used to derive an interval weight vector, and at same time, a new interval matrix is got. Next, a matrix with the approximate optimum consistency is searched by NISA from the new interval matrix. This experimental strategy is called S3. Fourth, the starting matrix is set as the matrix which is searched from the corresponding set-matrix by ISA, and then the NISA is used to search the matrix with the approximate optimum matrix in reciprocal interval matrix, and compute the weight vector. This experimental strategy is called S4. Fifth, the reciprocal interval matrix is set as the final interval matrix which is got at deriving the interval weight vector, and the starting matrix is the same matrix in fourth strategy. The NISA is used to search the matrix with the approximate optimum consistency, and compute the weight vector. This strategy is called S5.

b) Experimental Results Analysis

In experiment, we get ten linguistic term discrete region comparison matrices. For each linguistic term discrete region matrix, it can be transformed into a set-matrix and a reciprocal interval matrix, and the 5 experimental strategies are used to compute the weight vectors and the interval weight vector. For example, Fig.1 is a linguistic term discrete region matrix.

The results for all experimental strategies about this linguistic term discrete region matrix are shown in table 3. In table 3, the initialize interval weight vector is the one derived

TABLE 4 CR FOR ALL EXPERIMENTAL STRATEGIES

No.	S1	S2	S3	S4	S5
1	0.076208	0.077116	0.075452	0.075546	0.075452
2	0.092742	0.093501	0.092583	0.092506	0.092506
3	0.107555	0.108285	0.105107	0.105553	0.105107
4	0.082859	0.083691	0.083691	0.081876	0.081477
5	0.067010	0.068179	0.066669	0.066703	0.066669
6	0.070203	0.071348	0.068794	0.069173	0.068794
7	0.093818	0.094433	0.092802	0.092933	0.092802
8	0.084092	0.087781	0.083970	0.083975	0.083970
9	0.113185	0.116485	0.112937	0.112949	0.112937
10	0.087050	0.086691	0.085943	0.085988	0.085943

TABLE 5 THE AVERAGE ITERATION TIMES FOR ALGORITHMS

order	5	6	7	8	9	10	11	12	13	14	15	16
NISA	5.94	5.78	5.65	5.49	5.27	5.23	5.07	4.90	4.78	4.74	4.66	4.59
ISOIWA	21.51	22.09	22.39	22.43	21.00	20.74	19.96	19.40	19.00	18.51	18.07	17.69

in ISOIWA from the original reciprocal interval matrix, and the optimum interval weight vector is the interval vector which is got at the end of ISOIWA. Weight$i$ $(i=0,2,\cdots,6)$ is the components of the interval weight vector and the weight vector respectively. From table 3, we see that each interval of the initialize interval weight vector includes the corresponding interval of the optimum interval weight vector. The components of weight vectors for all experimental strategies are located in the corresponding intervals of the optimum interval weight vector, and they are approximately similar with each other. We calculate the cosine similarities for the weight vector of S1 comparing with the others. The minimum of the cosine similarities is 0.9996. This shows that the used approaches in the experiment are convergence. Furthermore, the preference order of components of the optimum interval weight vector is same with the order of components of the corresponding weight vectors computed by used 5 experimental strategies. For example, to the optimum interval weight vector and weight vectors in the TABLE 3, the preference orders of the weight vectors for 5 experimental strategies are that weight 4>weight2> weight6> weight3> weight1> weight0> weight5. According to the definition 4 in section preliminary knowledge, we can deduce that the preference order of the optimum interval weight vector is $\text{weight }4 \overset{100\%}{\succ} \text{weight}2 \overset{77.56\%}{\succ} \text{weight}6 \overset{59.57\%}{\succ} \text{weight}3 \overset{100\%}{\succ} \text{weight}1 \overset{99.54\%}{\succ} \text{weight}0 \overset{100\%}{\succ} \text{weight}5$. This is same with the preference order of the corresponding weight vectors.

The approximate optimum consistencies got by the five experimental strategies for all linguistic term discrete region matrices in the experiment are shown in table 4. The "No." is the sequence of the researcher invited in the experiment. To each linguistic term discrete region matrix, the approximate optimum consistencies (corresponding to a row in table 4) got by the five experimental strategies are very close to each other, and the standard variances about them is computed. The computed results about the standard variances are shown in Fig.2. The maximum of the standard variance is 0.001512.

B. Experiment 2

$$\begin{bmatrix} [s_4] & [s_6,s_8] & [s_2] & [s_0,s_4] & [s_2,s_3] & [s_2,s_3] & [s_2] \\ & [s_4] & [s_1,s_3] & [s_2,s_2] & [s_5] & [s_5,s_6] & [s_2,s_4] \\ & & [s_4] & [s_5,s_7] & [s_0,s_1] & [s_8] & [s_3] \\ & & & [s_4] & [s_0] & [s_6,s_7] & [s_6,s_7] \\ & & & & [s_4] & [s_7] & [s_4,s_8] \\ & & & & & [s_4] & [s_0,s_2] \\ & & & & & & [s_4] \end{bmatrix}$$

Fig.1 A DISCRETE REGION MATRIX FOR LINGUISTIC TERM

In this experiment, 1000 random linguistic comparison matrices are generated for each order matrix, and the orders of matrix are from 5 to 16. Each linguistic comparison matrix is transformed into an interval reciprocal matrix, and NISA and ISOIWA are used to search the approximate optimal matrix and the interval weight vector respectively. The iterate times of NISA and ISOIWA is counted, and the result is shown in TABLE 5.

From the results, we can see that the averages of iteration times are a constant. The iteration times of NISA are less than the order of matrix. The iteration times of ISOIWA are about 20 for all order matrices. As a result, the computing performance of two algorithms is good.

V. CONCLUSION AND FUTURE DISCUSSION

In this paper, we propose two new approaches that are able to improve the consistency of the pair-wise comparison reciprocal matrix and derive the interval weight vector for reciprocal interval matrix, respectively. The main contributions of our approaches include: transforming the decision maker's linguistic term region pair-wise comparison matrix into the reciprocal interval matrix, this enables the decision makers express their fuzzy cognition with linguistic term discrete region rather than specific values. As a result, it can increase the flexibility and reduce the difficulty of judgment by the decision maker making. Next, we design a new algorithm (NISA) to search a matrix with approximate optimal consistency from the reciprocal interval matrix. This matrix with approximate optimal consistency not only represents the decision maker' opinion, but also has well consistent. Furthermore, we proposed another algorithm to derive the interval weight vector for the reciprocal interval matrix. This algorithm is based on the same theory with NISA. The interval weight vector derived by this algorithm includes the weight vector by NISA derived to the same reciprocal interval matrix. Finally, in the experiment, 5 experimental strategies is designed, and the results show that the weight

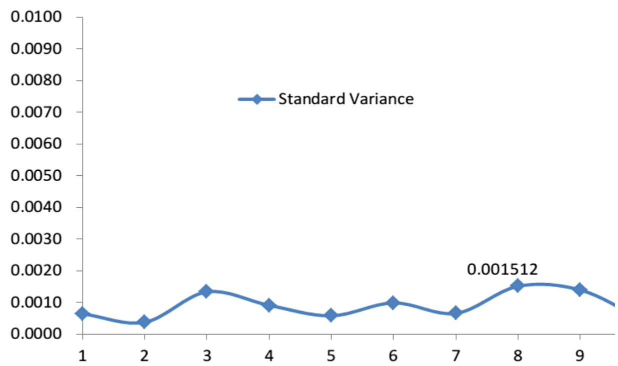

Fig.2 STANDARD VAVIANCE OF THE APPROXIMATE OPTIMUM CONSISTENCY FOR 5 EXPERIMENTAL STRATEGIES

vectors by ISA and NISA derived according to the set-matrix and corresponding reciprocal interval matrix are very similarly with each other. These two weight vectors are also included in the interval weight vector derived from the same reciprocal interval matrix. Through the random experiment, we know that the two algorithms have a good computing performance.

In the future, we will consider other consistence indexes proposed in literatures. We believe NISA can be adapted to improve these consistency indices as well. The application of our approach is also an importance problem, we will further research.

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