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Interpolative GCD Aggregators

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Abstract—**This paper investigates the structure and properties of interpolative generalized conjunction/disjunction (GCD) aggregators. The main advantage of interpolative aggregators is the possibility to include suitable properties and to exclude inconvenient properties of aggregators. Using this method we can create new forms of logic aggregators by interpolating between heterogeneous base aggregators, i.e. aggregators that belong to different families and have different logic properties. The resulting interpolative aggregators then extend the desirable properties inherited from the base aggregator classes. We propose a general interpolative GCD aggregator (IGCD) and its important uniform special case. IGCD can be used to build compound aggregators and complex logic aggregation structures with suitable logic properties.**

I. INTRODUCTION

AGGREGATION operators are important components of many decision models. In the case of decision models that are used for evaluation and comparison of complex systems and alternatives the basic properties of logic aggregators include monotonicity, internality, idempotency and compensativeness. In addition, such aggregators must express different degrees of importance of individual inputs, and consequently they must be commutative only in the special case of equal importance, and in all other cases they must be noncommutative. Logic aggregators that have such properties are regularly implemented using means [12][4] and called averaging functions or averaging aggregators [1][13].

In mathematical literature [15][13] the process of aggregation is considered using any nonempty real interval, bounded or not. In the case of logic aggregation we are only interested in aggregation within the unit hypercube. So, we have a vector of continuous logic variables $\mathbf{x}=(x_1,...,x_n)$, $x_i \in I,\quad i=1,...,n,\quad I=[0,1]$ and the aggregation function $f: I^n \to I,\ n>1$. Not surprisingly, there are many families of aggregation functions and to include all of them the aggregation functions are defined in [1] and [13] very generally as functions that are nondecreasing in each argument and satisfy two boundary conditions as follows:

$$\forall \mathbf{x} \in I^n,\ \forall \mathbf{y} \in I^n,\ n>1,\ \mathbf{x} \le \mathbf{y} \quad \Rightarrow \quad f(\mathbf{x}) \le f(\mathbf{y})$$

$$f(0,0,...,0)=0, \quad f(1,1,...,1)=1\ .$$

($\mathbf{x} \le \mathbf{y}$ denotes $x_i \le y_i,\ i=1,...,n$; however, if $\forall i:\ x_i \le y_i$ and $\exists j \in \{1,...,n\}:\ x_j < y_j$, that is denoted $\mathbf{x} \le \mathbf{y}$, $\mathbf{x} \ne \mathbf{y}$).

Two significant families of aggregation functions that are closely related to logic operations are conjunctive and disjunctive aggregators implemented using t-norms and t-conorms and the averaging aggregators that are implemented using means. Both families provide models of simultaneity and substitutability (replaceability) [10]. However, models of simultaneity and substitutability implemented using t-norms and t-conorms are strong in the sense that *conjunctive* aggregation functions satisfy $f(\mathbf{x}) \le \min(\mathbf{x})$ and *disjunctive* aggregation functions satisfy $f(\mathbf{x}) \ge \max(\mathbf{x})$ [1] [13], where *min* and *max* denote the pure/full conjunction $\min(\mathbf{x})=\min(x_1,...,x_n)$ and the pure/full disjunction $\max(\mathbf{x})=\max(x_1,...,x_n)$. As opposed to that, the averaging aggregation functions are weaker in the sense that an aggregator is considered *predominantly conjunctive* if $\min(\mathbf{x}) \le f(\mathbf{x}) \le \mathrm{mid}(\mathbf{x})$ and *predominantly disjunctive* if $\mathrm{mid}(\mathbf{x}) \le f(\mathbf{x}) \le \max(\mathbf{x})$, where *mid* denotes the arithmetic mean $\mathrm{mid}(\mathbf{x})=(x_1+...+x_n)/n$, and $f(\mathbf{x})=\mathrm{mid}(\mathbf{x})$ only if $x_1=...=x_n$ and in all other cases $f(\mathbf{x}) \lesseqgtr \mathrm{mid}(\mathbf{x})$. A related fundamental difference between these families is that averaging functions support internality and idempotency, and t-norms and t-conorms are not idempotent. In the evaluation area idempotency is a necessary property: if all components of a system have the same value (same suitability degree), then that value is the overall value of the whole system. Since the averaging functions by definition satisfy internality $\min(\mathbf{x}) \le f(\mathbf{x}) \le \max(\mathbf{x})$, then if $x \in I$ and $\mathbf{x}=(x,x,...,x)$ we have $f(x,x,...,x)=x$ (idempotency is a consequence of internality and monotonicity). The last fundamental difference between the averaging aggregators and t-norms/conorms is the strict monotonicity of averaging aggregators inside I^n . Interpolative logic aggregators are applicable primarily in evaluation and consequently, in this paper we focus on the family of averaging aggregators.

The paper is organized as follows. In Section II of the paper we introduce a set of definitions that characterize logic aggregators. In Section III we analyze additive properties of andness and orness indicators. Development and properties of interpolative GCD aggregators are presented in Section IV and Section V.

II. PROPERTIES OF LOGIC AGGREGATORS

In the context of logic aggregation of degrees of preference (or suitability, or degrees of fuzzy membership), we use logic aggregators $A: I^n \to I,\ n>1$ that generalize classic Boolean

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logic operations and have the following properties:

Definition 1. A logic aggregator $A: I^n \to I$ is an averaging function that satisfies (a) strict monotonicity inside I^n, (b) continuity, (c) internality, (d) idempotency, (e) compensativeness, and (f) boundary conditions, as follows:

(a) $\forall \mathbf{x} \in I^n,\ \forall \mathbf{y} \in I^n,\ \mathbf{x} \le \mathbf{y} \ \Rightarrow\ A(\mathbf{x}) \le A(\mathbf{y})$

$\forall \mathbf{x} \in]0,1[^n,\ \forall \mathbf{y} \in]0,1[^n,\ \mathbf{x} \le \mathbf{y},\ \mathbf{x} \ne \mathbf{y} \ \Rightarrow\ A(\mathbf{x}) < A(\mathbf{y})$

(b) $\forall \mathbf{x} \in I^n,\ \forall \mathbf{y} \in I^n,\ \mathbf{x} \le \mathbf{y},\ \forall c \in [A(\mathbf{x}), A(\mathbf{y})] \ \Rightarrow$

$\exists \mathbf{z} \in I^n,\ \mathbf{x} \le \mathbf{z} \le \mathbf{y},\ A(\mathbf{z}) = c$

(c) $\forall \mathbf{x} \in I^n \ \Rightarrow\ \min(x_1,...,x_n) \le A(\mathbf{x}) \le \max(x_1,...,x_n)$

(d) $\forall x \in I \ \Rightarrow\ A(x,x,...,x) = x$

(e) $\forall x_i \in]0,1[,\ \forall x_j \in]0,1[,\ i \ne j,\ \exists \delta_{ij} \in]0,1[\ \Rightarrow$

$A(x_1,...,x_i - \varepsilon_i,...,x_j + \varepsilon_j,...,x_n) =$

$A(x_1,...,x_i,...,x_j,...,x_n), \quad 0 < \varepsilon_i \le \delta_{ij}, \quad 0 < x_j + \varepsilon_j \le 1$

(f) $A(0,0,...,0) = 0, \quad A(1,1,...,1) = 1$.

Note 1. For this type of aggregators, idempotency is a direct consequence of internality and the boundary conditions are a direct consequence of idempotency.

Definition 2. Logic aggregators are symmetric (commutative) if they generate the same result for any permutation of arguments; if not, they are asymmetric (noncommutative):

$\forall i \in \{1,...,n\},\ \forall j \in \{1,...,n\},\ i \ne j,\ x_i \ne x_j \ \Rightarrow$

$A(...,x_i,...,x_j,...) = A(...,x_j,...,x_i,...)$, or

$A(...,x_i,...,x_j,...) \lessgtr A(...,x_j,...,x_i,...)$.

Note 2. Asymmetry is indispensable for supporting semantic aspects of aggregation, primarily for modeling different degrees of importance or arguments. Asymmetry is usually realized using different weights of aggregators and in such cases the symmetry is a special case that corresponds to equal weights.

Definition 3. A logic aggregator A is conjunctive if it is concave in each argument:

$\forall t \in I,\ \forall a \in I,\ \forall b \in I,\ a \ne b,$

$\forall i \in \{1,...,n\},\ x_i = ta + (1-t)b \ \Rightarrow$

$A(x_1,...,x_i,...,x_n) \ge tA(x_1,...,a,...,x_n) + (1-t)A(x_1,...,b,...,x_n)$

Definition 4. A logic aggregator A is disjunctive if it is convex in each argument:

$\forall t \in I,\ \forall a \in I,\ \forall b \in I,\ a \ne b,$

$\forall i \in \{1,...,n\},\ x_i = ta + (1-t)b \ \Rightarrow$

$A(x_1,...,x_i,...,x_n) \le tA(x_1,...,a,...,x_n) + (1-t)A(x_1,...,b,...,x_n)$

Definition 5. A logic aggregator A is neutral if it is both concave and convex:

$\forall t \in I,\ \forall a \in I,\ \forall b \in I,\ a \ne b,$

$\forall i \in \{1,...,n\},\ x_i = ta + (1-t)b \ \Rightarrow$

$A(x_1,...,x_i,...,x_n) = tA(x_1,...,a,...,x_n) + (1-t)A(x_1,...,b,...,x_n)$

Definition 6. A logic aggregator A has annihilator $a \in I$ in a specific argument $x_i, i \in \{1,...,n\}$ if for $x_i = a$ and $\forall x_j \in I$, $j \ne i$ we have $A(x_1,...,x_i,...,x_n) = a$. Otherwise, A is an aggregator without annihilator in argument $x_i, i \in \{1,...,n\}$.

Definition 7. A logic aggregator A has *homogeneous annihilators* if it is without annihilator in all arguments, or if it has the annihilator a in all arguments ($\forall i \in \{1,...,n\},\ x_i = a$ and $\forall x_j \in I,\ j \ne i \ \Rightarrow\ A(\mathbf{x}) = a$).

Definition 8. A logic aggregator A has *heterogeneous annihilators* if it has the annihilator a in a subset of arguments, and it is without annihilator in all other arguments.

Definition 9. A conjunctive logic aggregator A with homogeneous annihilators is *hard*, if it is has the annihilator 0 in each argument:

$\forall i \in \{1,...,n\},\ x_i = 0,\ x_j \ge 0,\ j \ne i \ \Rightarrow\ A(x_1,...,x_i,...,x_n) = 0$

A hard conjunctive logic aggregator is called a *Hard Partial Conjunction (HPC)* if $A(\mathbf{x}) > \min(\mathbf{x})$, $\mathbf{x} \ne \underline{\mathbf{x}} = (x,...,x)$, $x \in I$ (the arguments are not all equal).

Definition 10. A conjunctive logic aggregator A with homogeneous annihilators is *soft*, if it is without the annihilator 0 in all arguments:

$\forall i \in \{1,...,n\},\ x_i > 0,\ x_j = 0,\ j \ne i \ \Rightarrow\ A(x_1,...,x_i,...,x_n) > 0$

A soft conjunctive logic aggregator is called a *Soft Partial Conjunction (SPC)* if $A(\mathbf{x}) < \mathrm{mid}(\mathbf{x})$, $\mathbf{x} \ne \underline{\mathbf{x}} = (x,...,x)$, $x \in I$.

Definition 11. A parameterized conjunctive logic aggregator A that has homogeneous annihilators and can be either hard or soft is called a *Partial Conjunction (PC)*. PC satisfies the condition $\min(\mathbf{x}) < A(\mathbf{x}) < \mathrm{mid}(\mathbf{x})$, $\mathbf{x} \ne \underline{\mathbf{x}} = (x,...,x)$, $x \in I$. A symmetric PC is symbolically denoted $A(\mathbf{x}) = x_1 \Delta ... \Delta x_n$ and the asymmetric version is $A(\mathbf{x};\mathbf{W}) = W_1 x_1 \Delta ... \Delta W_n x_n$.

Definition 12. A disjunctive logic aggregator A with homogeneous annihilators is *hard*, if it has the annihilator 1 in each argument:

$\forall i \in \{1,...,n\},\ x_i = 1,\ x_j \le 1,\ j \ne i \ \Rightarrow\ A(x_1,...,x_i,...,x_n) = 1$

A hard disjunctive logic aggregator is called a *Hard Partial Disjunction (HPD)* if $A(\mathbf{x}) < \max(\mathbf{x})$, $\mathbf{x} \ne \underline{\mathbf{x}} = (x,...,x)$, $x \in I$.

Definition 13. A disjunctive logic aggregator A with homogeneous annihilators is *soft*, if it is without the annihilator 1 in all arguments:

$\forall i \in \{1,...,n\},\ x_i < 1,\ x_j = 1,\ j \ne i \ \Rightarrow\ A(x_1,...,x_i,...,x_n) < 1$

A soft disjunctive logic aggregator is called a *Soft Partial Disjunction (SPD)* if $A(\mathbf{x}) < \max(\mathbf{x})$, $\mathbf{x} \ne \underline{\mathbf{x}} = (x,...,x)$, $x \in I$.

Definition 14. A parameterized disjunctive logic aggregator A that has homogeneous annihilators and can be either hard or soft is called a *Partial Disjunction (PD)*. PD satisfies the condition $\mathrm{mid}(\mathbf{x}) < A(\mathbf{x}) < \max(\mathbf{x})$, $\mathbf{x} \ne \underline{\mathbf{x}} = (x,...,x)$, $x \in I$. A symmetric PD is symbolically denoted $A(\mathbf{x}) = x_1 \nabla ... \nabla x_n$ and the asymmetric version is $A(\mathbf{x};\mathbf{W}) = W_1 x_1 \nabla ... \nabla W_n x_n$.

Definition 15. A conjunction degree or andness $\alpha \in I$ is a degree of similarity between a symmetric (commutative) logic aggregator $A(\mathbf{x})$ with homogeneous annihilators and the pure conjunction; its fixed values are:

$$\alpha = \begin{cases} 0, & \text{if } A(\mathbf{x}) = \max(\mathbf{x}) \\ ½, & \text{if } A(\mathbf{x}) = \text{mid}(\mathbf{x}) \\ 1, & \text{if } A(\mathbf{x}) = \min(\mathbf{x}) \end{cases}$$

Andness can also be interpreted as a degree of membership of a logic aggregator in the fuzzy set of conjunctive aggregators.

Definition 16. A disjunction degree or orness $\omega \in I$ is a degree of similarity between a symmetric (commutative) logic aggregator $A(\mathbf{x})$ with homogeneous annihilators and the pure disjunction; its fixed values are:

$$\omega = \begin{cases} 0, & \text{if } A(\mathbf{x}) = \min(\mathbf{x}) \\ ½, & \text{if } A(\mathbf{x}) = \text{mid}(\mathbf{x}) \\ 1, & \text{if } A(\mathbf{x}) = \max(\mathbf{x}) \end{cases}$$

Orness can also be interpreted as a degree of membership of a logic aggregator in the fuzzy set of disjunctive aggregators.

Definition 17. Simultaneity and substitutability (replaceability) are complementary properties. Consequently, andness and orness are complementary indicators: $\alpha + \omega = 1$.

Definition 18. The threshold andness α_θ is the lowest andness of the hard partial conjunction.

Definition 19. The threshold orness ω_θ is the lowest orness of the hard partial disjunction.

Definition 20. The partial conjunction has the andness in the range $½ < \alpha < 1$. In the range $½ < \alpha < \alpha_\theta$ the partial conjunction is soft and in the range $\alpha_\theta \le \alpha < 1$ the partial conjunction is hard.

Definition 21. The partial disjunction has the orness in the range $½ < \omega < 1$. In the range $½ < \omega < \omega_\theta$ the partial disjunction is soft and in the range $\omega_\theta \le \omega < 1$ the partial disjunction is hard.

Definition 22. A logic aggregator is called the *Generalized Conjunction/disjunction (GCD)* if it has homogeneous annihilators and supports all values of andness $\alpha \in I$ and all values of orness $\omega \in I$. Thus, GCD supports the full (pure) conjunction (min(**x**)), PC, neutrality (mid(**x**)), PD, and the full (pure) disjunction (max(**x**)). The continuous transition from conjunction to disjunction is realized by selecting appropriate values of andness/orness.

Note 3: GCD can be symmetric or asymmetric. Andness and orness are defined for symmetric version of GCD. The arguments of GCD have weights, and in the case of symmetric version of GCD all weights are equal: $W_1 = \ldots = W_n = 1/n$. Asymmetric versions of GCD are realized using two or more weights different from $1/n$. A symmetric version of GCD is symbolically denoted $A(\mathbf{x}) = x_1 \Diamond \ldots \Diamond x_n$ and the asymmetric version is $A(\mathbf{x}; \mathbf{W}) = W_1 x_1 \Diamond \ldots \Diamond W_n x_n$.

Definition 23. GCD is *uniform* if each of its principal special cases (HPC, SPC, SPD, and HPD) partitions the region of andness/orness in equal parts (1/4 for each of them). In other words, the uniform GCD is characterized by $\alpha_\theta = \omega_\theta = 3/4$. If HPC, SPC, SPD, and HPD partition the region of andness/orness in parts that are not equal, then such a version of GCD is *nonuniform.*

III. Additivity of Andness and Orness

Three basic definitions of andness and orness are:

(1) *The local andness/orness* [5]:

$$\alpha_\ell(\mathbf{x}) = \frac{x_1 \vee \ldots \vee x_n - A(\mathbf{x})}{x_1 \vee \ldots \vee x_n - x_1 \wedge \ldots \wedge x_n},$$

$$\omega_\ell(\mathbf{x}) = 1 - \alpha_\ell(\mathbf{x}) = \frac{A(\mathbf{x}) - x_1 \wedge \ldots \wedge x_n}{x_1 \vee \ldots \vee x_n - x_1 \wedge \ldots \wedge x_n}$$

(2) *The mean local andness/orness* [5]:

$$\begin{aligned}\bar{\alpha}_\ell &= \int_{I^n} \alpha_\ell(\mathbf{x})\, dx_1 \ldots dx_n \\ &= \int_{I^n} \frac{x_1 \vee \ldots \vee x_n - A(\mathbf{x})}{x_1 \vee \ldots \vee x_n - x_1 \wedge \ldots \wedge x_n} dx_1 \ldots dx_n\end{aligned}$$

$$\bar{\omega}_\ell = 1 - \bar{\alpha}_\ell = \int_{I^n} \frac{A(\mathbf{x}) - x_1 \wedge \ldots \wedge x_n}{x_1 \vee \ldots \vee x_n - x_1 \wedge \ldots \wedge x_n} dx_1 \ldots dx_n$$

(3) *The global andness/orness* [7]:

$$\alpha_g = \frac{\int_{I^n} (x_1 \vee \ldots \vee x_n) dx_1 \ldots dx_n - \int_{I^n} A(\mathbf{x}) dx_1 \ldots dx_n}{\int_{I^n} (x_1 \vee \ldots \vee x_n) dx_1 \ldots dx_n - \int_{I^n} (x_1 \wedge \ldots \wedge x_n) dx_1 \ldots dx_n}$$

$$\begin{aligned}\omega_g &= 1 - \alpha_g \\ &= \frac{\int_{I^n} A(\mathbf{x}) dx_1 \ldots dx_n - \int_{I^n} (x_1 \wedge \ldots \wedge x_n) dx_1 \ldots dx_n}{\int_{I^n} (x_1 \vee \ldots \vee x_n) dx_1 \ldots dx_n - \int_{I^n} (x_1 \wedge \ldots \wedge x_n) dx_1 \ldots dx_n}\end{aligned}$$

According to [6] we have

$$\begin{aligned}\int_{I^n} (x_1 \wedge \ldots \wedge x_n) dx_1 \ldots dx_n &= \frac{1}{n+1} \\ \int_{I^n} (x_1 \vee \ldots \vee x_n) dx_1 \ldots dx_n &= \frac{n}{n+1}\end{aligned} \qquad (1)$$

Consequently, using (1) the global andness/orness can be written as follows:

$$\begin{aligned}\alpha_g &= \frac{n - (n+1)\int_{I^n} A(\mathbf{x}) dx_1 \ldots dx_n}{n-1} \\ \omega_g &= 1 - \alpha_g = \frac{(n+1)\int_{I^n} A(\mathbf{x}) dx_1 \ldots dx_n - 1}{n-1}\end{aligned} \qquad (2)$$

Let us now consider compound logic aggregators that are a linear combination of k component aggregators:

$$A(\mathbf{x}) = \sum_{i=1}^{k} p_i A_i(\mathbf{x}), \quad 0 < p_i < 1, \ \sum_{i=1}^{k} p_i = 1$$

$$\alpha_{\ell i}(\mathbf{x}) = \frac{x_1 \vee \ldots \vee x_n - A_i(\mathbf{x})}{x_1 \vee \ldots \vee x_n - x_1 \wedge \ldots \wedge x_n},$$

$$\bar{\alpha}_{\ell i} = \int_{I^n} \frac{x_1 \vee \ldots \vee x_n - A_i(\mathbf{x})}{x_1 \vee \ldots \vee x_n - x_1 \wedge \ldots \wedge x_n} dx_1 \ldots dx_n,$$

$$\alpha_{gi} = \frac{n - (n+1)\int_{I^n} A_i(\mathbf{x}) dx_1 \ldots dx_n}{n-1}, \quad i = 1, \ldots, k.$$

The local andness/orness, the mean local andness/orness, and the global andness/orness are additive indicators. Let us now prove the andness/orness additivity theorem.

Theorem. For compound logic aggregators that are a linear combination of $k>1$ component (base) aggregators the compound andness/orness is the same linear combination of k component andness/orness indicators:

$$
\begin{aligned}
&A(\mathbf{x})=\sum_{i=1}^{k} p_i A_i(\mathbf{x}),\quad 0<p_i<1,\ \sum_{i=1}^{k} p_i=1\\
&\alpha_\ell(\mathbf{x})=\sum_{i=1}^{k} p_i\alpha_{\ell i}(\mathbf{x}),\quad \omega_\ell(\mathbf{x})=\sum_{i=1}^{k} p_i\omega_{\ell i}(\mathbf{x}),\\
&\bar{\alpha}_\ell=\sum_{i=1}^{k} p_i\bar{\alpha}_{\ell i}\,,\qquad \bar{\omega}_\ell=\sum_{i=1}^{k} p_i\bar{\omega}_{\ell i}\,,\\
&\alpha_g=\sum_{i=1}^{k} p_i\alpha_{gi}\,,\qquad \omega_g=\sum_{i=1}^{k} p_i\omega_{gi}\quad .
\end{aligned}
\tag{3}
$$

Proof.

$$
\begin{aligned}
\alpha_\ell(\mathbf{x})&=\frac{x_1\vee\ldots\vee x_n-A(\mathbf{x})}{x_1\vee\ldots\vee x_n-x_1\wedge\ldots\wedge x_n}\\
&=\frac{x_1\vee\ldots\vee x_n-\sum_{i=1}^{k} p_i A_i(\mathbf{x})}{x_1\vee\ldots\vee x_n-x_1\wedge\ldots\wedge x_n}\\
&=\frac{\sum_{i=1}^{k} p_i(x_1\vee\ldots\vee x_n)-\sum_{i=1}^{k} p_i A_i(\mathbf{x})}{x_1\vee\ldots\vee x_n-x_1\wedge\ldots\wedge x_n}\\
&=\sum_{i=1}^{k}\left(p_i\frac{x_1\vee\ldots\vee x_n-A_i(\mathbf{x})}{x_1\vee\ldots\vee x_n-x_1\wedge\ldots\wedge x_n}\right)=\sum_{i=1}^{k} p_i\alpha_{\ell i}(\mathbf{x})
\end{aligned}
$$

Furthermore, in the case of mean local andness we have

$$
\begin{aligned}
\bar{\alpha}_\ell&=\int_{I^n}\sum_{i=1}^{k}\left(p_i\frac{x_1\vee\ldots\vee x_n-A_i(\mathbf{x})}{x_1\vee\ldots\vee x_n-x_1\wedge\ldots\wedge x_n}\right)dx_1\ldots dx_n\\
&=\sum_{i=1}^{k} p_i\int_{I^n}\frac{x_1\vee\ldots\vee x_n-A_i(\mathbf{x})}{x_1\vee\ldots\vee x_n-x_1\wedge\ldots\wedge x_n}dx_1\ldots dx_n=\sum_{i=1}^{k} p_i\bar{\alpha}_{\ell i}
\end{aligned}
$$

According to (2), in the case of global andness we have

$$
\begin{aligned}
\alpha_g&=\frac{n}{n-1}-\frac{n+1}{n-1}\int_{I^n}\sum_{i=1}^{k} p_i A_i(\mathbf{x})dx_1\ldots dx_n\\
&=\sum_{i=1}^{k} p_i\frac{n}{n-1}-\sum_{i=1}^{k} p_i\frac{n+1}{n-1}\int_{I^n}A_i(\mathbf{x})dx_1\ldots dx_n\\
&=\sum_{i=1}^{k} p_i\left(\frac{n}{n-1}-\frac{n+1}{n-1}\int_{I^n}A_i(\mathbf{x})dx_1\ldots dx_n\right)=\sum_{i=1}^{k} p_i\alpha_{gi}\quad .
\end{aligned}
$$

Since orness is the complement of andness the proofs for orness directly follow from the proofs for andness; e.g.:

$$
\begin{aligned}
\alpha_g&=1-\omega_g=\sum_{i=1}^{k} p_i\alpha_{gi}=\sum_{i=1}^{k} p_i(1-\omega_{gi})=\sum_{i=1}^{k} p_i-\sum_{i=1}^{k} p_i\omega_{gi}\\
&=1-\sum_{i=1}^{k} p_i\omega_{gi}\qquad\Rightarrow\qquad \omega_g=\sum_{i=1}^{k} p_i\omega_{gi}\quad .
\end{aligned}
$$

IV. Interpolative Logic Aggregators

The additive local, mean local, and global andness/orness are different metrics of simultaneity and substitutability. The global andness/orness is the simplest and the most frequently used indicator. For simplicity, let us now assume that α and ω denote the global andness and orness. An aggregator A, parameterized with the global andness α and denoted $A(\mathbf{x};\alpha)$, can be interpolated between the base aggregators $A_1(\mathbf{x};\alpha_1)$ and $A_2(\mathbf{x};\alpha_2)$ as follows:

$$
\begin{aligned}
&A(\mathbf{x};\alpha)=pA_1(\mathbf{x};\alpha_1)+(1-p)A_2(\mathbf{x};\alpha_2),\quad p\in I\\
&\alpha=p\alpha_1+(1-p)\alpha_2\,,\quad \omega=p\omega_1+(1-p)\omega_2\,.
\end{aligned}
$$

In this case the base aggregators $A_1(\mathbf{x};\alpha_1)$ and $A_2(\mathbf{x};\alpha_2)$ are interpreted as fixed bounds of an interval and (assuming $\alpha_1<\alpha_2$) inside the interval we use the linear interpolation yielding $\alpha_1\le\alpha\le\alpha_2$, $\omega_2\le\omega\le\omega_1$. Thus, $A(\mathbf{x};\alpha)$ can be interpreted as an *interpolative aggregator* with the range of andness $\alpha\in[\alpha_1,\alpha_2]$.

In many applications it is convenient to use andness and orness as parameters of interpolative logic aggregators. If we define $p=(\alpha_2-\alpha)/(\alpha_2-\alpha_1)$, $1-p=(\alpha-\alpha_1)/(\alpha_2-\alpha_1)$, then a parameterized interpolative logic aggregator can be defined using andness (or orness) as its parameter, as follows:

$$
A(\mathbf{x};\alpha)=\frac{(\alpha_2-\alpha)A_1(\mathbf{x};\alpha_1)+(\alpha-\alpha_1)A_2(\mathbf{x};\alpha_2)}{\alpha_2-\alpha_1},\quad \alpha_1\le\alpha\le\alpha_2 \tag{4}
$$

For example, $A(x_1,x_2;3/4)$ can be interpolated between $A_1(x_1,x_2;2/3)=\sqrt{x_1x_2}$ and $A_2(x_1,x_2;1)=x_1\wedge x_2$ as follows:

$$
\begin{aligned}
A(x_1,x_2;3/4)&=\frac{(1-3/4)\sqrt{x_1x_2}+(3/4-2/3)(x_1\wedge x_2)}{1-2/3}\\
&=\frac{3}{4}\sqrt{x_1x_2}+\frac{1}{4}(x_1\wedge x_2);\quad \alpha=\frac{3}{4}\alpha_1+\frac{1}{4}\alpha_2=\frac{3}{4}\cdot\frac{2}{3}+\frac{1}{4}\cdot 1=\frac{3}{4}
\end{aligned}
$$

V. Interpolative GCD

The selection of threshold andness and threshold orness are fundamental decisions in the development of GCD. The most frequently used model of GCD is based on weighted power mean (WPM) $M^{[r]}(\mathbf{x};\mathbf{W})=\lim_{s\to r}(W_1x_1^s+\ldots+W_nx_n^s)^{1/s}$, $-\infty\le r\le+\infty$, $\mathbf{W}=(W_1,\ldots,W_n)$, $0<W_i<1$, $i=1,\ldots,n$, $\sum_1^n W_i=1$ [4][10]. In this case $\alpha_\theta=2/3$, and $\omega_\theta=1$, i.e. the value of α_θ is rather low and the value of ω_θ is too high, so that the only hard partial disjunction is the pure disjunction $\max(x_1,\ldots,x_n)$. Therefore, WPM is a nonuniform aggregator. Evaluators can be trained to efficiently use WPM as a logic aggregator in applications [8], but there is no evidence that this is the most suitable among many means [1][4][12] that can be used as logic aggregators. A natural way to approach this problem is to first investigate what is the distribution of threshold andness in human reasoning and then to specify requirements that the GCD aggregators should satisfy.

An empirical analysis of the distribution of threshold andness in intuitive human reasoning [11] shows that 80% of both experts and non-experts suggest the range $0.71\le\alpha_\theta\le0.91$ and the mean value $\alpha_\theta=0.81$. So, there is a clear interest to have an *interpolative GCD aggregator* (denoted IGCD) that provides independently adjustable values of the threshold andness and the threshold orness.

In a general case of n attributes we can use the interpolative aggregator (4) to first create a threshold aggregator that is interpolated between the base aggregators of the geometric mean $A_{geo}(\mathbf{x};\alpha_{geo})$ and the pure conjunction $A_{con}(\mathbf{x};1)$:

$$A_\theta(\mathbf{x};\alpha_\theta) = \frac{(1-\alpha_\theta)A_{geo}(\mathbf{x};\alpha_{geo}) + (\alpha_\theta - \alpha_{geo})A_{con}(\mathbf{x};1)}{1-\alpha_{geo}},$$

$$\forall i \in \{1,...,n\},\ x_i = 0,\ x_j \geq 0,\ j \neq i \ \Rightarrow$$

$$A_\theta(x_1,...,x_i,...,x_n;\alpha_\theta) = 0, \quad \alpha_{geo} \leq \alpha_\theta \leq 1$$

The threshold aggregator has the annihilator 0 and provides a wide range of threshold andness; using (1) and (2) we have

$$A_{geo}(\mathbf{x};\alpha_{geo}) = x_1^{1/n}\cdots x_n^{1/n}, \quad n>1$$

$$\alpha_{geo} = \frac{n-(n+1)\int_{I^n} x_1^{1/n}\cdots x_n^{1/n}dx_1...dx_n}{n-1}$$

$$= \frac{n}{n-1} - \frac{(n+1)}{n-1}\cdot\left(\frac{n}{n+1}\right)^n = \frac{n}{n-1}\left[1-\left(\frac{n}{n+1}\right)^{n-1}\right]$$

$$1-\alpha_{geo} = \omega_{geo} = \frac{n}{n-1}\left[\left(\frac{n}{n+1}\right)^{n-1} - \frac{1}{n}\right]$$

$$A_\theta(\mathbf{x};\alpha_\theta) = \frac{(1-\alpha_\theta)x_1^{1/n}\cdots x_n^{1/n} + (\alpha_\theta - \alpha_{geo})(x_1 \wedge ... \wedge x_n)}{\omega_{geo}}$$

$$\frac{n}{n-1}\left[1-\left(\frac{n}{n+1}\right)^{n-1}\right] \leq \alpha_\theta \leq 1$$

In general asymmetric cases the threshold aggregator is using weights that are not all equal to $1/n$:

$$A_\theta(\mathbf{x};\mathbf{W},\alpha_\theta) = \frac{(1-\alpha_\theta)x_1^{W_1}\cdots x_n^{W_n} + (\alpha_\theta - \alpha_{geo})(x_1 \wedge ... \wedge x_n)}{\omega_{geo}}$$

$$\alpha_{geo} \leq \alpha_\theta \leq 1 \tag{5}$$

We can now use the arithmetic mean, the threshold aggregator (5) and the pure conjunction as the base aggregators for an interpolative GCD. Let $C_s(\mathbf{x};\mathbf{W},\alpha,\alpha_\theta)$, $½ < \alpha < \alpha_\theta$ denote the SPC with the threshold andness α_θ. By interpolating between the neutrality (weighted arithmetic mean) and the weighted threshold aggregator $A_\theta(\mathbf{x};\mathbf{W},\alpha_\theta)$ we have:

$$C_s(\mathbf{x};\mathbf{W},\alpha,\alpha_\theta) = \frac{(\alpha_\theta - \alpha)\mathrm{mid}(\mathbf{x};\mathbf{W}) + (\alpha - 1/2)A_\theta(\mathbf{x};\mathbf{W},\alpha_\theta)}{\alpha_\theta - 1/2}$$

$$\mathrm{mid}(\mathbf{x};\mathbf{W}) = W_1x_1 + ... + W_nx_n \tag{6}$$

$$0 < W_i < 1, \quad i = 1,...,n, \quad W_1 + ... + W_n = 1; \ ½ < \alpha < \alpha_\theta.$$

Using similar notation, the HPC $C_h(\mathbf{x};\mathbf{W},\alpha,\alpha_\theta)$, $\alpha_\theta \leq \alpha < 1$ should be interpolated between the weighted threshold aggregator $A(\mathbf{x};\mathbf{W},\alpha_\theta)$ and the pure conjunction:

$$C_h(\mathbf{x};\mathbf{W},\alpha,\alpha_\theta) = \frac{(1-\alpha)A_\theta(\mathbf{x};\mathbf{W},\alpha_\theta) + (\alpha - \alpha_\theta)(x_1 \wedge ... \wedge x_n)}{1-\alpha_\theta}$$

$$\alpha_\theta \leq \alpha < 1 \tag{7}$$

According to (2), the global andness of an interpolative aggregator $A(\mathbf{x};\mathbf{W},\alpha)$ has the following properties

$$\frac{n-(n+1)\int_{I^n} A(\mathbf{x};\underline{\mathbf{W}},\alpha)dx_1...dx_n}{n-1} = \alpha, \quad \underline{\mathbf{W}} = (1/n,...,1/n)$$

$$\frac{n-(n+1)\int_{I^n} A(\mathbf{1}-\mathbf{x};\underline{\mathbf{W}},\alpha)dx_1...dx_n}{n-1} = \alpha$$

Since $(n+1)\int_{I^n} A(\mathbf{x};\underline{\mathbf{W}},\alpha)dx_1...dx_n = n-(n-1)\alpha$, we also have

$$\frac{n-(n+1)\int_{I^n} (1-A(\mathbf{x};\underline{\mathbf{W}},\alpha))dx_1...dx_n}{n-1}$$

$$= \frac{n-(n+1)+(n+1)\int_{I^n} A(\mathbf{x};\underline{\mathbf{W}},\alpha)dx_1...dx_n}{n-1}$$

$$= \frac{-1+n-(n-1)\alpha}{n-1} = 1-\alpha \tag{8}$$

$$\frac{n-(n+1)\int_{I^n} (1-A(\mathbf{1}-\mathbf{x};\underline{\mathbf{W}},\alpha))dx_1...dx_n}{n-1} = 1-\alpha$$

$$= \frac{n-(n+1)\int_{I^n} A(\mathbf{x};\underline{\mathbf{W}},1-\alpha)dx_1...dx_n}{n-1}$$

From (8) we get De Morgan duality for GCD:

$$\begin{aligned} A(\mathbf{x};\underline{\mathbf{W}},1-\alpha) &= 1-A(\mathbf{1}-\mathbf{x};\underline{\mathbf{W}},\alpha) \\ A(\mathbf{x};\underline{\mathbf{W}},\alpha) &= 1-A(\mathbf{1}-\mathbf{x};\underline{\mathbf{W}},1-\alpha) \end{aligned} \tag{9}$$

Partial disjunction can be developed using the same method as the partial conjunction. The simplest way is to select a desired value of the threshold orness ω_θ and then to create the PD $D(\mathbf{x};\mathbf{W},\alpha,\omega_\theta)$ as De Morgan dual of the PC $C(\mathbf{x};\mathbf{W},\alpha,\alpha_\theta)$ that has the threshold andness $\alpha_\theta = \omega_\theta = \theta$:

$$\begin{aligned} D(\mathbf{x};\mathbf{W},\alpha,\theta) &= 1-C(\mathbf{1}-\mathbf{x};\mathbf{W},1-\alpha,\theta), \ 0<\alpha<½ \\ C(\mathbf{x};\mathbf{W},\alpha,\theta) &= 1-D(\mathbf{1}-\mathbf{x};\mathbf{W},1-\alpha,\theta), \ ½<\alpha<1 \end{aligned} \tag{10}$$

E.g., for desired threshold θ , $\alpha_{geo} \leq \theta \leq 1$ we can use WPM and create the SPD $D_s(\mathbf{x};\mathbf{W},\alpha,\theta)$ and the HPD $D_h(\mathbf{x};\mathbf{W},\alpha,\theta)$ using $C_s(\mathbf{x};\mathbf{W},\alpha,\theta)$ and $C_h(\mathbf{x};\mathbf{W},\alpha,\theta)$ as follows:

$$\begin{aligned} D_s(\mathbf{x};\mathbf{W},\alpha,\theta) &= 1-C_s(\mathbf{1}-\mathbf{x};\mathbf{W},1-\alpha,\theta), \ 1-\theta<\alpha<½ \\ D_h(\mathbf{x};\mathbf{W},\alpha,\theta) &= 1-C_h(\mathbf{1}-\mathbf{x};\mathbf{W},1-\alpha,\theta), \ 0<\alpha\leq 1-\theta \end{aligned} \tag{11}$$

De Morgan duality is used in (10) and (11) for $\alpha \in I \setminus \{0,½,1\}$ but it obviously holds in the whole range $\alpha \in I$. Using formulas (5)-(11) we can now define a general interpolative GCD aggregator $\mathrm{IGCD}(\mathbf{x};\mathbf{W},\alpha,\alpha_\theta,\omega_\theta)$ which has independently adjustable threshold andness α_θ and threshold orness ω_θ as shown in formula (12).

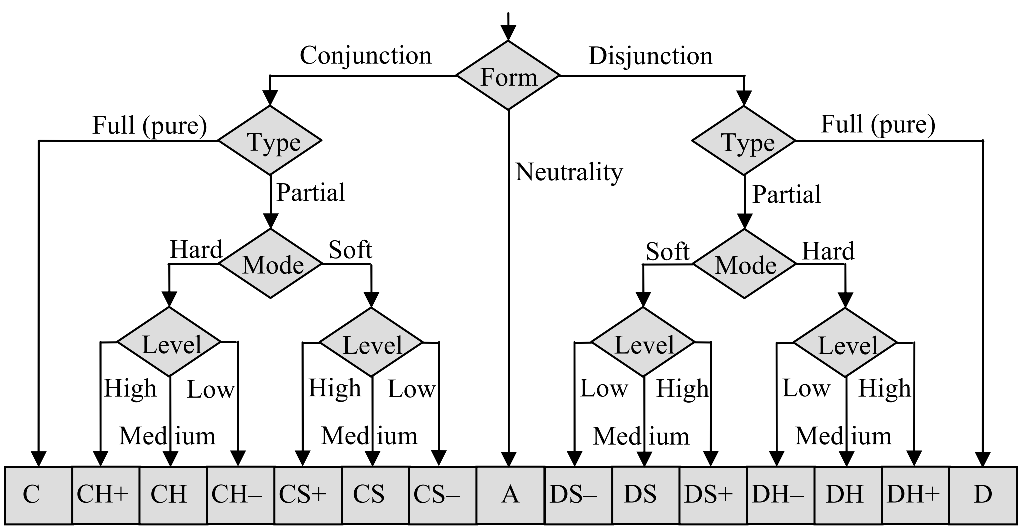
$$
\mathrm{IGCD}(\mathbf{x};\mathbf{W},\alpha,\alpha_\theta,\omega_\theta)
$$

$$
=\begin{cases}
x_1 \vee \ldots \vee x_n\,, & \alpha = 0 & \text{[Pure disjunction]}\\
1-\dfrac{\alpha}{1-\alpha_{geo}}\prod_{i=1}^{n}(1-x_i)^{W_i}+\dfrac{1-\alpha-\alpha_{geo}}{1-\alpha_{geo}}\bigwedge_{i=1}^{n}(1-x_i)\,, & 0<\alpha\le 1-\omega_\theta & \text{[Hard partial dis.]}\\
1-\dfrac{\omega_\theta-1+\alpha}{\omega_\theta-½}\sum_{i=1}^{n}W_i(1-x_i)+\dfrac{½-\alpha}{\omega_\theta-½}\left(\dfrac{1-\omega_\theta}{1-\alpha_{geo}}\prod_{i=1}^{n}(1-x_i)^{W_i}+\dfrac{\omega_\theta-\alpha_{geo}}{1-\alpha_{geo}}\bigwedge_{i=1}^{n}(1-x_i)\right), & 1-\omega_\theta<\alpha<½ & \text{[Soft partial dis.]}\\
W_1x_1+\ldots+W_nx_n & \alpha = ½ & \text{[Neutrality]}\\
\dfrac{\alpha_\theta-\alpha}{\alpha_\theta-½}\sum_{i=1}^{n}W_ix_i+\dfrac{\alpha-½}{\alpha_\theta-½}\left(\dfrac{1-\alpha_\theta}{1-\alpha_{geo}}\prod_{i=1}^{n}x_i^{W_i}+\dfrac{\alpha_\theta-\alpha_{geo}}{1-\alpha_{geo}}\bigwedge_{i=1}^{n}x_i\right), & ½<\alpha<\alpha_\theta & \text{[Soft partial con.]}\\
\dfrac{1-\alpha}{1-\alpha_{geo}}\prod_{i=1}^{n}x_i^{W_i}+\dfrac{\alpha-\alpha_{geo}}{1-\alpha_{geo}}\bigwedge_{i=1}^{n}x_i\,, & \alpha_\theta\le\alpha<1 & \text{[Hard partial con.]}\\
x_1\wedge\ldots\wedge x_n\,, & \alpha = 1 & \text{[Pure conjunction]}
\end{cases}
$$

$$
\alpha_{geo}=\frac{n}{n-1}\left[1-\left(\frac{n}{n+1}\right)^{n-1}\right],\quad \alpha_{geo}\le\alpha_\theta\le 1,\quad \alpha_{geo}\le\omega_\theta\le 1;\quad 0<W_i<1,\ i=1,\ldots,n,\ W_1+\ldots+W_n=1,\ n>1 \tag{12}
$$

TABLE I.
DECOMPOSITION AND SPECIAL CASES OF THE UNIFORM INTERPOLATIVE GCD

Aggregator decomposition and special cases	UNIFORM INTERPOLATIVE GENERALIZED CONJUNCTION/DISJUNCTION														
	Conjunction (models of simultaneity)							Neutrality	Disjunction (models of replaceability)						
	Pure Con.	Partial conjunction							Partial disjunction						Pure Dis.
		Hard partial con.			Soft partial con.				Soft partial dis.			Hard partial dis.			
Symbol	C	CH+	CH	CH-	CS+	CS	CS-	A	DS-	DS	DS+	DH-	DH	DH+	D
Andness	1	$\frac{13}{14}$	$\frac{6}{7}$	$\frac{11}{14}$	$\frac{5}{7}$	$\frac{9}{14}$	$\frac{4}{7}$	$\frac{1}{2}$	$\frac{3}{7}$	$\frac{5}{14}$	$\frac{2}{7}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{14}$	0
Orness	0	$\frac{1}{14}$	$\frac{1}{7}$	$\frac{3}{14}$	$\frac{2}{7}$	$\frac{5}{14}$	$\frac{3}{7}$	$\frac{1}{2}$	$\frac{4}{7}$	$\frac{9}{14}$	$\frac{5}{7}$	$\frac{11}{14}$	$\frac{6}{7}$	$\frac{13}{14}$	1



GCD decision steps:

1. **Form:** conjunction or disjunction or neutrality?
2. **Type:** partial or full (pure)?
3. **Mode:** hard or soft?
4. **Level:** low or medium or high?

Fig. 1. A simple form-type-mode-level decision process for selecting a uniform interpolative GCD aggregator.

Note that $\text{IGCD}(\mathbf{x};\mathbf{W},\alpha,\alpha_\theta,\omega_\theta)$ must be idempotent, and it is a weighted sum of idempotent functions:

$$\text{IGCD}(\mathbf{x};\mathbf{W},\alpha,\alpha_\theta,\omega_\theta)=Q\sum_{i=1}^{n}W_i x_i+R\prod_{i=1}^{n}x_i^{W_i}+T\bigwedge_{i=1}^{n}x_i \quad (13)$$

Consequently, if we insert $\mathbf{x}=\underline{\mathbf{x}}=(x,...,x),\ x\in I$ in (13), then we have that the sum of weights must be 1:

$$\text{IGCD}(\underline{\mathbf{x}};\mathbf{W},\alpha,\alpha_\theta,\omega_\theta)=x=Q(\alpha,\alpha_\theta,\omega_\theta)x+R(\alpha,\alpha_\theta,\omega_\theta)x$$
$$+T(\alpha,\alpha_\theta,\omega_\theta)x$$
$$\therefore\ Q(\alpha,\alpha_\theta,\omega_\theta)+R(\alpha,\alpha_\theta,\omega_\theta)+T(\alpha,\alpha_\theta,\omega_\theta)=1$$

In addition, if we insert $\mathbf{W}=\underline{\mathbf{W}}=\left(\frac{1}{n},...,\frac{1}{n}\right)$ and use (3) to compute the andness of (13) we have the following:

$$\alpha=\tfrac{1}{2}Q(\alpha,\alpha_\theta,\omega_\theta)+\frac{n}{n-1}\left[1-\left(\frac{n}{n+1}\right)^{n-1}\right]R(\alpha,\alpha_\theta,\omega_\theta)$$
$$+T(\alpha,\alpha_\theta,\omega_\theta)$$

A natural next question is related the selection of threshold andness and threshold orness, and the definition of special cases of GCD.

Generally, the region of hard partial conjunction can have a different size than the region of soft partial conjunction $(\alpha_\theta\neq 3/4)$. Similarly, the region of hard partial disjunction can have a different size than the region of soft partial disjunction $(\omega_\theta\neq 3/4)$. In addition, the conjunctive special cases of GCD need not necessarily be symmetric to the disjunctive special cases $(\alpha_\theta\neq\omega_\theta)$.

In the absence of convincing arguments justifying the need for nonuniform distribution of hard and soft partial conjunction and disjunction, or the need for asymmetric conjunctive and disjunctive properties of GCD, a default form of GCD should have *symmetric PC/PD and uniform soft and hard ranges*: $\alpha_\theta=\omega_\theta=3/4$. This value is close to the empirical mean value $\bar{\alpha}_\theta=0.81$ and inside the interval [0.71, 0.91] defined as $[\bar{\alpha}_\theta-\sigma,\bar{\alpha}_\theta+\sigma]$ where σ denotes the standard deviation of the empirical distribution of α_θ.

A *Uniform Interpolative GCD* (or UIGCD) is presented in Table I. The proposed UIGCD uses seven levels of simultaneity and seven levels of replaceability, providing easy and reliable choice of the most appropriate level [14]. In most cases the selection process consists of four easy steps illustrated in Fig. 1. Therefore, there are two binary decisions and two ternary decisions, what might an ultimate level of simplicity in selecting the most appropriate version of a GCD aggregator. The UIGCD aggregator should be a default version of GCD, suitable for a wide variety of users, including those that are not professional decision analysts.

In the special case of two variables UIGCD takes the following form:

$$\text{UIGCD}(\mathbf{x};\mathbf{W},\alpha,\tfrac{3}{4},\tfrac{3}{4})$$
$$=\begin{cases}1-3\alpha(1-x_1)^{W_1}(1-x_2)^{W_2}-(1-3\alpha)[1-(x_1\vee x_2)]\,, & 0\le\alpha\le\frac{1}{4}\\ 1-(4\alpha-1)(1-W_1x_1-W_2x_2)-\frac{3-6\alpha}{2}(1-x_1)^{W_1}(1-x_2)^{W_2}-\\ \quad-\frac{1-2\alpha}{2}[1-(x_1\vee x_2)]\,, & \frac{1}{4}\le\alpha\le\frac{1}{2}\\ (3-4\alpha)(W_1x_1+W_2x_2)+\frac{6\alpha-3}{2}x_1^{W_1}x_2^{W_2}+\frac{2\alpha-1}{2}(x_1\wedge x_2)\,, & \frac{1}{2}\le\alpha\le\frac{3}{4}\\ (3-3\alpha)x_1^{W_1}x_2^{W_2}+(3\alpha-2)(x_1\wedge x_2)\,, & \frac{3}{4}\le\alpha\le 1\end{cases}$$
$$0<W_1<1,\ \ 0<W_2<1,\ \ W_1+W_2=1 \quad (14)$$

According to (14), the 15 special cases of UIGCD aggregators presented in Table I are the following:

$$\text{C}(x_1,x_2)=x_1\wedge x_2$$
$$\text{CH}^+(x_1,x_2)=\tfrac{3}{14}x_1^{W_1}x_2^{W_2}+\tfrac{11}{14}(x_1\wedge x_2)$$
$$\text{CH}(x_1,x_2)=\tfrac{3}{7}x_1^{W_1}x_2^{W_2}+\tfrac{4}{7}(x_1\wedge x_2)$$
$$\text{CH}^-(x_1,x_2)=\tfrac{9}{14}x_1^{W_1}x_2^{W_2}+\tfrac{5}{14}(x_1\wedge x_2)$$
$$\text{CS}^+(x_1,x_2)=\tfrac{1}{7}(W_1x_1+W_2x_2)+\tfrac{9}{14}x_1^{W_1}x_2^{W_2}+\tfrac{3}{14}(x_1\wedge x_2)$$
$$\text{CS}(x_1,x_2)=\tfrac{3}{7}(W_1x_1+W_2x_2)+\tfrac{3}{7}x_1^{W_1}x_2^{W_2}+\tfrac{1}{7}(x_1\wedge x_2)$$
$$\text{CS}^-(x_1,x_2)=\tfrac{5}{7}(W_1x_1+W_2x_2)+\tfrac{3}{14}x_1^{W_1}x_2^{W_2}+\tfrac{1}{14}(x_1\wedge x_2)$$
$$\text{A}(x_1,x_2)=W_1x_1+W_2x_2$$
$$\text{DS}^-(x_1,x_2)=1-\tfrac{5}{7}(1-W_1x_1-W_2x_2)+$$
$$+\tfrac{3}{14}(1-x_1)^{W_1}(1-x_2)^{W_2}+\tfrac{1}{14}[1-(x_1\vee x_2)]$$
$$\text{DS}(x_1,x_2)=1-\tfrac{3}{7}(1-W_1x_1-W_2x_2)+$$
$$+\tfrac{3}{7}(1-x_1)^{W_1}(1-x_2)^{W_2}+\tfrac{1}{7}[1-(x_1\vee x_2)]$$
$$\text{DS}^+(x_1,x_2)=1-\tfrac{1}{7}(1-W_1x_1-W_2x_2)+$$
$$+\tfrac{9}{14}(1-x_1)^{W_1}(1-x_2)^{W_2}+\tfrac{3}{14}[1-(x_1\vee x_2)]$$
$$\text{DH}^-(x_1,x_2)=1-\tfrac{9}{14}(1-x_1)^{W_1}(1-x_2)^{W_2}-\tfrac{5}{14}[1-(x_1\vee x_2)]$$
$$\text{DH}(x_1,x_2)=1-\tfrac{3}{7}(1-x_1)^{W_1}(1-x_2)^{W_2}-\tfrac{4}{7}[1-(x_1\vee x_2)]$$
$$\text{DH}^+(x_1,x_2)=1-\tfrac{3}{14}(1-x_1)^{W_1}(1-x_2)^{W_2}-\tfrac{11}{14}[1-(x_1\vee x_2)]$$
$$\text{D}(x_1,x_2)=x_1\vee x_2$$
$$0<W_1<1,\ \ 0<W_2<1,\ \ W_1+W_2=1 \quad (15)$$

The presented aggregators illustrate a situation where the threshold andness and the threshold orness do not need to be below the value of α_{geo}. Experiments with human decision makers indicate that this is almost always acceptable. The selection of basic aggregators that we used (the arithmetic mean, the geometric mean, and the minimum function) is convenient because of the availability of function $\alpha_{geo}(n)$

for all values of n.

In cases where the computational simplicity is a primary goal, it is suitable to use the harmonic mean as the threshold andness aggregator:

$$W_1x_1\Delta_\alpha W_2x_2 = \begin{cases} \frac{\alpha_\theta-\alpha}{\alpha_\theta-0.5}(W_1x_1+W_2x_2)+\frac{\alpha-0.5}{\alpha_\theta-0.5}\left(\frac{W_1}{x_1}+\frac{W_2}{x_2}\right)^{-1}, & 0.5\le\alpha\le\alpha_\theta \\ \frac{1-\alpha}{1-\alpha_\theta}\left(\frac{W_1}{x_1}+\frac{W_2}{x_2}\right)^{-1}+\frac{\alpha-\alpha_\theta}{1-\alpha_\theta}(x_1\wedge x_2), & \alpha_\theta\le\alpha\le 1,\ \ \alpha_\theta=\ln 16-2 \end{cases}$$

$$W_1x_1\nabla_\omega W_2x_2 = 1-W_1(1-x_1)\Delta_\omega W_2(1-x_2),\quad 0.5\le\omega\le 1,\quad \omega_\theta=\alpha_\theta. \qquad (16)$$

In this case $\alpha_\theta = \ln 16 - 2 = 0.7726 > 3/4$ and the region of HPC/HPD is slightly smaller than the region of SPC/SPD. So, this aggregator provides the symmetry between PC and PD, but not the perfectly uniform soft and hard ranges.

Of course, there are many other candidates that can be used as basic aggregators [9]. The simplest way to create a three-base UIGCD is to use the weighted power mean $A_\theta(\mathbf{x};\mathbf{W},\alpha_\theta) = \left(W_1x_1^r+...+W_nx_n^r\right)^{1/r}$ directly as a threshold aggregator with $\alpha_\theta = 3/4$. Using numerical integration the corresponding values of exponent r for n=2,3,4,5 should respectively be -0.7201, -0.7317, -0.7205, and -0.7054. Another approach, yielding ultimate computational simplicity, is to use the harmonic mean $A_\theta(\mathbf{x};\mathbf{W},\alpha_\theta) = 1/\left(W_1/x_1+...+W_n/x_n\right)$ as the threshold aggregator and to interpolate soft partial conjunction between the arithmetic and harmonic means. A spectrum of threshold aggregators can also be designed using multiplicative techniques introduced in [9].

The concept of interpolative aggregators is not limited to idempotent aggregators only. Interpolative aggregators are a powerful technique for refinement and expansion of all types of aggregators. Interpolation can be efficiently applied to t-norms, t-conorms, and overlap and grouping functions [3]. E.g., the global andness of the min norm $T_M(x,y) = x\wedge y$ is $\alpha = 1$ and for the product norm $T_P(x,y) = xy$ is $\alpha = 5/4$. Using $T(x,y) = (5-4\alpha)(x\wedge y)+4(\alpha-1)xy,\ 1\le\alpha\le 5/4$ we can realize a continuous transition between these two norms. The same holds for t-conorms. In addition, interpolative aggregators can provide a seamless transition between the region of idempotent aggregators (means) and the region of nonidempotent aggregators [2][3]. This opens a rather wide area for research of new aggregators and their applications, particularly in the field of image processing.

VI. Conclusions

Uniform and nonuniform versions of GCD are fundamental logic averaging aggregators used to model human evaluation reasoning. IGCD and UIGCD are simple and efficient implementations of GCD. Generally, logic averaging aggregators need to satisfy several fundamental conditions: (1) monotonicity, (2) continuous transition from conjunction to disjunction based on adjustable andness/orness, (3) soft and hard aggregation properties in order to include/exclude annihilators, (4) the use of weights that express relative importance of inputs, and (5) idempotency, which is needed in the vast majority of evaluation decision models. Such logic aggregators are means, and the search for suitable aggregators is most promising among monotonic special cases of the Bajraktarević mean [1][4]:

$$B(\mathbf{x}) = F^{-1}\left(\frac{\sum_{i=1}^n w_i(x_i)F(x_i)}{\sum_{i=1}^n w_i(x_i)}\right) \qquad (17)$$

$w_i : I \to [0,+\infty[\,,\quad F: I\to[-\infty,+\infty]$, strictly monotonic

No special case of $B(\mathbf{x})$ is known to perfectly satisfy all conditions that logic aggregators must satisfy, but for each fundamental condition there is a special case of (17) that satisfies the condition. Interpolative logic aggregators IGCD and UIGCD provide a way to integrate convenient special cases of (17) in a single interpolative form, and to provide all desirable logic properties in a single aggregator. In particular, interpolative logic aggregators provide independent control of both the threshold andness and the threshold orness.

References

[1] G. Beliakov, A. Pradera , and T. Calvo, *Aggregation Functions: A Guide for Practitioners*, Springer 2007.

[2] B. Bedregal, G.Pereira Dimuro, H. Bustince, and E. Barrenechea, New Results on Overlap and Grouping Functions. Information Sciences 249, pp.148-170, 2013.

[3] H. Bustince, M. Pagola, R. Mesiar, E, Hüllermeier, and F, Herrera, Grouping, Overlap, and Generalized Bientropic Functions for Fuzzy Modeling of Pairwise Comparisons. IEEE TFS, Vol. 20, No.3, 405-415, 2012.

[4] P.S. Bullen, *Handbook of means and their inequalities*, Kluwer, 2003.

[5] J.J. Dujmović, A Generalization of Some Functions in Continuous Mathematical Logic – Evaluation Function and its Applications, (In Serbo-Croatian). Proceedings of the Informatica Conference, paper d27, Bled, Yugoslavia, 1973.

[6] J.J. Dujmović, Two Integrals Related to Means. *Journal of the University of Belgrade EE Dept., Series Mathematics and Physics*, No. 457, pp. 231-232, 1973.

[7] J.J. Dujmović, Weighted conjunctive and disjunctive means and their application in system evaluation. *Journal of the University of Belgrade, EE Dept., Series Mathematics and Physics*, No. 483, 1974, pp. 147-158 (1974).

[8] J.J. Dujmović, Preference logic for system evaluation. IEEE TFS, Vol. 15, No. 6, 1082-1099, Dec. 2007.

[9] J.J. Dujmović, Characteristic Forms of Generalized Conjunction/Disjunction, Proceedings of the IEEE World Congress on Computational Intelligence, pp. 1075-1080b, IEEE Press, ISBN 978-1-4244-1819-0, Hong Kong, June 2008.

[10] J.J. Dujmović and H.L. Larsen, "Generalized conjunction/disjunction." *International J. Approx. Reas.* 46, pp. 423-446. 2007

[11] J.J. Dujmović, Aggregation Operators and Observable Properties of Human Reasoning. Proceedings of the 7th International Summer School on Aggregation Operators, AGOP 2013.

[12] C. Gini in collaboration with G. Barbensi, L. Galvani, S. Gatti, and E. Pizzetti, *Le Medie*. Unione Tipografico-Editrice Torinese, Torino, 1958.

[13] M. Grabisch, J-L. Marichal, R. Mesiar, and E. Pap, *Aggegation Functions*. Cambridge Univ. Press, 2009.

[14] G.A. Miller, The Magical Number Seven, Plus or Minus Two: Some Limits on Our Capacity for Processing Information. The Psychological Review, Vol. 63, pp. 81-97 (1956).

[15] E. Pap, Aggregation functions for decision making. *Proceedings of EUROFUSE 2005*, edited by B. De Baets, J. Fodor, and D. Radojević, pp. 179-193. ISBN 86-7172-022-5, Inst. "M. Pupin", Belgrade, 2005.