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Sensitivity Analysis of the Weighted Generalized Mean Aggregation Operator and its Application to Fuzzy Signatures

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***Abstract*— In this paper we give bounds on the changing of the weighted generalized mean in terms of vector norms of the changing of the variables. Applying this result we characterize the sensitivity of fuzzy signatures which equipped with weighted generalized mean operators in their nodes. Finally, a practical example from civil engineering is also examined.**

I. Introduction

FUZZY SIGNATURES are hierarchical representations of data structuring into vectors of fuzzy values [1]. A fuzzy signature is defined as a special multidimensional fuzzy data structure, which is a generalization of vector valued fuzzy sets [2], [3], [4]. Vector valued fuzzy sets are special cases of L-fuzzy sets which were introduced in [5]. A fuzzy signature is denoted by

$$A\colon X \to S^{(n)},$$

where $1 \leq n$ and

$$S^{(n)} = \times_{i=1}^{n} S_i \qquad S_i = \begin{cases} [0,1] \\ S^{(m)} \end{cases}$$

We can represent a fuzzy signature by nested vector value fuzzy sets and also by a tree graph, which is much more understandable [6].

The goal of this article is to discuss how the membership value of the whole fuzzy set changes if the membership values in the nested vectors change. In other words, if we think of the tree graph representation, how the membership value of the root changes if the membership values of leaves change. To answer this question we have to know how to compute a membership value of a subgraph from the leaves. In this article we assume that all the operators applied on membership values in the signature are from the class of weighted generalized mean aggregation operators.

The paper organized as follows: in Section II we recall the weighted generalized mean (WGM) and some of the mathematical tools, in Section III and IV the sensitivity of WGM is discussed, in Section V and VI we determine the sensitivity of fuzzy signatures, and in Section VII we analyse the sensitivity of a fuzzy signature applied in civil engineering.

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II. The weighted generalized mean

The generalized mean and its generalization, the weighted generalized mean form a very large class of aggregation operators. Their various special cases often arise also in theoretical and practical problems.

Definition 1 (Generalized mean): Let $x_1, \dots, x_n$ be nonnegative real numbers and $p \in \mathbb{R}$ ($p \neq 0$). Then their generalized mean with parameter p:

$$M_p(x_1, \dots, x_n) = \left[\frac{1}{n} \sum_{k=1}^{n} x_k^p \right]^{\frac{1}{p}}$$

Some special cases in p:

- $p = 1$ arithmetic mean
- $p = 2$ quadratic mean
- $p = -1$ harmonic mean

Definition 2 (Weighted generalized mean; WGM): Let $x_1, \dots, x_n$ and $w_1, \dots, w_n$ be nonnegative real numbers, $w_i \geq 0, \sum_{i=1}^{n} w_i = 1$ and $p \in \mathbb{R}$ ($p \neq 0$). Then the weighted generalized mean of $x_1, \dots, x_n$ with weights $w_1, \dots, w_n$ and with parameter p:

$$M_p^w(x_1, \dots, x_n) = \left[\sum_{k=1}^{n} w_k x_k^p \right]^{\frac{1}{p}}$$

The generalized mean is a special case of the weighted generalized mean with weights $w_k = \frac{1}{n}$. The definitions above work well if $p > 0$. If $p < 0$ and any of x_k equals zero the formulae above are meaningless, so we have to define their values in these points. We will use the following, which is consistent with the case of $p > 0$:

Definition 3: If $p < 0$ then the weighted generalized mean of $x_1, \dots, x_n$ with weights $w_1, \dots, w_n$ and with parameter p:

$$M_p^w(x_1, \dots, x_n) = \begin{cases} \left[\sum_{k=1}^{n} w_k x_k^p \right]^{\frac{1}{p}} & \text{if } \forall x_k > 0 \\ 0 & \text{otherwise} \end{cases}$$

The limits at $\pm\infty$ regardless to the weights:

$$\lim_{p \to \infty} \left[\sum_{k=1}^{n} w_k x_k^p \right]^{\frac{1}{p}} = \max(x_i)$$

$$\lim_{p \to -\infty} \left[\sum_{k=1}^{n} w_k x_k^p \right]^{\frac{1}{p}} = \min(x_i)$$

The limit if $p \to 0$ is the weighted geometric mean:

$$\lim_{p\to 0}\left[\sum_{k=1}^{n} w_k x_k^p\right]^{\frac{1}{p}} = \prod_{i=1}^{n} x_i^{w_i}$$

Our goal is to give an upper bound on the changing of M if we know the changing of the input values $x_1,\ldots,x_n$. Let we introduce the following notations:

$$\begin{array}{lll} \underline{x} = (x_1,\ldots,x_n) & \underline{x}^* = (x_1^*,\ldots,x_n^*) & \Delta\underline{x} = \underline{x}^* - \underline{x} \\ M = M_p^w(\underline{x}) & M^* = M(x_1^*,\ldots,x_n^*) & \Delta M = M^* - M \end{array}$$

We search for such a bound for $|\Delta M|$ which depends on $\Delta\underline{x}$, more exactly, on a kind of vector norm of $\Delta\underline{x}$. First we recall the definition of the p-norm. Unfortunately the usual naming is p-norm, but this p is not necesseraliy identical with the p of the generalized mean. For this reason this p will be denoted by p'.

Definition 4 (p-norm): Let $p' \geq 1$ a real number and $\underline{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n$. Then the p'-norm of $\underline{x}$

$$\|\underline{x}\|_{p'} = \left(\sum_{k=1}^{n} |x_k|^{p'}\right)^{\frac{1}{p'}}$$

Some widely used p-norms:

- $p' = 1$ (taxicab norm) $\|\underline{x}\|_1 = |x_1| + \ldots + |x_n|$
- $p' = 2$ (euclidean norm) $\|\underline{x}\|_2 = \sqrt{x_1^2 + \ldots + x_n^2}$
- $p' = \infty$ (maximum norm) $\|\underline{x}\|_\infty = \max(|x_1|,\ldots,|x_n|)$

For estimation of $|\Delta M|$ we use the multivariate case of Lagrange's mean value theorem and its corollaries:

Theorem 5: Let G be an open subset of $\mathbb{R}^n$ and let $f\colon G \subset \mathbb{R}^n \to \mathbb{R}$. If $\underline{x}$, $\underline{y} \in G$ and f is differentiable at each point of the line segment $\underline{xy}$, then there exists on that line segment a point $\underline{\xi}$ between $\underline{x}$ and $\underline{y}$ such that

$$f(\underline{y}) - f(\underline{x}) = \nabla f(\underline{\xi}) \cdot (\underline{y} - \underline{x})$$

or in other form:

$$f(\underline{y}) - f(\underline{x}) = \sum_{i=1}^{n} \frac{\partial f(\underline{\xi})}{\partial x_i} \cdot (y_i - x_i)$$

Corollary 6: From the theorem above we can derive upper bounds on the changing of f in terms of p-norms:

- Applying the CAUCHY-BUNYAKOVSKY-SCHWARZ inequality we get the following:

$$\begin{aligned}\left|f(\underline{y}) - f(\underline{x})\right| &\leq \sqrt{\sum_{i=1}^{n}\left(\frac{\partial f(\underline{\xi})}{\partial x_i}\right)^2} \cdot \sqrt{\sum_{i=1}^{n}(y_i - x_i)^2} \\ &= \|\nabla f(\underline{\xi})\|_2 \cdot \|\underline{y} - \underline{x}\|_2\end{aligned}$$

- From the triangular inequality we get that

$$\begin{aligned}\left|f(\underline{y}) - f(\underline{x})\right| &\leq \sum_{i=1}^{n}\left|\frac{\partial f(\underline{\xi})}{\partial x_i}\right| \cdot \max|y_i - x_i| \\ &= \|\nabla f(\underline{\xi})\|_1 \cdot \|\underline{y} - \underline{x}\|_\infty\end{aligned}$$

- Again from the triangular inequality we get that

$$\begin{aligned}\left|f(\underline{y}) - f(\underline{x})\right| &\leq \max\left|\frac{\partial f(\underline{\xi})}{\partial x_i}\right| \cdot \sum_{i=1}^{n}|y_i - x_i| \\ &= \|\nabla f(\underline{\xi})\|_\infty \cdot \|\underline{y} - \underline{x}\|_1\end{aligned}$$

$$M = M_p^w(x_1,\ldots,x_n) = \left[\sum_{k=1}^{n} w_k x_k^p\right]^{\frac{1}{p}}$$

We look for a bound on the change of the output in the terms of the change of the input, so we look for a $K_{p'}$ for which the inequality

$$|\Delta M| \leq K_{p'} \cdot \|\Delta\underline{x}\|_{p'}$$

holds for a p'-norm.

III. SENSITIVITY ANALYSIS OF WGM

In this section we introduce the values of $K_{p'}$-s for the values $p' = 1$, 2 and $p' = \infty$, and for the whole range of the parameter p.

A. In $\|\cdot\|_2$ norm of $\Delta\underline{x}$

Let we have

$$M = M_p^w(x_1,\ldots,x_n) = \left[\sum_{k=1}^{n} w_k x_k^p\right]^{\frac{1}{p}}$$

then the first order partial derivative:

$$\frac{\partial M}{\partial x_i} = \left[\sum_{k=1}^{n} w_k x_k^p\right]^{\frac{1}{p}-1} \cdot w_i \cdot x_i^{p-1}$$

The sum of squares of the first order partial derivatives:

$$G = \sum_{i=1}^{n}\left(\frac{\partial M}{\partial x_i}\right)^2 = \frac{\sum_{i=1}^{n} w_i^2 x_i^{2p-2}}{\left[\sum_{k=1}^{n} w_k x_k^p\right]^{2-\frac{2}{p}}}$$

Analyzing the expression of G we can conclude the following:

- if $p \geq 2$ then $G \leq \max\left(w_i^{\frac{2}{p}}\right) = (\max w_i)^{\frac{2}{p}}$
- if $1.5 \leq p < 2$ then $G \leq (\max w_i)^{\frac{2}{p}} \cdot n^{\frac{2}{p}-1}$
- if $1 < p < 1.5$ then $G \leq \max(w_i)$
- if $p = 1$ then $G = \sum_{i=1}^{n} w_i^2$
- if $0 \leq p < 1$ then G is unbounded
- if $p < 0$ then $G \leq \max\left(w_i^{\frac{2}{p}}\right) = (\min w_i)^{\frac{2}{p}}$

Since G is not bounded in case of $0 \leq p < 1$, we cannot give such a K_2 for which the inequality $|\Delta M| \leq K_2 \cdot \|\Delta\underline{x}\|_2$ holds for arbitrary $\Delta\underline{x}$. The K_2 coefficients are listed in Table I, where we used the notation $\underline{w}^{1/p} = (w_1^{1/p},\ldots,w_n^{1/p})$.

TABLE I

VALUES OF THE COEFFICIENT K_2 FOR $|\Delta M| \le K_2 \cdot \|\Delta \underline{x}\|_2$.

value of p	K_2	if $w_i = 1/n$
$p < 0$	$\max\left\{w_i^{1/p}\right\} = \|\underline{w}^{1/p}\|_\infty$	$n^{-1/p}$
$p = 0$	-	-
$0 < p < 1$	-	-
$p = 1$	$\|\underline{w}\|_2$	$n^{-1/2}$
$1 < p < 1.5$	$\max\{w_i\}^{1/2} = \|\underline{w}^{1/2}\|_\infty$	$n^{-1/2}$
$1.5 \le p < 2$	$\max\left\{w_i^{1/p}\right\} \cdot n^{1/p-1/2}$	$n^{-1/2}$
$2 \le p$	$\max\left\{w_i^{1/p}\right\} = \|\underline{w}^{1/p}\|_\infty$	$n^{-1/p}$

B. In $\|\cdot\|_\infty$ norm of $\Delta\underline{x}$

The sum of the absolute values of the first order partial derivatives:

$$H = \sum_{i=1}^{n} \left|\frac{\partial M}{\partial x_i}\right| = \frac{\sum_{i=1}^{n} w_i x_i^{p-1}}{\left[\sum_{k=1}^{n} w_k x_k^p\right]^{1-\frac{1}{p}}}$$

Analyzing H we arrive at the following statements:

- if $p \ge 2$ then $H \le \max\left(w_i^{\frac{1}{p}}\right) \cdot n^{\frac{1}{p}}$
- if $1 < p < 2$ then $H \le 1$
- if $p = 1$ then $H = 1$
- if $0 \le p < 1$ then H is unbounded
- if $p < 0$ then $H \le \max\left(w_i^{\frac{1}{p}}\right) = (\min w_i)^{\frac{1}{p}}$

As in the previous case we cannot give a bound if $0 \le p < 1$. The values of K_∞ are listed in Table II.

TABLE II

VALUES OF THE COEFFICIENT K_∞ FOR $|\Delta M| \le K_\infty \cdot \|\Delta \underline{x}\|_\infty$.

value of p	K_∞	if $w_i = 1/n$
$p < 0$	$\max\left\{w_i^{1/p}\right\} = \|\underline{w}^{1/p}\|_\infty$	$n^{-1/p}$
$0 \le p < 1$	-	-
$1 \le p < 2$	1	1
$2 \le p$	$\max\left\{w_i^{1/p}\right\} \cdot n^{1/p} = \|\underline{w}^{1/p}\|_\infty \cdot n^{1/p}$	1

C. In $\|\cdot\|_1$ norm of $\Delta\underline{x}$

The maximal value of the absolute values of $\frac{\partial M}{\partial x_i}$:

$$F = \max_i \left|\frac{\partial M}{\partial x_i}\right| = \max\left\{\left[\sum_{k=1}^{n} w_k x_k^p\right]^{\frac{1}{p}-1} \cdot w_i \cdot x_i^{p-1}\right\}$$

From the partial derivative we can conclude that if $p > 1$ then $\frac{\partial M}{\partial x_i}$ is increasing in x_i and decreasing in all other variables. It reaches its maximal value if $x_i = 1$ and $x_k = 0 \quad (k \ne i)$. Its maximal value is $w_i^{1/p}$. If $p = 1$ then $\frac{\partial M}{\partial x_i} = w_i$. If $0 \le p < 1$ then the partial derivative is unbounded. Finally, if $p < 0$, then $\frac{\partial M}{\partial x_i}$ is decreasing in x_i and increasing in all other variables. It reaches its maximal value if $x_i = 0$ and $x_k = 1 \quad (k \ne i)$. After some transformation we get that its maximal value is $w_i^{1/p}$. The values of F

- if $p > 1$ then $F \le \max\left(w_i^{\frac{1}{p}}\right)$
- if $p = 1$ then $F = \max(w_i)$
- if $0 \le p < 1$ then F is unbounded
- if $p < 0$ then $F \le \max\left(w_i^{\frac{1}{p}}\right)$

As in the case of $\|\cdot\|_2$ and $\|\cdot\|_\infty$ we again cannot give a bound if $0 \le p < 1$. The values of K_1 are listed in Table III.

TABLE III

VALUES OF THE COEFFICIENT K_1 FOR $|\Delta M| \le K_1 \cdot \|\Delta \underline{x}\|_1$.

value of p	K_1	if $w_i = 1/n$
$p < 0$	$\max\left\{w_i^{1/p}\right\} = \|\underline{w}^{1/p}\|_\infty$	$n^{-1/p}$
$0 \le p < 1$	-	-
$p = 1$	$\max(w_i)$	n^{-1}
$p > 1$	$\max\left\{w_i^{1/p}\right\} = \|\underline{w}^{1/p}\|_\infty$	$n^{-1/p}$

IV. SPECIAL CASES IN p

In this section we give the bounds for special cases in p which occur more often in mathematics. In these cases the weights are equal, namely $w_k = \frac{1}{n}$ for all $1 \le k \le n$.

A. Arithmetic mean ($p = 1$)

$$|\Delta M| \le n^{-1/2} \cdot \|\Delta\underline{x}\|_2 = \frac{1}{\sqrt{n}} \cdot \|\Delta\underline{x}\|_2$$
$$|\Delta M| \le \frac{1}{n} \cdot \|\Delta\underline{x}\|_1$$
$$|\Delta M| \le \|\Delta\underline{x}\|_\infty$$

B. Harmonic mean ($p = -1$)

$$|\Delta M| \le n \cdot \|\Delta\underline{x}\|_2$$
$$|\Delta M| \le n \cdot \|\Delta\underline{x}\|_1$$
$$|\Delta M| \le n \cdot \|\Delta\underline{x}\|_\infty$$

C. Quadratic mean ($p = 2$)

$$|\Delta M| \le \frac{1}{\sqrt{n}} \cdot \|\Delta\underline{x}\|_2$$
$$|\Delta M| \le \frac{1}{\sqrt{n}} \cdot \|\Delta\underline{x}\|_1$$
$$|\Delta M| \le \|\Delta\underline{x}\|_\infty$$

D. Extreme values of p

1) $p \to \infty$:

$$M_\infty = \lim_{p\to\infty} M = \lim_{p\to\infty}\left[\sum_{k=1}^{n} w_k x_k^p\right]^{\frac{1}{p}} = \max_i(x_i)$$

$$\begin{aligned}
|\Delta M_\infty| &\le \lim_{p\to\infty}\left(\frac{1}{n}\right)^{\frac{2}{p}} \cdot \|\Delta\underline{x}\|_2 = \|\Delta\underline{x}\|_2 \\
|\Delta M_\infty| &\le \|\Delta\underline{x}\|_1 \\
|\Delta M_\infty| &\le \|\Delta\underline{x}\|_\infty
\end{aligned}$$

2) $p \to -\infty$:

$$M_{-\infty} = \lim_{p\to-\infty} M = \lim_{p\to-\infty}\left[\sum_{k=1}^{n} w_k x_k^p\right]^{\frac{1}{p}} = \min_i(x_i)$$

$$\begin{aligned}
|\Delta M_{-\infty}| &\le \lim_{p\to-\infty}\left(\frac{1}{n}\right)^{\frac{2}{p}} \cdot \|\Delta\underline{x}\|_2 = \|\Delta\underline{x}\|_2 \\
|\Delta M_{-\infty}| &\le \|\Delta\underline{x}\|_1 \\
|\Delta M_{-\infty}| &\le \|\Delta\underline{x}\|_\infty
\end{aligned}$$

V. Sensitivity of fuzzy signatures

Based on the results shown in the previous section we analyse the sensitivity of fuzzy signatures in which the values are determined by a WGM operator in every nodes. The sensitivity bound of the whole fuzzy signature is derived from the bounds of the WGM-s, according to the graph structure of the signature.

A. In $\|\cdot\|_1$ norm of the input vector

Let us denote by K_{11} the bound for the WGM applied in the root of the signature and by $\Delta\underline{x}_{11}$ of the changing of its input vector; the bounds for their WGM operators are $K_{21}, \ldots, K_{2n_2}$ (n_2 is the number of vertices to the root), the changing of their inputs are $\Delta\underline{x}_{21}, \ldots, \Delta\underline{x}_{2n_2}$ etc., till the end of the graph. Then changing of the output value can be estimated by the following way:

$$\begin{aligned}
|\Delta f| &\le K_{11} \cdot \|\Delta\underline{x}_{11}\|_1 \le \\
&\le K_{11} \cdot \left(K_{21} \cdot \|\Delta\underline{x}_{21}\|_1 + \cdots + K_{2n_2} \cdot \|\Delta\underline{x}_{2n_2}\|_1\right) \\
&\vdots \\
&\le \sum_{i=1}^{N} K_i \cdot |\Delta x_i| \le \max\left(K_i\right) \cdot \sum_{i=1}^{N} |\Delta x_i| \\
&= \max(K_i) \cdot \|\Delta\underline{x}\|_1
\end{aligned}$$

where K_i is the product of the bounds form the root to the i-th leaf.

B. In $\|\cdot\|_2$ norm of the input vector

Now it is more convenient to deal with $|\Delta f|^2$ instead of $|\Delta f|$. The estimation works quite similar as in the previous case. The C_{**}-s denote the squares of the bounds for the WGM operators. The estimation:

$$\begin{aligned}
|\Delta f|^2 &\le C_{11}^2 \cdot \|\Delta\underline{x}_{11}\|_2^2 \\
&\le C_{11}^2 \cdot \left(C_{21}^2 \cdot \|\Delta\underline{x}_{21}\|_2^2 + \cdots + C_{2n_2}^2 \cdot \|\Delta\underline{x}_{2n_2}\|_2^2\right) \\
&\vdots \\
&\le \sum_{i=1}^{N} C_i^2 \cdot |\Delta x_i|^2 \le \max(C_i^2) \cdot \sum_{i=1}^{N} |\Delta x_i|^2 \\
&= \max(C_i^2) \cdot \|\Delta\underline{x}\|_2^2
\end{aligned}$$

where C_i^2 is the product of the squares of the bounds from the root to the i-th leaf.

C. In $\|\cdot\|_\infty$ norm of the input vector

This case differs a bit form the others because of the $\max$ operator. The D_{**}-s are the bounds for the WGM operators.

$$\begin{aligned}
|\Delta f| &\le D_{11} \cdot \|\Delta\underline{x}_{11}\|_\infty \\
&\le D_{11} \cdot \max\left(D_{21} \cdot \|\Delta\underline{x}_{21}\|_\infty, \ldots, D_{2n_2} \cdot \|\Delta\underline{x}_{2n_2}\|_\infty\right) \\
&\vdots \\
&\le \\
&= \max(D_i) \cdot \|\Delta\underline{x}\|_\infty
\end{aligned}$$

where D_i is the product of the greatest bounds at every level.

VI. Sensitivity of homogeneous fuzzy signatures

The sensitivity analysis of a fuzzy signature becomes much more simple if the value of the parameter p is the same for all of the WGM operators applied in the nodes. If this condition holds, the output vale of the signature is the weighted generalized mean of the input values with parameter p, where the weights are the product of the weights form the root to the leaves.

Definition 7: A fuzzy signature is called homogeneous if all of the aggregation operators in the nodes are weighted generalized mean operators with the same value of p.

Lemma 1: The WGM of $y_1, \ldots, y_k$ with weights $v_1, \ldots, v_k$ and with parameter p where all of the y_i-s are WGM's of x_{ji}-s with weights $w_{1j}, \ldots, w_{n_i j}$ and with the same parameter of p, is the WGM of the x-s with weights $v_i \cdot w_{ji}$

Proof:

$$\begin{aligned}
\left[\sum_{i=1}^{k} v_i \cdot y_i^p\right]^{\frac{1}{p}} &= \left[\sum_{i=1}^{k} v_i \cdot \left[\left[\sum_{j=1}^{n_i} w_{ji} \cdot x_{ji}^p\right]^{\frac{1}{p}}\right]^p\right]^{\frac{1}{p}} \\
= \left[\sum_{i=1}^{k} v_i \cdot \sum_{j=1}^{n_i} w_{ji} \cdot x_{ji}^p\right]^{\frac{1}{p}} &= \left[\sum_{i=1}^{k}\sum_{j=1}^{n_i} v_i \cdot w_{ji} \cdot x_{ji}^p\right]^{\frac{1}{p}}
\end{aligned}$$

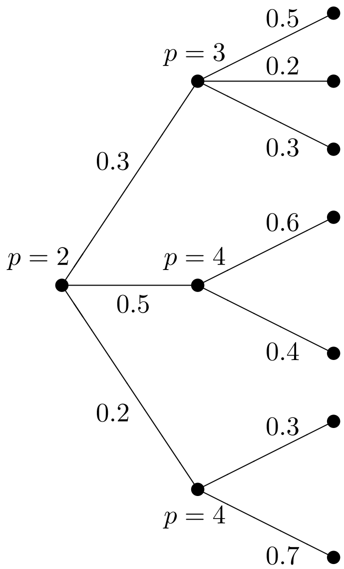

Fig. 1. Non-homogeneous fuzzy signature graph.

A. Example for non-homogeneous fuzzy signature

Consider the fuzzy signature in Figure 1, which is a non-homogeneous fuzzy signature. The estimations of the changing of the output are given by:

$$|\Delta f| \leq 0.5^{1/2} \cdot 0.7^{1/3} \cdot \|\Delta \underline{x}\|_1 \approx 0.63 \cdot \|\Delta \underline{x}\|_1$$
$$|\Delta f| \leq 0.5^{1/2} \cdot 0.7^{1/3} \cdot \|\Delta \underline{x}\|_2 \approx 0.63 \cdot \|\Delta \underline{x}\|_2$$
$$|\Delta f| \leq 0.5^{1/2} \cdot 3^{1/2} \cdot 0.7^{1/3} \cdot 2^{1/3} \cdot \|\Delta \underline{x}\|_\infty \approx 1.37 \cdot \|\Delta \underline{x}\|_\infty$$

B. Example for homogeneous fuzzy signature

If it were a homogeneous fuzzy signature with parameter $p = 3$, then the weights for the leaves (from top to bottom)

$$w_1 = 0.15 \quad w_2 = 0.06 \quad w_3 = 0.09 \quad w_4 = 0.30$$
$$w_5 = 0.20 \quad w_6 = 0.06 \quad w_7 = 0.14$$

From these weights we can give an estimation for the changing of the output:

$$|\Delta f| \leq 0.3^{1/3} \cdot \|\Delta \underline{x}\|_1 \approx 0.67 \cdot \|\Delta \underline{x}\|_1$$
$$|\Delta f| \leq 0.3^{1/3} \cdot \|\Delta \underline{x}\|_2 \approx 0.67 \cdot \|\Delta \underline{x}\|_2$$
$$|\Delta f| \leq 0.3^{1/3} \cdot 7^{1/3} \cdot \|\Delta \underline{x}\|_\infty \approx 1.28 \cdot \|\Delta \underline{x}\|_\infty$$

If we not use the fact of homogeneity our the estimation is the following:

$$|\Delta f| \leq 0.5^{1/3} \cdot 0.7^{1/3} \cdot \|\Delta \underline{x}\|_1 \approx 0.705 \cdot \|\Delta \underline{x}\|_1$$
$$|\Delta f| \leq 0.5^{1/3} \cdot 0.7^{1/3} \cdot \|\Delta \underline{x}\|_2 \approx 0.705 \cdot \|\Delta \underline{x}\|_2$$
$$|\Delta f| \leq 0.5^{1/3} \cdot 3^{1/3} \cdot 0.5^{1/3} \cdot 3^{1/3} \cdot \|\Delta \underline{x}\|_\infty \approx 1.31 \cdot \|\Delta \underline{x}\|_\infty$$

VII. An example from civil engineering

In this section we give the sensitivity analysis of a fuzzy signature which was applied for status-determining and ranking buildings of similar age and structural arrangement. For detailed description of the methodology of status-determining and ranking see [7], [8] and [9].

The structure of the signature is shown in Figure 2. The names and meanings of the input and internal variables are listed below.

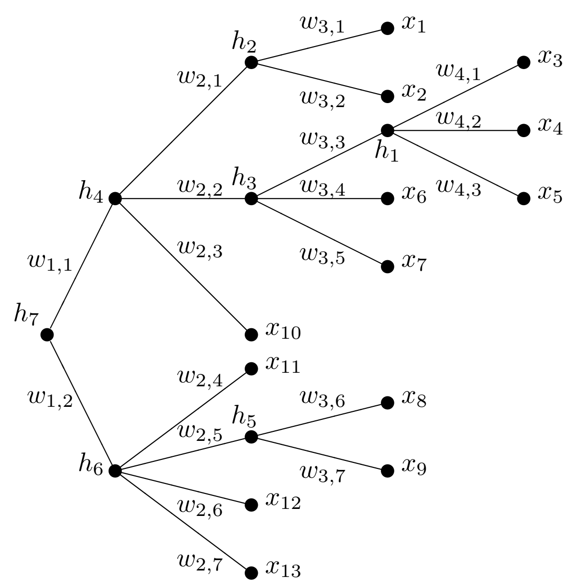

Fig. 2. A fuzzy signature for status-determining and ranking buildings.

The input variables:

x_1 : foundation structures
x_2 : wall structures
x_3 : cellar floor
x_4 : intermediate floor
x_5 : cover floor
x_6 : side corridor structures
x_7 : step structures
x_8 : facade
x_9 : footing
x_{10} : roof structures
x_{11} : roof covering
x_{12} : tin structures
x_{13} : insulation against soil moisture and ground water

The internal variables:

h_1 : floor structures
h_2 : vertical load-bearing structures
h_3 : horisontal load-bearing structures
h_4 : primary structures
h_5 : surface formation
h_6 : secondary structures
h_7 : primary and secondary structures

This is a homogeneous fuzzy signatures with parameter $p = 1$ and with the following weights:

$$
\begin{array}{ll}
w_{1,1} = 0.75 & w_{1,2} = 0.25 \\
w_{2,1} = 0.4 & w_{2,2} = \dfrac{0.6 \cdot n}{n+1} \\
w_{2,3} = \dfrac{0.6}{n+1} & w_{2,4} = \dfrac{0.4}{0.8 + 0.2 \cdot n} \\
w_{2,5} = \dfrac{0.2 \cdot n}{0.8 + 0.2 \cdot n} & w_{2,6} = \dfrac{0.2}{0.8 + 0.2 \cdot n} \\
w_{2,7} = \dfrac{0.2}{0.8 + 0.2 \cdot n} & w_{3,1} = 0.55 - 0.05 \cdot n \\
w_{3,2} = 0.45 + 0.05 \cdot n & w_{3,3} = \dfrac{0.65}{0.8 + 0.2 \cdot f} \\
w_{3,4} = \dfrac{0.2 \cdot f}{0.8 + 0.2 \cdot f} & w_{3,5} = \dfrac{0.15}{0.8 + 0.2 \cdot f} \\
w_{3,6} = 1 - \dfrac{0.5}{n} & w_{3,7} = \dfrac{0.5}{n}
\end{array}
$$

$$
\begin{aligned}
w_{4,1} &= \frac{0.35 \cdot m}{0.2 + 0.45 \cdot (n-1) + 0.35 \cdot m} \\
w_{4,2} &= \frac{0.45 \cdot (n-1)}{0.2 + 0.45 \cdot (n-1) + 0.35 \cdot m} \\
w_{4,3} &= \frac{0.2}{0.2 + 0.45 \cdot (n-1) + 0.35 \cdot m}
\end{aligned}
$$

The possible values of the parameters:

- $n = 2, 3, 4, 5$ (number of the storeys of the building)
- $0 \leq m \leq 1$ (extend of the cellar built)
- $f = 0$ or 1 (building with or without side corridor)

The input values (x_i-s) are real numbers between 0 and 1 according to the opinion of a human expert about the status of the i-th partial structure. The final output is the membership value of h_7. If a building is surveyed by different experts then their opinion about the status of partial structures may result different values of h_7. The crucial question is the following: may a small deviation in the input yields a large changing in the output or not?

From the weights and using the property of homogeneity we get that:

$$
\begin{aligned}
|\Delta h_7| &\leq 0.28 \cdot \|\Delta \underline{x}\|_1 \\
|\Delta h_7| &\leq 0.4 \cdot \|\Delta \underline{x}\|_2 \\
|\Delta h_7| &\leq \|\Delta \underline{x}\|_\infty
\end{aligned}
$$

The result shows that this signature is not sensitive, namely a relatively small change in the input does not have an enermous effect (so the butterfly effect will not happen). For example if two experts examine the same building and they score the partial structures within a 0.1 error, then the difference of their final decisions will not more than 0.1.

VIII. Conclusions

We discussed the sensitivity of the weighted generalized mean aggregation operator in various vector norms, which depends on the maximal weight and on the p parameter of the WGM. Based on this result we described the sensitivity of fuzzy signatures equipped with WGM-s. A special case, when all of the WGM-s have the same parameter was also discussed, and its advantage is obvious.

As an example the sensitivity of a method for status-determining and ranking buildings was analysed, and we established that the applied fuzzy signature is not sensitive not only in the sense of mathematics, but also in practical sense.

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