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Upper and Lower Generalized Factoraggregations Based on Fuzzy Equivalence Relation

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***Abstract*—We develop the concept of a general factoraggregation operator introduced by the authors on the basis of an equivalence relation and applied in two recent papers for analysis of bilevel linear programming solving parameters. In the paper this concept is generalized by using a fuzzy equivalence relation instead of the crisp one. By using a left-continuous t-norm and its residuum we define and investigate two modifications of such generalized construction: upper and lower generalized factoraggregations. These generalized factoraggregations can be used for construction of extensional fuzzy sets.**

I. Introduction

The paper deals with some generalization of the concept of factoraggregation introduced, studied and applied by the authors in [11] and [12]. Factoraggregation is a special construction of a general aggregation operator based on an equivalence relation. The idea of factoraggregation is based on factorization, which allows to aggregate fuzzy subsets taking into account the classes of equivalence, i.e. the partition generated by an equivalence relation. The factoraggregation operator was specially designed in the context of bilevel linear programming in order to analyse the satisfaction degree of objectives on each level and to choose solving parameters values.

In this paper we develop the concept of a factoraggregation operator by using a left-continuous t-norm T, its residuum $\vec{T}$ and a T-fuzzy equivalence relation E instead of the crisp one. We define upper and lower generalized T-fuzzy factoraggregations with respect to E and consider their properties. Taking into account that fuzzy equivalence relations represent the fuzzification of equivalence relations and extensional fuzzy subsets play the role of fuzzy equivalence classes we consider the generalized fuzzy factoraggregations in the context of extensional fuzzy sets.

Within the theory of fuzzy logic the first researcher, who pointed the relevance of extensional fuzzy sets, was L.A. Zadeh [15]. Extensional fuzzy sets are a key concept in the comprehension of the universe set under the effect of a T-fuzzy equivalence relation as they correspond with the observable sets or granules of the universe set. Upper and lower T-fuzzy factoraggregations generalize upper and lower extensional fuzzy sets (see, e.g., [9]). The results of upper and lower generalized T-fuzzy factoraggregations corresponding to E are extensional with respect to T-fuzzy equivalence E.

The last property mentioned above is important in the context of a fuzzy equivalence relation and could be used in the following way. First of all, aggregating extensional fuzzy sets it could be necessary to obtain as a result an extensional fuzzy set as well. It could be effectively done by generalized factoraggregations, while an ordinary aggregation does not provide us this advantage. Second, upper and lower factoraggregations could be treated as upper and lower approximations of a general aggregation operator acting on the class of all fuzzy sets. Aggregation of these approximations by applying generalized factoraggregation could improve the approximation results obtained in the class of extensional fuzzy sets.

The paper is structured as follows. The second section is devoted to extensional fuzzy sets. We recall the definition of a T-fuzzy equivalence relation E and consider upper and lower extensional with respect to E fuzzy sets.

In the third section we recall the definitions of an ordinary aggregation operator and of a general aggregation operator acting on fuzzy structures. Then we give the definition of a factoraggregation operator corresponding to an equivalence relation, which is a case of the general aggregation operator.

The fourth section is devoted to the concepts of upper and lower generalized T-fuzzy factoraggregations with respect to a T-fuzzy equivalence relation E. We show that all properties of the definition of a general aggregation operator such as the boundary conditions and the monotonicity hold for the generalized T-fuzzy factoraggregation operators also.

The fifth section contains some numerical examples.

Finally in the sixth section we specify the definition of upper and lower generalized T-fuzzy factoraggregations for the case of a crisp equivalence relation.

II. Extensional fuzzy sets

Fuzzy equivalence relations were introduced in 1971 by L.A. Zadeh [15] for the strongest t-norm T_M and later were developed and applied by several authors in more general cases.

Definition 1: Let T be a t-norm and E be a fuzzy relation on a set D, i.e. E is a fuzzy subset of $D \times D$. A fuzzy relation E is a T-fuzzy equivalence relation if and only if for all $x, y, z \in D$

it holds
$(E1)$ $E(x,x) = 1$ *(reflexivity);*
$(E2)$ $E(x,y) = E(y,x)$ *(symmetry);*
$(E3)$ $T(E(x,y), E(y,z)) \leq E(x,z)$ *(T-transitivity).*

Dealing with fuzzy equivalence relations usually extensional fuzzy sets attracts an additional attention. These sets correspond to the fuzzification of classical classes of equivalence, they play the role of fuzzy equivalence classes altogether with their intersections and unions.

Definition 2: Let T be a t-norm and E be a T-fuzzy equivalence relation on a set D. A fuzzy subset $\mu \in [0,1]^D$ is called extensional with respect to E if and only if:

$$T(E(x,y), \mu(y)) \leq \mu(x) \text{ for all } x, y \in D.$$

Extensional fuzzy subsets have been widely studied in the literature [2], [6], [7].

We recall two approximation operators ϕ_E and ψ_E, which appear in a natural way in the theory of fuzzy rough sets (see, e.g., [4], [10], [14]). Fuzzy sets $\phi_E(\mu)$ and $\psi_E(\mu)$ were introduced to provide upper and lower approximation of a fuzzy set μ by extensional fuzzy sets with respect to fuzzy equivalence relation E.

Definition 3: Let T be a left-continuous t-norm and E be a T-fuzzy equivalence relation on a set D. The maps $\phi_E\colon [0,1]^D \to [0,1]^D$ and $\psi_E\colon [0,1]^D \to [0,1]^D$ are defined for by:

$$\phi_E(\mu)(x) = \sup_{y \in D} T(E(x,y), \mu(y)),$$

$$\psi_E(\mu)(x) = \inf_{y \in D} \overrightarrow{T}(E(x,y)|\mu(y)),$$

for all $x \in D$ and for all $\mu \in [0,1]^D$, where $\overrightarrow{T}$ is the residuum of T defined for all $x, y \in [0,1]$ by

$$\overrightarrow{T}(x|y) = \sup\{\alpha \in [0,1] \mid T(\alpha, x) \leq y\}.$$

We recall the following basic properties of the residuum, which will be used later in the paper:
$(\overrightarrow{T}1)$ $\overrightarrow{T}(x|y) = 1$ if and only if $x \leq y$;
$(\overrightarrow{T}2)$ $\overrightarrow{T}(1|y) = y$ for all $y \in [0,1]$;
$(\overrightarrow{T}3)$ $\overrightarrow{T}(0|y) = 1$ for all $y \in [0,1]$;
$(\overrightarrow{T}4)$ if $y = 0$ and $x \neq 0$, then $\overrightarrow{T}(x|y) = 0$;
$(\overrightarrow{T}5)$ residuum is a non-increasing function with respect to the first argument and a non-decreasing function with respect to the second argument:

$$x_1 \leq x_2 \Longrightarrow \overrightarrow{T}(x_1|y) \geq \overrightarrow{T}(x_2|y);$$

$$y_1 \leq y_2 \Longrightarrow \overrightarrow{T}(x|y_1) \leq \overrightarrow{T}(x|y_2).$$

III. FACTORAGGREGATION

In this section we recall the definition of a factoraggregation operator, which is based on a crisp equivalence relation. This concept was introduced and studied in [11], [12]. Let us start with the classical notion of an aggregation operator (see, e.g., [1], [3], [5]).

Definition 4: A mapping $A : \bigcup_n [0,1]^n \to [0,1]$ is called an aggregation operator if and only if the following conditions hold:
$(A1)$ $A(0,\ldots,0) = 0$*;*
$(A2)$ $A(1,\ldots,1) = 1$*;*
$(A3)$ *for all $n \in \mathbb{N}$ for all $x_1,\ldots,x_n, y_1,\ldots,y_n \in [0,1]$:*

$$x_1 \leq y_1, \ldots, x_n \leq y_n \Longrightarrow$$

$$\Longrightarrow A(x_1,\ldots,x_n) \leq A(y_1,\ldots,y_n).$$

Conditions (A1) and (A2) are called the boundary conditions of A, but (A3) means the monotonicity of A.

The general aggregation operator $\tilde{A}$ acting on $[0,1]^D$, where $[0,1]^D$ is the set of all fuzzy subsets of a set D, was introduced in 2003 by A. Takaci [13]. We denote an order on $[0,1]^D$ by $\preceq$. The least and the greatest elements of this order are denoted by $\tilde{0}$ and $\tilde{1}$, which are indicators of $\varnothing$ and D respectively, i.e.

$$\tilde{0}(x) = 0 \text{ and } \tilde{1}(x) = 1 \text{ for all } x \in D.$$

Definition 5: A mapping $\tilde{A}\colon \bigcup_n ([0,1]^D)^n \to [0,1]^D$ is called a general aggregation operator if and only if the following conditions hold:
$(\tilde{A}1)$ $\tilde{A}(\tilde{0},\ldots,\tilde{0}) = \tilde{0}$*;*
$(\tilde{A}2)$ $\tilde{A}(\tilde{1},\ldots,\tilde{1}) = \tilde{1}$*;*
$(\tilde{A}3)$ *for all $n \in \mathbb{N}$ for all $\mu_1,...,\mu_n, \eta_1,...,\eta_n \in [0,1]^D$:*

$$\mu_1 \preceq \eta_1, \ldots, \mu_n \preceq \eta_n \Longrightarrow$$

$$\Longrightarrow \tilde{A}(\mu_1,\ldots,\mu_n) \preceq \tilde{A}(\eta_1,\ldots,\eta_n).$$

We consider the case:

$$\mu \preceq \eta \text{ if and only if } \mu(x) \leq \eta(x) \text{ for all } x \in D,$$

for $\mu, \eta \in [0,1]^D$.

There exist several approaches to construct a general aggregation operator $\tilde{A}$ based on an ordinary aggregation operator A. The most simplest one is the pointwise extension of an aggregation operator A:

$$\tilde{A}(\mu_1,...,\mu_n)(x) = A(\mu_1(x),...,\mu_n(x)),$$

where $\mu_1,...,\mu_n \in [0,1]^D$ are fuzzy sets and $x \in D$.

A widely used approach to constructing a general aggregation operator $\tilde{A}$ is the T - extension [13], whose idea comes from the classical extension principle and uses a t-norm T (see, e.g., [8]):

$$\tilde{A}(\mu_1,\mu_2,...,\mu_n)(x) =$$

$$= \sup_{x=A(u_1,u_2,...,u_n)} T(\mu_1(u_1), \mu_2(u_2), ..., \mu_n(u_n)).$$

Here $\mu_1, \mu_2, \ldots, \mu_n \in [0,1]^D$ and $x, u_1, u_2, \ldots, u_n \in D$, where $D = [0,1]$.

Another method of constructing a general aggregation operator is factoraggregation [11], [12]. This method is based on an equivalence relation ρ defined on a set D and allows to aggregate fuzzy subsets of D taking into account the classes of equivalence ρ, i.e. the corresponding partition of D.

Definition 6: Let $A\colon \bigcup_n [0,1]^n \to [0,1]$ be an ordinary *aggregation operator and ρ be an equivalence relation defined on a set D. An operator*

$$\tilde{A}_\rho \colon \bigcup_n ([0,1]^D)^n \to [0,1]^D$$

such as

$$\tilde{A}_\rho(\mu_1, \mu_2, \ldots, \mu_n)(x) =$$
$$= \sup_{u \in D : (u,x) \in \rho} A(\mu_1(u), \mu_2(u), \ldots, \mu_n(u)),$$

where $x \in D$ and $\mu_1, \mu_2, \ldots, \mu_n \in [0,1]^D$, is called a factoraggregation operator corresponding to ρ.

The motivation of using the name factoraggregation for $\tilde{A}_\rho$ is that ρ factorizes D into the classes of equivalence. Operator $\tilde{A}_\rho$ aggregates fuzzy sets $\mu_1, \mu_2, \ldots, \mu_n$ in accordance with these classes of equivalence. In this construction for evaluation of $\tilde{A}_\rho(\mu_1, \mu_2, \ldots, \mu_n)(x)$ we take the supremum of aggregation A of values $\mu_1(u), \mu_2(u), \ldots, \mu_n(u)$ on the set of all points u, which are equivalent to x with respect to ρ, i.e. we consider all elements $u \in D$ such that $(u, x) \in \rho$.

In our previous papers [11], [12] this construction was used for the analysis of solving parameters for bilevel linear programming problems with one objective on the upper level P^U with the higher priority in optimization than multiple objectives on the lower level $P^L = (P_1^L, P_2^L, ..., P_n^L)$:

$$P^U: \quad y_0(x) = c_{01}x_1 + ... + c_{0k}x_k \longrightarrow \min$$
$$P_i^L: \quad y_i(x) = c_{i1}x_1 + ... + c_{ik}x_k \longrightarrow \min, \; i = \overline{1,n}$$

$$D: \begin{cases} a_{j1}x_1 + ... + a_{jk}x_k \le b_j, \; j = \overline{1,m}, \\ x_l \ge 0, \; l = \overline{1,k}, \end{cases}$$

where $k, l, m, n \in \mathbb{N}$, $a_{jl}, b_j, c_{il} \in \mathbb{R}$, $j = \overline{1,m}$, $l = \overline{1,k}$, $i = \overline{0,n}$,

and $x = (x_1, ..., x_k) \in \mathbb{R}^k$, $D \subset \mathbb{R}^k$ is non-empty and bounded.

The factoraggregation was applied to the membership functions of the objectives, which characterise how the corresponding objective function is close to its optimal value (see [16]):

$$\mu_i(x) = \begin{cases} 1, & y_i(x) < y_i^{min}, \\ \dfrac{y_i(x) - y_i^{max}}{y_i^{min} - y_i^{max}}, & y_i^{min} \le y_i(x) \le y_i^{max}, \\ 0, & y_i(x) > y_i^{max}, \end{cases}$$

where y_i^{min} and y_i^{max} are the individual minimum and the individual maximum of the objective y_i subject to the given constraints, $i = \overline{0,n}$. The introduced operator aggregates the membership functions on the lower level considering the classes of equivalence generated by the membership function on the upper level:

$$\tilde{A}(\mu_1, ..., \mu_n)(x) = \max_{\mu_0(x) = \mu_0(u)} A(\mu_1(u), ..., \mu_n(u)), \quad x \in D.$$

In this case μ_0 generates the equivalence relation ρ_{μ_0}:

$$(u, v) \in \rho_{\mu_0} \Longleftrightarrow \mu_0(u) = \mu_0(v).$$

The role of this equivalence in the construction of factoraggregation follows from the hierarchy between the objectives, it was explained in details in [11], [12].

The factoraggregation operator is a general aggregation operator. In the next section we generalize this construction by using a t-norm T and a T-fuzzy equivalence relation E instead of the crisp one.

IV. Upper and Lower Generalized Factoraggregations

In this paper we modify the construction of factoraggregation by using a T-fuzzy equivalence relation E in order to obtain a T-fuzzy generalization of factoraggregation.

Definition 7: Let $A \colon \bigcup_n [0,1]^n \to [0,1]$ be an ordinary aggregation operator, T be a t-norm and E be a T-fuzzy equivalence relation defined on D. An operator

$$\tilde{A}_{E,T} \colon \bigcup_n ([0,1]^D)^n \to [0,1]^D$$

such as

$$\tilde{A}_{E,T}(\mu_1, \mu_2, \ldots, \mu_n)(x) =$$
$$= \sup_{u \in D} T(E(x,u), A(\mu_1(u), \mu_2(u), \ldots, \mu_n(u))), \tag{1}$$

where $x \in D$ and $\mu_1, \mu_2, \ldots, \mu_n \in [0,1]^D$, is called an upper generalized T-fuzzy factoraggregation corresponding to E.

Let us prove that the construction (1) gives us a general aggregation operator. We must show that conditions $(\tilde{A}1)$, $(\tilde{A}2)$ and $(\tilde{A}3)$ are satisfied.

Proposition 8: Let $A \colon \bigcup_n [0,1]^n \to [0,1]$ be an ordinary aggregation operator, T be a t-norm and E be a T-fuzzy equivalence relation defined on D. Operator $\tilde{A}_{E,T}$ given by (1) is a general aggregation operator.

Proof: First we prove the boundary conditions:

1)
$$\tilde{A}_{E,T}(\tilde{0}, \ldots, \tilde{0})(x) =$$
$$= \sup_{u \in D} T(E(x,u), A(\tilde{0}(u), \ldots, \tilde{0}(u))) =$$
$$= \sup_{u \in D} T(E(x,u), A(0, \ldots, 0)) =$$
$$= \sup_{u \in D} T(E(x,u), 0) = \tilde{0}(x);$$

2)
$$\tilde{A}_{E,T}(\tilde{1}, \ldots, \tilde{1})(x) =$$
$$= \sup_{u \in D} T(E(x,u), A(\tilde{1}(u), \ldots, \tilde{1}(u))) =$$
$$= \sup_{u \in D} T(E(x,u), A(1, \ldots, 1)) =$$
$$= \sup_{u \in D} T(E(x,u), 1) = \sup_{u \in D} E(x,u) = \tilde{1}(x).$$

To prove the monotonicity of $\tilde{A}_{E,T}$ we use the monotonicity of A and T:

$$\mu_i \preceq \eta_i, \quad i = 1, 2, \ldots, n \implies$$
$$\implies A(\mu_1(u), \ldots, \mu_n(u)) \leq$$
$$\leq A(\eta_1(u), \ldots, \eta_n(u)) \text{ for all } u \in D \implies$$
$$\implies T(E(x,u), A(\mu_1(u), \ldots, \mu_n(u))) \leq$$
$$\leq T(E(x,u), A(\eta_1(u), \ldots, \eta_n(u)))$$
$$\text{for all } x \in D \text{ and for all } u \in D \implies$$
$$\implies \sup_{u \in D} T(E(x,u), A(\mu_1(u), \ldots, \mu_n(u))) \leq$$
$$\leq \sup_{u \in D} T(E(x,u), A(\eta_1(u), \ldots, \eta_n(u))) \text{ for } x \in D \implies$$
$$\implies \tilde{A}_{E,T}(\mu_1, \ldots, \mu_n) \preceq \tilde{A}_{E,T}(\eta_1, \ldots, \eta_n).$$

Now we will define another general aggregation operator analogously to an upper generalized T-fuzzy factoraggregation, where instead of $\sup$ and T we use $\inf$ and $\overrightarrow{T}$.

Definition 9: Let $A\colon \bigcup_n [0,1]^n \to [0,1]$ be an ordinary aggregation operator, $\overrightarrow{T}$ be the residuum of a left-continuous t-norm T, and E be a T-fuzzy equivalence relation defined on D. An operator

$$\tilde{A}_{E,\overrightarrow{T}}\colon \bigcup_n ([0,1]^D)^n \to [0,1]^D$$

such as

$$\tilde{A}_{E,\overrightarrow{T}}(\mu_1, \mu_2, \ldots, \mu_n)(x) =$$
$$= \inf_{u \in D} \overrightarrow{T}(E(x,u) | A(\mu_1(u), \mu_2(u), \ldots, \mu_n(u))), \tag{2}$$

where $\mu_1, \mu_2, \ldots, \mu_n \in [0,1]^D$ and $x \in D$, is called a lower generalized T-fuzzy factoraggregation corresponding to E.

Again, let us prove that the construction (2) is a general aggregation operator.

Proposition 10: Let $A\colon \bigcup_n [0,1]^n \to [0,1]$ be an ordinary aggregation operator, T be a left-continuous t-norm with the residuum $\overrightarrow{T}$, and E be a T-fuzzy equivalence relation defined on D. Operator $\tilde{A}_{E,\overrightarrow{T}}$ given by (2) is a general aggregation operator.

Proof: First we prove the boundary conditions, using the basic properties of $\overrightarrow{T}$:

1)
$$\tilde{A}_{E,\overrightarrow{T}}(\tilde{0}, \ldots, \tilde{0})(x) =$$
$$= \inf_{u \in D} \overrightarrow{T}(E(x,u) | A(\tilde{0}(u), \ldots, \tilde{0}(u))) =$$
$$= \inf_{u \in D} \overrightarrow{T}(E(x,u) | A(0, \ldots, 0)) =$$
$$= \inf_{u \in D} \overrightarrow{T}(E(x,u) | 0) = \tilde{0}(x);$$

2)
$$\tilde{A}_{E,\overrightarrow{T}}(\tilde{1}, \ldots, \tilde{1})(x) =$$
$$= \inf_{u \in D} \overrightarrow{T}(E(x,u) | A(\tilde{1}(u), \ldots, \tilde{1}(u))) =$$
$$= \inf_{u \in D} \overrightarrow{T}(E(x,u) | A(1, \ldots, 1)) =$$
$$= \inf_{u \in D} \overrightarrow{T}(E(x,u) | 1) = \tilde{1}(x).$$

To prove the monotonicity of $\tilde{A}_{E,\overrightarrow{T}}$ we use the monotonicity of A and basic properties of $\overrightarrow{T}$:

$$\mu_i \preceq \eta_i, \quad i = 1, 2, \ldots, n \implies$$
$$\implies A(\mu_1(u), \ldots, \mu_n(u)) \leq$$
$$\leq A(\eta_1(u), \ldots, \eta_n(u)) \text{ for all } u \in D \implies$$
$$\implies \overrightarrow{T}(E(x,u) | A(\mu_1(u), \ldots, \mu_n(u))) \leq$$
$$\leq \overrightarrow{T}(E(x,u) | A(\eta_1(u), \ldots, \eta_n(u)))$$
$$\text{for all } x \in D \text{ and for all } u \in D \implies$$
$$\implies \inf_{u \in D} \overrightarrow{T}(E(x,u) | A(\mu_1(u), \ldots, \mu_n(u))) \leq$$
$$\leq \inf_{u \in D} \overrightarrow{T}(E(x,u) | A(\eta_1(u), \ldots, \eta_n(u))) \text{ for } x \in D \implies$$
$$\implies \tilde{A}_{E,\overrightarrow{T}}(\mu_1, \ldots, \mu_n) \preceq \tilde{A}_{E,\overrightarrow{T}}(\eta_1, \ldots, \eta_n).$$

It is clear, that for all $\mu_1, \ldots, \mu_n \in [0,1]^D$ and for all $x \in D$ it holds

$$\tilde{A}_{E,\overrightarrow{T}}(\mu_1, \ldots, \mu_n)(x) \leq A(\mu_1(x), \ldots, \mu_n(x)) \leq$$
$$\leq \tilde{A}_{E,T}(\mu_1, \ldots, \mu_n)(x).$$

Indeed,

$$A(\mu_1(x), \ldots, \mu_n(x)) =$$
$$= T(E(x,x), A(\mu_1(x), \ldots, \mu_n(x))) \leq$$
$$\leq \sup_{u \in D} T(E(x,u), A(\mu_1(u), \ldots, \mu_n(u))) =$$
$$= \tilde{A}_{E,T}(\mu_1, \ldots, \mu_n)(x);$$
$$A(\mu_1(x), \ldots, \mu_n(x)) =$$
$$= \overrightarrow{T}(E(x,x) | A(\mu_1(u), \ldots, \mu_n(u))) \geq$$
$$\geq \inf_{u \in D} \overrightarrow{T}(E(x,u) | A(\mu_1(u), \ldots, \mu_n(u))) =$$
$$= \tilde{A}_{E,\overrightarrow{T}}(\mu_1, \ldots, \mu_n)(x).$$

Let us note that the results of upper and lower generalized T-fuzzy factoraggregations corresponding to E are extensional fuzzy sets with respect to T-fuzzy equivalence E.

V. Numerical examples

Now we illustrate the generalized T-fuzzy factoraggregation with some particular numerical examples. Here and throughout the paper the numerical inputs are taken from [9].

Let us consider the discrete universe

$$D = \{x_1, x_2, x_3, x_4, x_5\}$$

and the following T_M-fuzzy (T_M is the minimum t-norm equivalence relation E, given in the matrix form:

$$E = \begin{pmatrix} 1 & 0.9 & 0.7 & 0.4 & 0.2 \\ 0.9 & 1 & 0.7 & 0.4 & 0.2 \\ 0.7 & 0.7 & 1 & 0.4 & 0.2 \\ 0.4 & 0.4 & 0.4 & 1 & 0.2 \\ 0.2 & 0.2 & 0.2 & 0.2 & 1 \end{pmatrix}.$$

This equivalence relation is also T_L-transitive and T_P-transitive, i.e. transitive with respect to the Lukasiewicz t-norm T_L and the product t-norm T_P respectively. Relation E has a noteworthy feature: elements x_4 and x_5 are equivalent to other elements with relatively lower degree, than elements x_1, x_2 and x_3. As will be seen from the further examples, this fact has significant impact on the result of factoraggregation.

Let us take the following fuzzy subsets of D:

$$\mu_1 = \begin{pmatrix} 0.9 \\ 0.5 \\ 0.6 \\ 0.8 \\ 0.3 \end{pmatrix}, \ \mu_2 = \begin{pmatrix} 0.2 \\ 0.0 \\ 0.2 \\ 0.6 \\ 0.9 \end{pmatrix}, \ \mu_3 = \begin{pmatrix} 0.7 \\ 0.5 \\ 0.1 \\ 0.8 \\ 0.6 \end{pmatrix}, \ \mu_4 = \begin{pmatrix} 0.1 \\ 0.9 \\ 0.2 \\ 0.8 \\ 0.5 \end{pmatrix}.$$

Now we consider the minimum aggregation operator

$$A = MIN$$

and obtain the following upper generalized T-fuzzy factoraggregation:

$$\tilde{A}_{E,T}(\mu_1, \mu_2, \mu_3, \mu_4)(x) =$$

$$= \max_{u \in D} T(E(x,u), \min(\mu_1(u), \mu_2(u), \mu_3(u), \mu_4(u))).$$

Taking $T = T_L$, $T = T_M$ and $T = T_P$ we obtain as results the fuzzy subsets μ_{T_L}, μ_{T_M} and μ_{T_P} respectively:

$$\mu_{T_L} = \begin{pmatrix} 0.1 \\ 0.0 \\ 0.1 \\ 0.6 \\ 0.3 \end{pmatrix}, \ \mu_{T_M} = \begin{pmatrix} 0.4 \\ 0.4 \\ 0.4 \\ 0.6 \\ 0.3 \end{pmatrix}, \ \mu_{T_P} = \begin{pmatrix} 0.24 \\ 0.24 \\ 0.24 \\ 0.6 \\ 0.3 \end{pmatrix}.$$

First, let us note, that the result μ_T at points x_4 and x_5 does not depend on the choice of the t-norm. It could be explained with low degrees of equivalence for these points with respect to other elements. Therefore the result of factoraggregation is not effected by these other elements. Second, the values of μ_T at points x_1, x_2 and x_3 depend on each other, since the degree of equivalence between any two of these points is relatively high, and at the same time the results of ordinary aggregation of $\mu_1(x)$, $\mu_2(x)$, $\mu_3(x)$ and $\mu_4(x)$ at these points are relatively small.

Taking as an ordinary aggregation operator the arithmetic mean aggregation operator $A = AVG$, we obtain the following upper generalized T-fuzzy factoraggregations respectively:

$$\tilde{A}_{E,T}(\mu_1, \mu_2, \mu_3, \mu_4)(x) =$$

$$= \max_{u \in D} T(E(x,u), AVG(\mu_1(u), \mu_2(u), \mu_3(u), \mu_4(u))).$$

Taking $T = T_L$, $T = T_M$ and $T = T_P$ we obtain as results the following fuzzy subsets:

$$\mu_{T_L} = \begin{pmatrix} 0.475 \\ 0.475 \\ 0.275 \\ 0.750 \\ 0.575 \end{pmatrix}, \ \mu_{T_M} = \begin{pmatrix} 0.475 \\ 0.475 \\ 0.475 \\ 0.750 \\ 0.575 \end{pmatrix}, \ \mu_{T_P} = \begin{pmatrix} 0.475 \\ 0.475 \\ 0.333 \\ 0.750 \\ 0.575 \end{pmatrix},$$

Here again, one can see, that the result μ_T at points x_4 and x_5 does not depend on the choice of the t-norm and is not effected by other points x_1, x_2 and x_3. The values of the upper generalized factoraggregation at points x_1 and x_2 depend on each other because of the high equivalence degree between these two elements. The dependence of the value of μ_T at point x_3 on the values at points x_1 and x_2 is effected by the choice of the t-norm.

Similarly, we will calculate several results for the following lower generalized T-fuzzy factoraggregation:

$$\tilde{A}_{E,\overrightarrow{T}}(\mu_1, \mu_2, \mu_3, \mu_4)(x) =$$

$$= \min_{u \in D} \overrightarrow{T}(E(x,u) | AVG(\mu_1(u), \mu_2(u), \mu_3(u), \mu_4(u))).$$

Residua of left-continuous t-norms T_L, T_M and T_P are given by

$$\overrightarrow{T}_L(x|y) = \begin{cases} 1, & x \le y, \\ 1 - x + y, & x > y; \end{cases}$$

$$\overrightarrow{T}_M(x|y) = \begin{cases} 1, & x \le y, \\ y, & x > y; \end{cases}$$

$$\overrightarrow{T}_P(x|y) = \begin{cases} 1, & x \le y, \\ y/x, & x > y. \end{cases}$$

As a result we obtain the fuzzy subsets $\mu_{\overrightarrow{T}_L}$, $\mu_{\overrightarrow{T}_M}$ and $\mu_{\overrightarrow{T}_P}$ respectively:

$$\mu_{\overrightarrow{T}_L} = \begin{pmatrix} 0.475 \\ 0.475 \\ 0.275 \\ 0.750 \\ 0.575 \end{pmatrix}, \ \mu_{\overrightarrow{T}_M} = \begin{pmatrix} 0.275 \\ 0.275 \\ 0.275 \\ 0.275 \\ 0.575 \end{pmatrix}, \ \mu_{\overrightarrow{T}_P} = \begin{pmatrix} 0.393 \\ 0.393 \\ 0.275 \\ 0.688 \\ 0.575 \end{pmatrix}.$$

In this case the low degree of equivalence has major impact only at point x_5. The result of the lower generalized T-fuzzy factoraggregation at point x_4 now is also effected by other elements x_1, x_2 and x_3, while changing the t-norm.

VI. THE CASE OF CRISP EQUIVALENCE RELATION

Let ρ be an equivalence relation defined on a set D. We take $E = E_\rho$, where

$$E_\rho(x,y) = \begin{cases} 1, & (x,y) \in \rho, \\ 0, & (x,y) \notin \rho, \end{cases}$$

and obtain $\tilde{A}_{E_\rho,T} = \tilde{A}_\rho$ for any t-norm T:

$$\tilde{A}_{E_\rho,T}(\mu_1,\ldots,\mu_n)(x) =$$
$$= \sup_{u\in D} T(E_\rho(x,u), A(\mu_1(u),\ldots,\mu_n(u))) =$$
$$= \sup_{u\in D:(u,x)\in\rho} T(1, A(\mu_1(u),\ldots,\mu_n(u))) =$$
$$= \sup_{u\in D:(u,x)\in\rho} A(\mu_1(u),\ldots,\mu_n(u)) =$$
$$= \tilde{A}_\rho(\mu_1,\ldots,\mu_n)(x)$$

for all $\mu_1,\ldots,\mu_n \in [0,1]^D$ and $x \in D$.

Numerical evaluation of the value $\tilde{A}_{E_\rho,T}(\mu_1,...,\mu_n)(x)$ can be reduced to the problem

$$\alpha \longrightarrow \min$$
$$\begin{cases} A(\mu_1(u),\ldots,\mu_n(u)) \leq \alpha, \\ (u,x) \in \rho, \quad u \in D. \end{cases} \tag{3}$$

If we apply the crisp equivalence relation E_ρ in $\tilde{A}_{E_\rho,\overrightarrow{T}}$, for any left-continuous t-norm T we obtain the following formula:

$$\tilde{A}_{E_\rho,\overrightarrow{T}}(\mu_1,\ldots,\mu_n)(x) =$$
$$= \inf_{u\in D} \overrightarrow{T}(E_\rho(x,u)|A(\mu_1(u),\ldots,\mu_n(u))) =$$
$$= \inf_{u\in D:(u,x)\in\rho} \overrightarrow{T}(1|A(\mu_1(u),\ldots,\mu_n(u))) =$$
$$= \inf_{u\in D:(u,x)\in\rho} A(\mu_1(u),\ldots,\mu_n(u))$$

for all $\mu_1,\ldots,\mu_n \in [0,1]^D$ and $x \in D$.

By analogy with the previous case, numerical evaluation of the value $\tilde{A}_{E_\rho,\overrightarrow{T}}(\mu_1,...,\mu_n)(x)$ can be reduced to the problem

$$\alpha \longrightarrow \max$$
$$\begin{cases} A(\mu_1(u),\ldots,\mu_n(u)) \geq \alpha, \\ (u,x) \in \rho, \quad u \in D. \end{cases} \tag{4}$$

Let us note that problems (3) and (4) in the context of our investigation presented in [11], [12] can be solved by linear programming methods. Dealing with bilevel linear programming problems (BLPP), the use of factoaggregation provided us a possibility to evaluate the degree of optimization of the lower level objectives on the set, where the upper level objective reaches its prior defined degree of optimization. This evaluation in [11], [12] was taken as a basis for further analysis and adjustment of BLPP solving parameters.

VII. CONCLUSION

In this paper we introduced two aggregation operators which act taking into account a fuzzy equivalence relation: upper and lower generalized factoraggregations. In the case of a crisp equivalence relation the upper factoraggregation operator was successfully applied for the analysis of bilevel linear programming solving parameters in [11], [12]. We hope that the proposed generalization of factoraggregation will help us to investigate bilevel linear programming problem in the context mentioned above for the case of imprecise information.

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