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L-Fuzzy Inference

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***Abstract*—In this paper, we present a complete inferencing framework based on L-fuzzy sets, comprising fuzzification, inferencing itself, and both linguistic and numeric defuzzification strategies. We present the algorithms for each step, and then present a range of worked examples to illustrate the methods. Finally, we compare the results with similar examples which carry out 'standard' Mandani-style inference. To the best of our knowledge, this is the first time that practical algorithms for complete L-fuzzy inference have been presented.**

***Index Terms*—L-Fuzzy sets, fuzzy inference systems, fuzzification, defuzzification, similarity.**

I. Introduction

Two years after Zadeh's original 1965 paper on fuzzy sets, Goguen (one of Zadeh's students) published a paper outlining the concept of L-fuzzy sets [1]. Goguen pointed out that the codomain of a fuzzy set could be any transitive partially ordered set (poset), denoted by L, rather than the unit interval $[0, 1]$ as per Zadeh's original definition, and he termed a fuzzy set with this more general definition an L-fuzzy set. Goguen went on to state that restricting the codomain to be a complete lattice, with distributivity, was probably more useful in practice and considered simply ordered sets as special cases (albeit still being generalisations of standard fuzzy sets). Although Goguen proposed many theoretical considerations and provided proofs for some of these, he did not provide practical algorithms for performing inference.

Whilst Zadeh's seminal papers in 1975 [2] considered inference processes, it was Mamdani who first provided a practical algorithm for carrying out an approximate form of inference for standard fuzzy sets [3]. Despite the fact that Mamdani's approach is pragmatic in the way it implements inferencing (essentially blurring the distinction between logical implication and conjunction), the methodology has become the most widely implemented form of fuzzy inference in practical algorithms. Of course, other inferencing methodologies have also been proposed (including TSK inference, ANFIS and others) along with practical algorithms for implementation. However, despite attracting some subsequent theoretical interest such as from Dubois and Prade [4], Wang and He [5], and others, practical inferencing methodologies for L-fuzzy sets, akin to Mamdani type inference for fuzzy sets, have yet to be proposed. Partially due to this lack of practical inferencing algorithms, L-fuzzy sets have perhaps not attracted as much interest as they might.

We feel that more attention should be given to the use of L-fuzzy sets featuring posets or simply ordered sets as membership grades, particularly as more natural expressions of modelling human reasoning (as opposed, for example, to fuzzy control). As an example, take the expression of people's ages. It is quite natural for a person to describe the membership of someone of 60 years of age as being 'somewhat' in the set of old people, describing someone of 70 as being 'probably' old, and someone of 90 years as being 'definitely' old, without providing clear indications of numeric values of such memberships.

Some authors, including ourselves, have resorted to utilising type-2 fuzzy sets (both interval type-2 and general) to represent the vagueness in such numeric membership values, but this still requires some numerical representation of membership on the unit interval [6]. In many cases, we know nothing about the order of membership grades other than that, for example, 'not at all' is less than 'somewhat', which in turn is less than 'definitely'. Putting arbitrary numeric values onto these labels is meaningless (or perhaps even misleading). It may be sufficient, more natural and, in some cases, a better reflection of reality to use simply ordered sets for these membership grade labels. Hence the attraction of L-fuzzy sets. Whilst we note that posets are the most general codomain of L-fuzzy sets, and may be useful in some contexts, we restrict our consideration to codomains which are simple ordered sets.

This paper continues as follows. In Section II, important theoretical notions pertaining to L-fuzzy sets and properties thereof are reviewed, followed by the proposal of some novel manipulations. In Section III, we then present a complete practical methodology, together with algorithmic details, necessary for carrying out L-fuzzy inference akin to Mamdani inference. In Section IV, we describe similar methods and algorithms for defuzzification. Once the methods and algorithms are described, in Section V we present a complete worked example to illustrate the methods in practice, comparing and contrasting with standard approaches. Finally, we discuss the implications of this work in Section VI and then present conclusions and opportunities for future work in Section VII.

II. L-Fuzzy Sets

Definition 1 An L-fuzzy set $\breve{A}$ on a set X is a function $\breve{A}: X \to L$, where L is a poset [1]. □

Remark 1 For the remainder of this paper, we will restrict L to being a simply ordered set. We write A rather than $\breve{A}$ to simplify notation where the context is clear. To distinguish this from the general case, we shall refer to a simply ordered

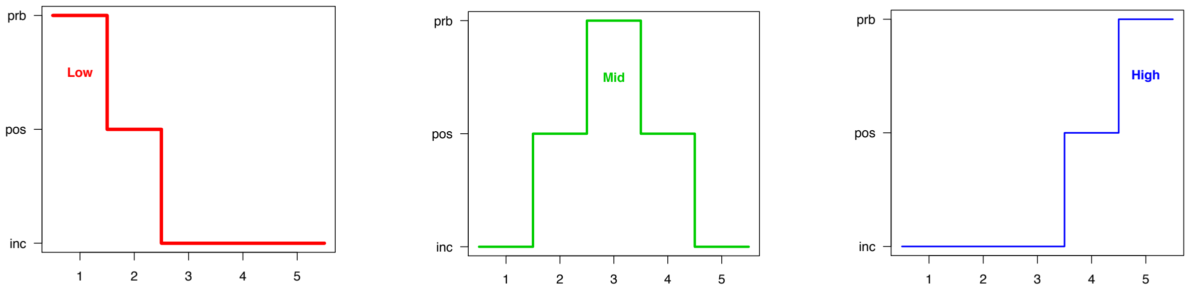

Fig. 1. Examples of L-fuzzy sets *Low*, *Mid*, and *High*. Note that the labels on the y-axis are arbitrary.

codomain as Y. That is, we will restrict the definition such that an L-fuzzy set A on a set X is a function $A : X \to Y$, where Y is a simply ordered set. Note that Y will be discrete and finite. □

Given that we are concerned with creating practical methodologies with algorithmic implementations, we will also restrict X to be a discrete, finite ordered set. That is, all L-fuzzy sets will have a discrete and finite universe of discourse. For cases where the universe of discourse is conceptually infinite (such as defined on continuous subsets of $\Re$), the universe will be discretised for representation and manipulation. We now introduce some notation.

Definition 2 Let A be a specific L-fuzzy set, $A : X^A \to Y^A$. X^A is the domain of A, discretised into m points, with elements x_i^A $(i = 1 \ldots m)$:

$$X^A = (x_1^A, x_2^A, \ldots, x_m^A)$$

where the parentheses are use to denote an ordered set. Also, let Y^A be the codomain of A, discretised into n points, with elements y_j^A $(j = 1 \ldots n)$:

$$Y^A = (y_1^A, y_2^A, \ldots, y_n^A)$$

then the specific L-fuzzy set A, with elements $a_i \in Y^A$, can be written as:

$$A = (a_1, a_2, \ldots, a_m).$$

Taken together, these can be written in equivalent Zadeh style notation as:

$$A = a_1/x_1^A + a_2/x_2^A + \cdots + a_m/x_m^A$$

where the codomain is implicit (in Zadeh's notation, the codomain is omitted, as it is always $[0, 1]$). □

Remark 2 We emphasise that the elements y_j^A of the codomain Y^A are represented by application dependent linguistic labels, such as { *incompatible*, *possible*, *probable* }, but that these linguistic labels are just identifying the elements. Thus, for example, whether we label three elements as *incompatible*, *possible* and *probable*, or as *possible*, *probable* and *definite*, or *false*, *half-true* and *true*, or even as '0.0', '0.5' and '1.0' does not matter — they are simply labels describing a rank-ordered list in which the position or **rank** (relative position or degree of value in the graded group) is the important matter. □

Example 1 Let *Low* be an L-fuzzy set defined on domain X^{Low}, being the integers $1 \ldots 5$, over a codomain Y^{Low}, being an ordered set with elements *incompatible* (*inc*), *possible* (*pos*), and *probable* (*prb*), i.e.:

$$X^{Low} = (1, 2, 3, 4, 5)$$
$$Y^{Low} = (inc, pos, prb)$$
$$Low = (prb, pos, inc, inc, inc)$$

or in Zadeh notation:

$$Low = prb/1 + pos/2 + inc/3 + inc/4 + inc/5.$$

Then, two different L-fuzzy sets *Mid* and *High* on the same domain $(X^{High} = X^{Mid} = X^{Low})$ and codomain $(Y^{High} = Y^{Mid} = Y^{Low})$ might be:

$$Mid = (inc, pos, prb, pos, inc) \quad \text{and}$$
$$High = (inc, inc, inc, pos, prb).$$

These sets are illustrated in Fig. 1. □

Note that any element of an L-fuzzy set may be undefined, which is quite distinct from being defined as any one of the values on the codomain including, for example, *inc* (*incompatible*). This will be written as '—'. In essence, — represents an undefined value, such that the membership could lie anywhere in the codomain; i.e. we have no information about the membership value. As an example, consider a fuzzy set representing the level of enjoyment at work, defined for each day of the week. So the domain X^A = (*Mon*, *Tue*, *Wed*, *Thu*, *Fri*, *Sat*, *Sun*), and the codomain Y^A represents various levels of enjoyment, such as (*None*, *Little*, *Some*, *Lots*). Given that a person only works *Mon* to *Fri*, the set may be defined as A = (*Little*, *Some*, *Some*, *Little*, *Lots*, —, —).

Later, it will be necessary to state explicitly the domain and codomain of the L-fuzzy set, this can be written as:

$${}^{X^A}A^{Y^A} \text{ or } {}^{X^A=(x_1^A,x_2^A,\ldots,x_m^A)}A^{Y^A=(y_1^A,y_2^A,\ldots,y_n^A)}$$

To simplify notation, we drop the superscript on the domain X (and even the domain itself) when obvious, and we drop the superscript A on the codomain Y and from the elements of the codomain y_j^A when obvious. Hence, we may write ${}^{X^A}A^{Y^A}$ as just A^Y, and y_j^A as just y_j, e.g.:

$${}^{X=(1,2,3,4,5)}Low^{Y=(inc,pos,prb)} \text{ or } {}^{X}Low^{Y}.$$

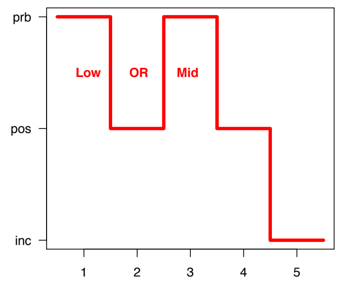

Fig. 2. Illustration of L-fuzzy union of sets *Low* and *Mid*, each as in Fig. 1.

A. Intersection and Union

Definition 3 For two L-fuzzy sets, A and B, defined on the same domain X, with the same codomain Y, the intersection is defined as:

$$\begin{aligned} A \cap B &\equiv \min(A, B) \\ &= \big(\min(a_1, b_1), \min(a_2, b_2), \ldots, \min(a_m, b_m)\big) \quad \square \end{aligned}$$

Definition 4 Similarly, the union of two L-fuzzy sets defined on the same universe X and codomain Y is defined as:

$$A \cup B \equiv \max(A, B) \quad \square$$

Example 2 Using the same sets as in the example above:

$$\begin{aligned} \textit{Low} \cap \textit{Mid} &= \min\big((\textit{prb}, \textit{pos}, \textit{inc}, \textit{inc}, \textit{inc}), \\ &\qquad (\textit{inc}, \textit{pos}, \textit{prb}, \textit{pos}, \textit{inc})\big) \\ &= (\textit{inc}, \textit{pos}, \textit{inc}, \textit{inc}, \textit{inc}). \end{aligned}$$

while:

$$\begin{aligned} \textit{Low} \cup \textit{Mid} &= \max\big((\textit{prb}, \textit{pos}, \textit{inc}, \textit{inc}, \textit{inc}), \\ &\qquad (\textit{inc}, \textit{pos}, \textit{prb}, \textit{pos}, \textit{inc})\big) \\ &= (\textit{prb}, \textit{pos}, \textit{prb}, \textit{pos}, \textit{inc}). \quad \square \end{aligned}$$

An illustration of the L-fuzzy union of sets *Low* and *Mid*, as given in Fig. 1, are shown in Fig. 2.

Before we can proceed to an inferencing framework, including fuzzification, implication and defuzzification, we need to define a number of operations that may be performed on L-fuzzy sets, namely *implication*, *L-similarity* and a *numeric defuzzification*.

B. Implication

When carrying out implication within a rule-based system in a practical Mamdani context, the truth value of a rule is evaluated, and then that truth value is used to 'fire' a consequent fuzzy set at the given truth value. In Mamdani terms, this is implemented by carrying out an intersection operation (using a tnorm) between the evaluated truth value and the elements of the consequent set. We need to formulate an equivalent process for L-fuzzy implication.

In standard fuzzy sets, in which the codomain is always fixed as the unit interval, membership values are simply expressed as a value in $[0, 1]$. In contrast, in L-fuzzy systems in which different sets may be defined over different codomains, a membership value is expressed as a specific element of a codomain. So, rather than simply stating that a membership value of a certain element in a standard fuzzy set is (say) 0.5 (where the codomain $[0, 1]$ is implicit), we may state that the membership of a certain element of an L-fuzzy set is *pos* on codomain $(\textit{inc}, \textit{pos}, \textit{prb})$ which is quite different, of course, from *pos* on codomain $(\textit{inc}, \textit{pos}, \textit{may(be)}, \textit{prb}, \textit{def(inite)})$. The essential process of implication is that the consequent set is fired at the evaluated level of the antecedent. That is, with an antecedent evaluated as *pos* on codomain $(\textit{inc}, \textit{pos}, \textit{prb})$, the fired consequent set is simply labelled as belonging to $\textit{pos}^{(\textit{inc},\textit{pos},\textit{prb})}$.

To represent this formally, assume we have an antecedent:

$$A^{Y^A=(y_1^A, y_2^A, \ldots, y_n^A)}$$

and a consequent:

$$C^{Y^C=(y_1^C, y_2^C, \ldots, y_n^C)}.$$

To emphasise, note that the codomain of the antecedent and consequent may differ — that is $Y^A \neq Y^C$, $n^A \neq n^C$, and $y_j^A \neq y_j^C$. To carry out implication, it is necessary to maintain the concept that the result of implication will be an L-fuzzy set that is one of a set of terms of a single linguistic variable.

Definition 5 A linguistic variable, denoted as $\mathcal{A}$, consists of a set of related L-fuzzy sets $\mathcal{A}_k \in \mathcal{A}$, defined over the same domain and codomain. $\square$

Definition 6 The result of implication $A \to C$ is a linguistic variable $\mathcal{C}$, consisting of a set of L-fuzzy set terms $\mathcal{C}_{y_i^A}$, where $y_i^A \in Y^A$, the codomain of the antecedent, and each $\mathcal{C}_{y_i^A}$ has the domain and codomain of the consequent set C. If a specific evaluation (firing strength) of the antecedent is denoted $a^{Y^A=(y_1^A, y_2^A, \ldots, y_n^A)}$, for example $\textit{pos}^{Y^A=(\textit{inc},\textit{pos},\textit{prb})}$, then the result of implication is the linguistic variable $\mathcal{C}$, in which $\mathcal{C}_a = C^{Y^C}$, with the remaining constituent sets having undefined (—) membership values. $\square$

Example 3 The process is best illustrated with an example. Assume the codomain of a consequent set C is:

$$Y^C = (\textit{inc}, \textit{pos}, \textit{may}, \textit{prb}, \textit{def})$$

and that a specific L-fuzzy set consequent of implication on the domain $1 \ldots 5$ is:

$$C = (\textit{inc}, \textit{inc}, \textit{may}, \textit{prb}, \textit{def}).$$

Note the fact that m ($= 5$, the number of elements in the domain) is equal to n ($= 5$, the number of elements in the codomain) is entirely coincidental, and without significance. The codomain of the antecedent A is:

$$Y^A = (\textit{inc}, \textit{pos}, \textit{prb})$$

and the specific antecedent of the implication evaluates as *pos*, i.e. $\textit{pos}^{(\textit{inc},\textit{pos},\textit{prb})}$. The term sets of the linguistic variable resulting from the implication $A \to C$ are:

$$\mathcal{C}_{inc} = (—,—,—,—,—),$$
$$\mathcal{C}_{pos} = (inc, pos, may, prb, def), \quad \text{and}$$
$$\mathcal{C}_{prb} = (—,—,—,—,—).$$

□

III. L-Fuzzy Inference

A. Inference Process

As usual in fuzzy rules, linguistic variables appear as rule antecedents and as rule consequents, each variable comprising a set of terms (fuzzy sets) defined over a common domain and codomain. Suppose we have a set of multi-input single-output rules and each rule, denoted as $R_{r\in R}$, is of the form:

R_1: **If** $\tilde{a}_1$ **is** $A_{1,1}$ **and/or** $\ldots \tilde{a}_2$ **is** $A_{1,2}$ **and/or** $\ldots$
Then $\mathbb{C}_1$ **is** C_1

R_2: **If** $\tilde{a}_1$ **is** $A_{2,1}$ **and/or** $\ldots \tilde{a}_2$ **is** $A_{2,2}$ **and/or** $\ldots$
Then $\mathbb{C}_2$ **is** C_2

$R_{|R|}$: **If** $\tilde{a}_1$ **is** $A_{|R|,1}$ **and/or** $\ldots \tilde{a}_2$ **is** $A_{|R|,2}$ **and/or** $\ldots$
Then $\mathbb{C}_{|R|}$ **is** $C_{|R|}$

where $\tilde{a}_i$ is the actual observed value of input i being evaluated, $A_{r,i}$ is the fuzzy antecedent set of the i^{th} input variable for the r^{th} rule, and $\mathbb{C}_r$ is the resulting linguistic variable of the r^{th} implication (the rule output). The overall result of the inference is $\mathbb{C} = \bigcup_{r\in R} \mathbb{C}_r$, that is, a single linguistic variable comprising the union of all the rule outputs.

As a pragmatic constraint, we require all input variables to be defined over the same codomain, while the consequent sets may have a different codomain (although each rule consequent is a set drawn from the same linguistic variable). Note that the domains of all variables are independent, as per usual.

The L-fuzzy inference process is essentially the same as in Mamdani inference. For each rule: each input variable is fuzzified by evaluating the membership value on the codomain at the value of the input on the domain; the overall value of the antecedent is formed by taking the union or intersection of the sub-clauses of the rule; the consequent of the rule is fired as specified by the implication process, above; and, finally, the individual rule outputs are combined by the union operation.

B. Inference Example

We provide a complete worked example of a very simple inference process, with two rules.

Example 4 Let there be two linguistic variables, $\mathcal{A}$ and $\mathcal{B}$, each with three terms *Low*, *Mid* and *High*, defined on the domain $1\ldots5$ and with a codomain (inc, pos, prb), as follows:

$$\mathcal{A}_{Low} = (prb, pos, inc, inc, inc),$$
$$\mathcal{A}_{Mid} = (inc, pos, prb, pos, inc),$$
$$\mathcal{A}_{High} = (inc, inc, inc, pos, prb),$$
$$\mathcal{B}_{Low} = (prb, prb, inc, inc, inc),$$
$$\mathcal{B}_{Mid} = (inc, prb, prb, prb, inc), \quad \text{and}$$
$$\mathcal{B}_{High} = (inc, inc, inc, prb, prb).$$

$\mathcal{A}$ comprises the three sets shown in Fig. 1 and is shown, together with $\mathcal{B}$ in Fig. 3. There is a third linguistic variable, $\mathcal{C}$, which also has three terms *Low*, *Mid* and *High*, defined on the domain $1\ldots5$, with a codomain $(inc, pos, may, prb, def)$:

$$\mathcal{C}_{Low} = (def, prb, may, inc, inc),$$
$$\mathcal{C}_{Mid} = (inc, pos, def, pos, inc), \quad \text{and}$$
$$\mathcal{C}_{High} = (inc, inc, may, prb, def).$$

and also shown in Fig. 3. There are two rules in this simple illustrative system:

R_1: **If** $\tilde{a}$ **is** $\mathcal{A}_{Low}$ **and** $\tilde{b}$ **is** $\mathcal{B}_{Mid}$ **Then** $\mathbb{C}_1$ **is** $\mathcal{C}_{High}$

R_2: **If** $\tilde{a}$ **is** $\mathcal{A}_{Mid}$ **and** $\tilde{b}$ **is** $\mathcal{B}_{Mid}$ **Then** $\mathbb{C}_2$ **is** $\mathcal{C}_{Mid}$

Now, assume the inference is fired with the crisp values $\tilde{a} = 2$ and $\tilde{b} = 3$. Then, for R_1:

$$\mathcal{A}_{Low}(2) = pos^{(inc,pos,prb)}$$
$$\mathcal{B}_{Mid}(3) = prb^{(inc,pos,prb)}$$

$$\mathcal{A}_{Low}(2) \cap \mathcal{B}_{Mid}(3) = pos^{(inc,pos,prb)} \cap prb^{(inc,pos,prb)} = pos^{(inc,pos,prb)}$$

Hence, the consequent $\mathcal{C}_{High}$ = (*inc*, *inc*, *may*, *prb*, *def*) is fired at $pos^{(inc,pos,prb)}$ and so, as detailed in Sec. II-B, the result $\mathbb{C}_1$ of R_1 is:

$$\mathbb{C}_{1,inc} = (—,—,—,—,—),$$
$$\mathbb{C}_{1,pos} = (inc, pos, may, prb, def),$$
$$\mathbb{C}_{1,prb} = (—,—,—,—,—).$$

It is straightforward to see that for R_2:

$$\mathcal{A}_{Mid}(2) \cap \mathcal{B}_{Mid}(3) = pos^{(inc,pos,prb)}$$

and hence that the result of R_2 is:

$$\mathbb{C}_{2,pos} = (inc, pos, def, pos, inc),$$
$$\mathbb{C}_{2,inc} = \mathbb{C}_{2,prb} = (—,—,—,—,—).$$

Taking the union of the two rules gives the final result of inference consisting of the three constituent L-Fuzzy sets:

$$\mathbb{C}_{pos} = (inc, pos, def, prb, def),$$
$$\mathbb{C}_{inc} = \mathbb{C}_{prb} = (—,—,—,—,—),$$

as shown in Fig. 4(a-c), for $\mathbb{C}_{inc}$, $\mathbb{C}_{pos}$ and $\mathbb{C}_{prb}$, respectively.

□

IV. Defuzzification

We now proceed with a set of operations for defuzzification of the result(s) of L-fuzzy inference. Note that these are not the only methods that may be used, in the same way that centre of gravity defuzzification is just one of a range of conventional defuzzification methods.

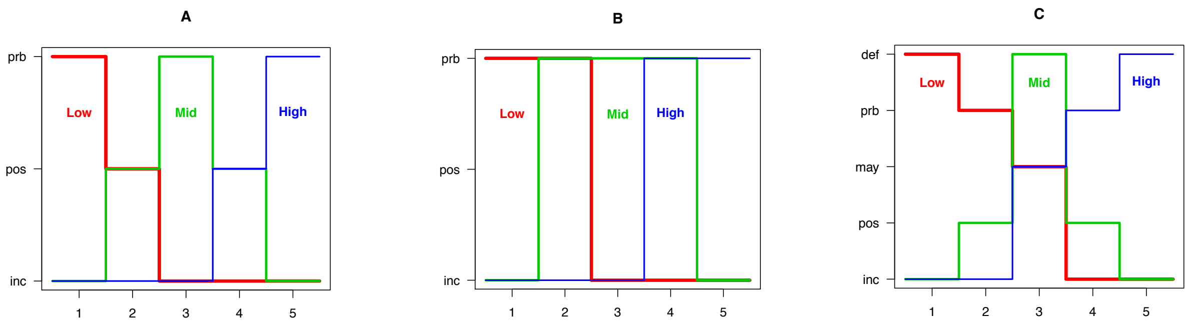

Fig. 3. Illustration of the three linguistic variables, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, each containing three L-fuzzy term sets, *Low*, *Mid* and *High*. Note that the terms are different in each case (despite having similar names) and that the codomain of $\mathcal{C}$ is different to that of $\mathcal{A}$ and $\mathcal{B}$.

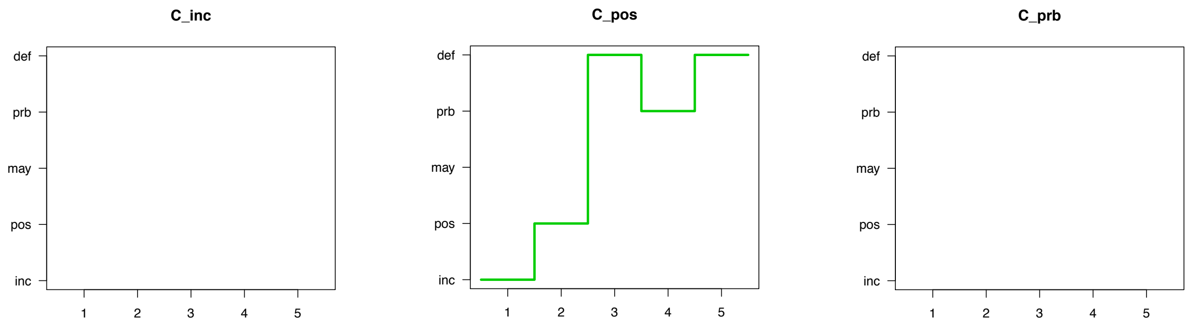

Fig. 4. Illustration of the final output of the two rule system presented in Example 4. Both $\mathbb{C}_{inc}$ and $\mathbb{C}_{prb}$ are undefined for all values of the domain, as a result of the fact that neither of the rules provided any information on these levels.

A. Similarity Matrix

As described in Sec. III, the result of an inference process on a set of rules is a single overall consequence linguistic variable $\mathbb{C}$ comprising one term for each level of the codomain of the rule antecedent(s). So, one can now form the similarity of each of the resultant consequence terms with each of the terms of the original consequent variable $\mathcal{C}$.

Definition 7 A *similarity matrix*, $\mathbf{S}$, may be constructed for the result of L-fuzzy inference, of size $|Y^A| \times |\mathcal{C}|$, with rows for each level in the codomain Y^A of the antecedent(s), and columns for each of the terms in the consequent linguistic variable $\mathcal{C}$. Each element is the matrix represents the similarity, *sim*, between the consequence term for the corresponding row and the original consequent term for the corresponding column. □

Example 5 Thus, continuing the previous example, the similarity matrix will be a 3×3 matrix with rows *inc*, *pos* and *prb* (for consequence terms $\mathbb{C}_{inc}$, $\mathbb{C}_{pos}$ and $\mathbb{C}_{prb}$), and columns *Low*, *Mid* and *High* (for consequent terms $\mathcal{C}_{Low}$, $\mathcal{C}_{Mid}$ and $\mathcal{C}_{High}$), with elements:

$$\begin{bmatrix} sim(\mathbb{C}_{inc}, \mathcal{C}_{Low}) & sim(\mathbb{C}_{inc}, \mathcal{C}_{Mid}) & sim(\mathbb{C}_{inc}, \mathcal{C}_{High}) \\ sim(\mathbb{C}_{pos}, \mathcal{C}_{Low}) & sim(\mathbb{C}_{pos}, \mathcal{C}_{Mid}) & sim(\mathbb{C}_{pos}, \mathcal{C}_{High}) \\ sim(\mathbb{C}_{prb}, \mathcal{C}_{Low}) & sim(\mathbb{C}_{prb}, \mathcal{C}_{Mid}) & sim(\mathbb{C}_{prb}, \mathcal{C}_{High}) \end{bmatrix}$$

□

B. L-Similarity

In order to calculate the similarity, *sim*, between two L-fuzzy sets, we propose a form of Jaccard similarity.

Definition 8 The similarity between two L-fuzzy sets A and B is given by:

$$sim(A, B) = \frac{|A \cap B|}{|A \cup B|}$$

□

While this may appear straightforward, this is not the case. The Jaccard similarity is essentially based on the relative size of the intersection and union. As the codomain of an L-fuzzy set is simply an ordered set without any numerical scale to the levels, there is no obvious, natural method to determine the size of such a set. To address this issue, we propose the most natural way to assess the difference between levels, namely counting the steps. That is, to move from the first level to the third level requires two steps. It is obvious that counting steps is the same as mapping the levels to the natural numbers and taking the absolute value of the difference (that is, from the first step to the third step is $3 - 1 = 2$ steps).

Definition 9 The codomain Y^A of L-fuzzy set A, with elements y_j^A $(j = 1 \ldots n)$ can be mapped onto the domain, Y'^A, of integers $1 \ldots n$:

$$y_j^A = (y_1^A, y_2^A, \ldots, y_n^A) \Rightarrow y_j'^A = (1, 2, \ldots, n) \qquad (1)$$

□

Definition 10 The cardinality, $|A|$, of L-fuzzy set A is given by the sum of the levels of the memberships of A *above* the lowest level of the codomain, once the codomain has been mapped to integers. That is:

$$|A| = \sum_{i=1}^{m} (y_j'^A(a_i) - 1)$$

Thus, the cardinality of L-fuzzy set $\mathcal{C}_{Mid}$ shown in Fig. 3 is $1 + 4 + 1 = 6$, while that of $\mathbb{C}_{pos}$ shown in Fig. 4 is 12. □

Now we have these definitions, for the same example, the similarity between $\mathcal{C}_{Mid}$ and $\mathbb{C}_{pos}$, as above, is:

$$
\begin{aligned}
sim(\mathbb{C}_{pos}, \mathcal{C}_{Mid}) &= \frac{|\mathbb{C}_{pos} \cap \mathcal{C}_{Mid}|}{|\mathbb{C}_{pos} \cup \mathcal{C}_{Mid}|} \\
&= \frac{|\mathcal{C}_{Mid}|}{|\mathbb{C}_{pos}|} = \frac{6}{12} = 0.5
\end{aligned}
$$

This gives a similarity matrix:

$$
\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 \\ 0.167 & 0.5 & 0.75 \\ 0 & 0 & 0 \end{bmatrix} \equiv \begin{array}{cccc} & \textit{Low} & \textit{Mid} & \textit{High} \\ \textit{inc} & 0 & 0 & 0 \\ \textit{pos} & 0.167 & 0.5 & 0.75 \\ \textit{prb} & 0 & 0 & 0 \end{array}
$$

This matrix may be interpreted as indicating the similarity between each of the consequent terms and the various levels of the antecedent codomain. Thus, the example similarity matrix above may be interpreted as indicating an output which is *Mid* to *High* at the level *pos* on the ordered scale (*inc*, *pos*, *prb*).

C. Numeric Defuzzification

While the similarity matrix above is a natural point to conclude defuzzification, there may be circumstances in which a complete numeric defuzzification of the results of L-fuzzy inference is required; for example, if the mapping of the codomain onto a numeric scale is known (or is somehow natural or obvious from the context). In order to arrive at a single numeric value for inference, two steps are necessary.

First, the locations of the term sets of the consequent variable on its domain are derived using a centroid-like calculation. Then, the similarity matrix is interpreted as the centroid of these locations, weighted by the absolute value of the matrix element and the level of the corresponding row on the codomain of the antecedent(s). In both of these steps, the codomain needs to be transformed from an ordered set onto a numeric scale, such as the unit interval $[0, 1]$, as usually featured in fuzzy sets. This may be achieved through the linear mapping provided in Eqs. (1) and (2), above. The location of the L-fuzzy set may then be obtained by any method of numeric defuzzification as for standard fuzzy sets; for example, by centroid (centre-of-gravity) defuzzification, in which case the location of set C is given by:

$$
\tilde{C} = \frac{\sum_{i=1}^{m} c_i \; \hat{y}_i^C}{\sum_{i=1}^{m} \hat{y}_i^C}
$$

As an example, consider the term set of the consequent variable $\mathcal{C}$, as shown in Fig. 3. The five elements of the codomain $(\textit{inc}, \textit{pos}, \textit{may}, \textit{prb}, \textit{def})$ are first mapped to $(0, 0.25, 0.5, 0.75, 1)$ such that the set $\textit{Low}^{(\textit{inc},\textit{pos},\textit{may},\textit{prb},\textit{def})}$ can be interpreted as the fuzzy set 1/1 + 0.75/2 + 0.5/3 + 0/4 + 0/5 (in standard Zadeh notation), which has a centroid at 1.778. Similarly, it can easily be derived that *Mid* has a centroid at 3, while *High* has a centroid at 4.222. These centroids are denoted $\bar{\mathcal{C}}_k$, one for each term of the consequent variable $\mathcal{C}$, that is, $\bar{\mathcal{C}}_{Low} = 1.778$, $\bar{\mathcal{C}}_{Mid} = 3.0$ and $\bar{\mathcal{C}}_{High} = 4.222$.

To perform numeric defuzzification, it is necessary to map the numerical codomain Y'^A onto the unit interval, as $\hat{Y}^A$. By default, this mapping is most naturally made onto equally spaced intervals on the unit interval, such that y'_1 is mapped to zero and y'_n is mapped to one:

$$
\begin{aligned}
y_j'^A = (1, 2, \ldots, n) \Rightarrow \hat{y}_j^A &= \left(\forall_{j=1}^{n} \frac{j-1}{n-1} \right) \\
&= (0, \frac{1}{n-1}, \ldots, 1) \qquad (2)
\end{aligned}
$$

Next, the rows of the matrix are weighted by the levels $\hat{y}^A$ of the codomain Y^A of the antecedent(s) mapped onto $[0, 1]$, so that the total weight of each column is given by:

$$
w_q = \sum_{p=1}^{|Y^A|} \hat{y}_q^A \; \mathbf{S}_{p,q}
$$

for each column q from 1 to $|\mathcal{C}|$.

Finally, the location of the entire similarity matrix, can be derived from the centroid of the consequent term locations based on these column weights:

$$
\tilde{\mathbf{S}} = \frac{\sum_{q=1}^{|\mathcal{C}|} \bar{\mathcal{C}}_q \; w_q}{\sum_{q=1}^{|\mathcal{C}|} w_q}
$$

where the $\bar{\mathcal{C}}_q$ are the indexed elements of $\bar{\mathcal{C}}$.

For the matrix $\mathbf{S}$ above, this simply reduces to $(0.167 \times 1.778 + 0.5 \times 3 + 0.75 \times 4.222)$ / $(0.167 + 0.5 + 0.75)$ $= 3.503$. That is, the similarity matrix $\mathbf{S}$ has an overall 'centroid' located at 3.503, that is just to the R.H.S. of the *Mid* set. Visually, this can be interpreted as the overall centroid of the three sets shown in Fig. 4, where the set $\mathbb{C}_{pos}$ lies half-way between $\mathbb{C}_{inc}$ and $\mathbb{C}_{prb}$. While other approaches may be adopted, nevertheless, this results seems intuitively appealing, maintaining the essence of the conventional centroid.

V. Restaurant Tipper Example

In order to demonstrate the complete L-fuzzy inference process, we present a complete worked example of an L-fuzzy equivalent of the restaurant tipper example as given in the MATLAB® Fuzzy Logic Toolbox® [7].

The example relates the size of the tip to be given in a restaurant based on two factors, the quality of the service and the quality of the food. Thus, there are two input linguistic variables (*service* and *food*) and one output linguistic variable (*tip*). The following rules are provided:

If *service* is *poor* or *food* is *rancid*, then *tip* is *cheap*
If *service* is *good*, then *tip* is *average*
If *service* is *excellent* or *food* is *delicious*, then *tip* is *generous*

with the understanding that *service* has three terms *poor*, *good* and *excellent*; *food* has two terms *rancid* and *delicious*; and *tip* has three terms *cheap*, *average* and *generous*. The membership functions for each of these terms can be found in the MATLAB® Fuzzy Logic Toolbox®.

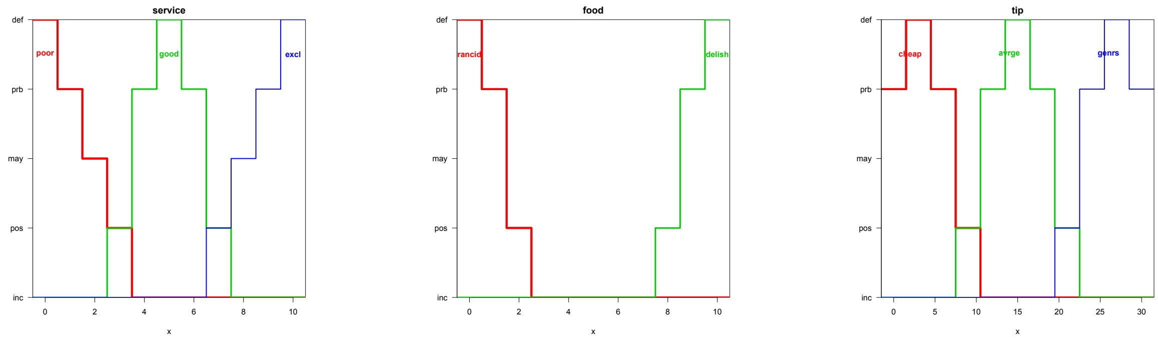

Fig. 5. Illustration of terms in the three linguistic variables, *service*, *food* and *tip*, in the coarse-grained restaurant tipper example.

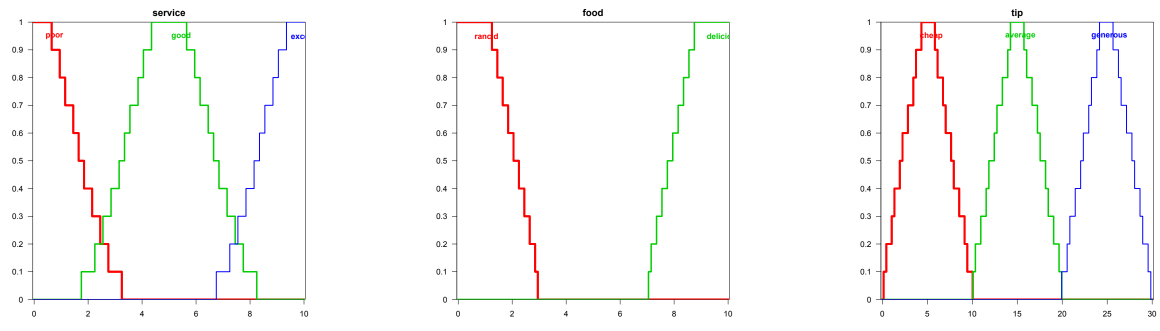

Fig. 6. Illustration of terms in the three linguistic variables, *service*, *food* and *tip*, in the fine-grained restaurant tipper example.

A. Coarse-Grained Tipper Example

We present two different L-fuzzy implementations of the tipper example. Firstly, a *coarse-grained* version in which all three linguistic variables have the same codomain $(inc, pos, may, prb, def)$ on integer domains discretised to 11 equally spaced points, over the intervals $[0, 10]$, $[0, 10]$ and $[0, 30]$, respectively. The L-Fuzzy sets are shown in Fig. 5. Note that, the coarse nature of the discretisation mean that these are only broadly similar to the MATLAB® versions.

B. Fine-Grained Tipper Example

In the *fine-grained* version, the three linguistic variables are all defined over the codomain $(0, 0.1, \ldots, 1.0)$ with 101 discretisations on real domains over the intervals $[0, 10]$, $[0, 10]$ and $[0, 30]$. The L-Fuzzy sets, modelled on the MATLAB® Fuzzy Logic Toolbox® sets, are shown in Fig. 6. Note that these sets are now much closer approximations to the triangular membership functions used in the MATLAB® example.

C. Restaurant Tipper Inference

The MATLAB® Fuzzy Logic Toolbox® version of tipper produces the centroid result of 7.79 when run with the crisp inputs *service* = 2 and *food* = 7. In comparison, the coarse-grained L-fuzzy version produces a similarity matrix of:

	cheap	*average*	*generous*
inc	0.031	0.545	0.5
pos	0.000	0.000	0.0
may	1.000	0.045	0.0
prb	0.000	0.000	0.0
def	0.000	0.000	0.0

which has a centroid of 4.04. While this centroid is quite far from the MATLAB® example, this is mainly attributable to the rough nature of the approximations in the terms, necessary due to the coarse discretisations used. When the fine-grained L-fuzzy model is used, with far closer approximations of the term sets, a much closer result is obtained. Surface plots of the original MATLAB® example, the coarse-grained model and the fine-grained model are shown in Fig. 7.

VI. Discussion

In this paper, we have defined a restricted form of L-fuzzy set, in which the codomain is a simply ordered, finite and discrete set. The domain of the L-fuzzy set is also a simply ordered, finite and discrete set. We go on to detail basic operations of intersection, union, and implication, and then present a complete inference framework for L-fuzzy rule-based inference systems, akin to Mamdani inference. Within the inference, we require all antecedent terms to be defined over the same codomain, while the rule consequents may be defined over a different codomain. While some of the restrictions on general L-fuzzy sets that we impose may appear quite onerous at first sight, they are actually a significant relaxation of conventional fuzzy sets and systems, in which the codomain is always the unit interval $[0, 1]$.

The simple two-rule example used through the paper illustrates the complete inference process, while the restaurant tipper example provides a complete L-fuzzy inference including numeric defuzzification. It can be seen from the results obtained that the numeric results of our operations can replicate conventional Mamdani inference to a reasonable accuracy. Although this is not the primary goal of these

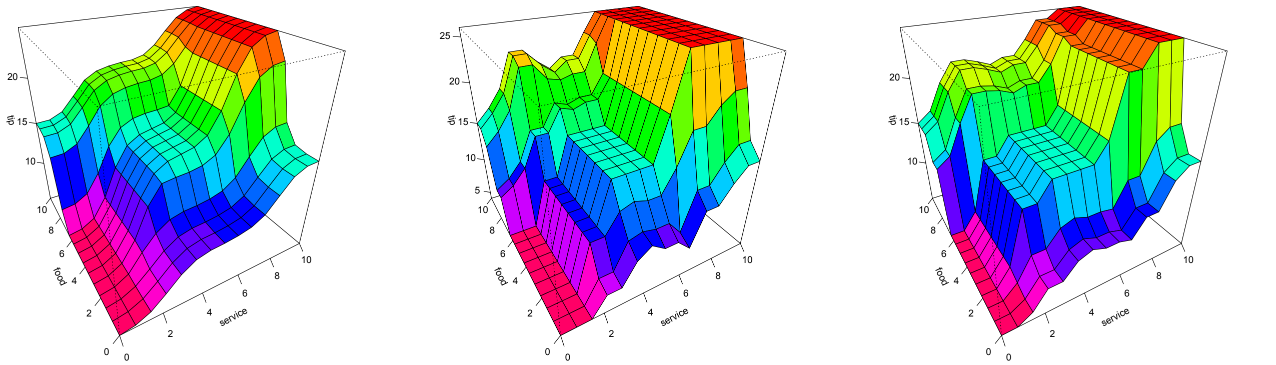

Fig. 7. The input-output surface for the tipper example, as given in the MATLAB® Fuzzy Logic Toolbox® [7]; for the coarse-grained L-fuzzy equivalent model; and for the fine-grained L-fuzzy model.

L-fuzzy methods, it nevertheless provides some confidence that the operation described are appropriate.

The primary result of our L-fuzzy inference is a set of L-Fuzzy sets, as shown in Fig 4. We provide defuzzification strategies in both linguistic and numeric forms. The linguistic defuzzification is based on finding the similarity of each of the resulting L-fuzzy sets, against the original term sets in the consequent variable, using a form of Jaccard similarity. In order to implement this step, it is necessary to map the simply ordered set of the codomain onto a numeric scale, for which we initial use the natural numbers. In doing so, we obtain a form of Manhattan distance between the two sets, that is the number of steps required to move from one ordering to the other, passing through the intermediate ordered elements on the way. As a subsequent step, in order to calculate an equivalent to the centroid, we map simply ordered codomains onto equally spaced intervals covering the unit interval.

There is an obvious contradiction in describing an entire inferencing process that can operate on simply ordered L-fuzzy sets, and then as a final step to introduce an arbitrary mapping of the codomains of the sets onto numeric scales, to permit similarity calculations and numeric defuzzification. However, we maintain that there is an advantage of doing so, Namely, that the mappings to numeric scales is delayed until the end of the inference process. Indeed, the final mapping to the unit interval is only required for the numeric defuzzification step.

As this is a separate self-contained step after inferencing has been completed, it is obviously possible to perform inferencing and *then* investigate a range of alternative numerical mappings that may be suitable to a specific application context. While we have not done this in the present paper, we are currently exploring this 'delayed mapping' concept. One application area in which a complete L-fuzzy approach may be more natural is, for example, in the context of expert (subject) agreements. Consider a situation in which we have five experts and we wish to model the expert agreement: the number of expert agreements obtained for a particular element of a set could be used as the codomain of an L-fuzzy set [8]. Then, the L-fuzzy methods presented in this paper may be used to perform inference, *without* having to place the various levels of agreement onto an arbitrary mapping on $[0, 1]$ (as needs to be done when using conventional fuzzy sets). Having determined the results of inference, various mappings may then be used to weight agreement of different numbers of experts. Again, this is the subject of current research.

VII. Conclusion

In this paper, we have presented a complete L-fuzzy inferencing system, with practical algorithms, from input fuzzification to output defuzzification, allowing inference without the need to arbitrarily map fuzzy sets onto $[0, 1]$. To the best of our knowledge, this is the first time a complete such system has been described. We have demonstrated that the fuzzification, inferencing and defuzzification strategies described, provide both natural linguistic consequences, and can provide numeric results that closely match a conventional 'Mamdani' approach.

In future, we intend to explore the use of L-fuzzy sets in example real-world applications, including using them when representing levels of expert agreement, as mentioned above. Suppose we need to represent the fact that m of n experts agree. For example, three experts (of five) agreeing has more worth than two agreeing and less than four agreeing, but these agreement levels do not have specific associated numerical values. Thus, this would be better represented by L-fuzzy sets as described herein, rather than standard fuzzy sets.

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