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Fuzzy Clustering Algorithm with H-operator Applied to Problems with Interval-based Data

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***Abstract*— The main advantage of using an interval-based distance for interval-based data lies on the fact that it preserves the underlying imprecision on intervals which is usually lost when real-valued distances are applied. One of the main problems when using interval-based distance in fuzzy clustering algorithms is the way to obtain the center of the groups. In this case, it is necessary to make adaptations in order to obtain those centers. Therefore, in this paper, we propose the use of the family of H-operator to proposed three approaches to transform the interval-based membership matrix into real-valued membership matrix and, as a consequence, to calculate the centers of the groups in interval-based fuzzy clustering algorithms. In this case, we will perform a comparative analysis using the three different approaches proposed in this paper, using seven interval-based datasets (four synthetic and three real datasets). As a result of this analysis, we will observe that the proposed approaches achieved better performance than all analyzed methods for interval-based methods.**

I. Introduction

DATA CLUSTERING is a field that has been largely studied in the machine learning subject. By definition, data clustering is a set of algorithms that aims to divide objects into groups, in such a way that objects (instances) in one cluster are very similar and objects in different clusters are very dissimilar. In other words, these algorithms have a goal to find an ideal division of the objects in which there is high similarity among objects of one group and high dissimilarity among objects of different groups[18].

There are several clustering algorithms proposed in the literature and they can be broadly divided into two categories: hard (crisp) and fuzzy (soft) clustering. In hard clustering, an instance belongs to one and only one cluster, while in fuzzy clustering, an instance may belong to more than one cluster with a certain membership degree (membership matrix). In this paper, we will work with fuzzy clustering algorithms.

The characterization of a clustering problem is represented by a dataset (set of organized information, resulting from an experiment or observation), which is frequently related to problems found daily. Recently, the scientific community has driven its attention to numeric data with interval nature, since closed intervals are data that represent numeric information endowed with impreciseness, which are normally captured from measures of real world.

One of the first formal work on interval-based data was reported in [26]. For interval-based similarity measures, for instance, recently the notion of interval-based distances were investigated on the studies as in [14] and [31]. However, none of those studies provided a wide investigation on the subject. In [28], the authors proposed the first notion of an interval-based metric, called i-metric. They also proposed a theoretical way to calculate the similarity (d_{km} distance) in interval-based data for this proposed metric (i-metric). In addition, they performed a theoretical investigation and provided suitable theoretical definitions, using interval-based data that includes definitions proposed in [31] and [14].

In [8], the authors have provided a practical framework based on the theoretical approach for interval-based distance proposed in [28]. They applied this framework in the context of clustering algorithms. As a result, they have adapted two well-known fuzzy clustering algorithms to interval-based data: Interval-based Fuzzy c-Means (IbFcM) and Interval-based ckMeans (IbckM). The authors have also proposed interval-based validation indexes. According to the authors, the empirical analysis showed the usefulness of applying the d_{km} distance in clustering problems.

However, one of the main problems when using interval-based distances in fuzzy clustering algorithms is the way to obtain the center of the groups. This is because it is not possible to use directly the interval-based membership matrix using the interval arithmetic proposed in [26] to calculate the centers of the groups (the imprecision of the interval-based data increases drastically). In this case, the interval-based membership matrix has to be transformed into a real-valued membership matrix (or simply membership matrix). Nevertheless, this transformation usually causes loss of information.

In this paper, we propose three different approaches to transform the interval-based membership matrix into real-valued membership matrix. These proposed approaches aim to smooth out the mentioned loss of information. To achieve that, we use H-operators to *guide* this transformation. In order to assess the performance of the proposed approach, an empirical analysis will be conducted. In this analysis, the proposed approaches will be applied to interval-based FCM (Fuzzy c-Means) algorithm. In addition, we will use seven datasets and evaluate the performance of the clustering algorithms through the use of an external index, called Correct Rand cluster validity measure. For comparison purposes, some existing distance measures and interval-based clustering algorithms will be used in this analysis.

This paper is organized into seven sections and it is structured in the following way: Section II introduces the notion of interval-based metrics. Section III presents the clustering algorithms. Section IV describes the proposed methods. Section V describes the experimental setting up for the empirical analysis and it illustrates the experimental

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results. Section VI describes the results and discussion while Section VII presents the final remarks of this paper.

II. Interval Mathematics

In this section, some important concepts of interval mathematics (**IM**) and interval-based metrics are presented.

A. Interval Arithmetic

Interval mathematics is based on the studies of Ramon Moore [22] and Sunaga [20]. They provided a way to deal with the lack of accuracy during numerical calculations which normally comes from the imprecision of data inputs, physical limitations of machines, rounding errors, among others. In order to deal with those types of imprecisions, many researches can be found in the literature, such as in [7] and [22].

In this section, we present some relevant concepts which can be found in the literature; e.g. [22]. Here, the set of real intervals will be denoted by $\mathbb{IR}$.

Definition 1: The set $X = \{x \in \mathbb{R} \mid \underline{x} \leq x \leq \overline{x}\}$, represented by $X = [\underline{x}; \overline{x}]$ is called closed interval with endpoints $\underline{x}$ and $\overline{x}$, or, here, simply interval. X is a non-negative interval, $X \geq 0$, whenever $\underline{x} \geq 0$. It is said to be negative, $X < 0$, if $\overline{x} < 0$. X is degenerate, if $\underline{x} = \overline{x}$.

Definition 2 (Arithmetic): Let $X, Y \in \mathbb{I}(\mathbb{R})$. The operations of addition, subtraction, multiplication and division on intervals are defined in the following way:

1) Addition: $X + Y = [\underline{x} + \underline{y}, \overline{x} + \overline{y}]$
2) Subtraction: $X + (-Y) = [\underline{x} - \overline{y}, \overline{x} - \underline{y}]$
3) Multiplication: $X \cdot Y = [\min(\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}), \max(\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy})]$;
4) Pseudo inverse Multiplication: $X^{-1} = \frac{1}{X} = [\frac{1}{\overline{x}}; \frac{1}{\underline{x}}]$ and $0 \notin X$
5) Division: $X/Y = [\underline{x}, x_2] \cdot (1/[\underline{y}, \overline{y}])$, where $1/[\underline{y}, \overline{y}] = [1/\overline{y}, 1/\underline{y}]$ if $0 \notin [\underline{y}, \overline{y}]$.

There are situations in which the division leads to an enlargement of the resulting intervals.

B. Metrics

There are several generalizations of the concept of metric. On the one hand, there are the generalizations made by modifying the metric axioms, see [1] or [30]. On the other hand, there are generalizations which modify the valuation of the metric; see [21] and [19]. The generalization used here was proposed by Santana and Santiago [28] and has both characteristics. In that paper, based on the theory developed in [13], the authors proposed an interval-based generalization for usual metrics, in such a way that instead of functions of the form $d : M \times M \to \mathbb{R}$, they provided functions with the signature: $d : M \times M \to \mathbb{I}(\mathbb{R})^+$, where $\mathbb{I}(\mathbb{R})^+$ is the set of closed intervals with non-negative endpoints. Those functions were proposed to be used to model measurements with impreciseness; for example: a digital measurement instrument that captures the distance between two points of M and stores the resulting value in a machine memory as intervals, $[a; b]$.

The authors in [28] introduced the notion of interval-based metrics, which is a function of the form $d : M \times M \to \mathbb{I}(\mathbb{R})^+$ satisfying some axioms (which will be presented below). The question was: What would be a suitable interval-based metric if $M = \mathbb{I}(\mathbb{R})$? Since intervals are entities which represent an imprecise real number together with an error information, it is reasonable that given two distinct intervals $[a; b]$ and $[c; d]$, $d([a; b], [c; d])$ be $\{d(x, y) \mid x \in [a; b] \wedge y \in [c; d]\}$. A function like this will be according to the Interval Representation paradigm investigated in [29]. The interval-based metric which satisfy this requirement is called km-metric and is described in the following section.

C. Km-Metric

All the items required to define km-metric is reproduced here and can also be found in [28] and [13]. Before we go further, let us introduce some well-known order-theoretic terminology.

Given a non-empty set A, a binary relation, $\leq$, on A is called partial order, whenever it is reflexive, symmetric and transitive. The pair $\langle A, \leq \rangle$ is called partially ordered set or just poset. If A has an element $\perp$ such that $\perp \leq x, \forall x \in A$, then this element is called the bottom element of A and the structure $\langle A, \leq, \perp \rangle$ is called poset with bottom element.

Definition 3 ([28]): Given a poset A and a non-empty subset $D \subseteq A$, D is called **d-directed set** if for every $x, y \in D$, there exist $z \in D$, such that $z \leq x, y$. A binary relation, R, on A is called **semi-auxiliary relation for** $\leq$ whenever:

1) If xRy, then $x \leq y$;
2) If $x \leq y$, xRz and $z \leq w$, then xRw.

A poset $\langle A, \leq, \perp \rangle$ endowed with a semi-auxiliary relation, R, has **separable bottom element**, whenever A is a d-directed set, such that for each pair of elements $a, b \in A$, with $\perp Ra$ and $\perp Rb$, there is $z \in A$, such that $z \leq a, b$ and $\perp Rz$.

Definition 4 (i-Distance Valuation,[13]): A **Valuation for i-Distances (VID)** is a partial order $\langle A, \leq, R, \perp \rangle$ with separable bottom element $\perp$.

Example 1: A well-known VID is the usual $\langle [0, +\infty), \leq, <, 0 \rangle$, which is the valuation of Euclidean metric.

The next definition states the generalization of the metric proposed by Santana in [13].

Definition 5 (i-Metric,[13]): Let M be a non-empty set and $\mathcal{V} = \langle A, \leq, R, \perp \rangle$ a VID. A function $d : M \times M \longrightarrow A$ is called **$\mathcal{V}$-valued i-metric** , or just **i-metric** when the context is clear, whenever:

1) $d(a, b) = \perp$ iff $a = b$;
2) $d(a, b) = d(b, a)$, for every $a, b \in M$;
3) If $d(a, b)R\varepsilon$, for some $\varepsilon \in A$ with $\perp R\varepsilon$, then there exists $\delta \in A$, with $\perp R\delta$, such that $d(b, c)R\delta \Rightarrow d(a, c)R\varepsilon$.

In this case, the triple $(M, d, \mathcal{V})$ is called **i-metric space**.

In [28], the authors proposed some VID's for intervals giving rise to idea of interval-based metrics, one of such VID is based on Kulish-Miranker order and proved to be suitable to provide a metric able to follow the Interval Representation

Paradigm [29]; namely: The km-metric. In what follows we reproduce those notions.

A VID of the form $\Omega = \langle \mathbb{I}(\mathbb{R})^+, \leq, R, [0,0] \rangle$ receives a special name, it is called **Interval Metric Valuation** or just **(IMV)**. An i-metric valued in this structure is named **interval metric**. Since in this paper we will use just this kind of distance, we will make an abuse of language and also call them as i-metrics (in this case meaning interval metrics). Rewriting the definitions for this kind of VID, an interval-based metric will have the following definition:

Definition 6: Let M be a non-empty set and $\Omega = \langle \mathbb{I}(\mathbb{R})^+, \leq, R, [0,0] \rangle$ an *IMV*. A function $d : M \times M \to \Omega$ is called an interval metric, i-metric, if it satisfies:

1) $d(x, y) = [0, 0]$ iff $x = y$;
2) $d(x, y) = d(y, x)$, for all $x, y \in M$;
3) If $d(x, y) R \epsilon$, for some $\epsilon \in \mathbb{I}(\mathbb{R})^+ - \{[0, 0]\}$, then exists $\delta \in \mathbb{I}(\mathbb{R})^+ - \{[0, 0]\}$ such that if $z \in \mathbb{I}(\mathbb{R})$ and $d(y, z) R \delta$, then $d(x, z) R \epsilon$.

In this case, the triple (M, d, Ω) is called i-metric space.

All the mathematics developed by Santana in [13] aimed to provide a distance which captures the important concept of interval representation [29]. To achieve that, the author built an IMV which is based on the Kulisch-Miranker order. The resulting interval metric was called km-metric.

Definition 7 (Kulisch-Miranker Order [28]): Given two intervals $X, Y \in \mathbb{IR}$, the Kulisch-Miranker order on $\mathbb{I}(\mathbb{R}), \leq_{km}$, is defined by $X \leq_{km} Y$iff $\underline{x} \leq \underline{y}$ and $\overline{x} \leq \overline{y}$.

Definition 8: [28] Let be the binary relation, $\ll$, on $\mathbb{I}(\mathbb{R})^+$:

1) $[0, 0] \ll X$, for all $X \in \mathbb{I}(\mathbb{R})^+$;
2) If $\overline{x}, \overline{y} > 0$, then $[0, \overline{x}] \ll [0, \overline{y}] \Leftrightarrow \overline{x} < \overline{y}$;
3) If $\underline{x}, \overline{x}, \underline{y}, \overline{y} > 0$, then $X \ll Y \Leftrightarrow (\underline{x} < \underline{y}) \wedge (\overline{x} < \overline{y})$.

Theorem 1: [28] The structure $\langle \mathbb{I}(\mathbb{R})^+, \leq_{km}, \ll, [0, 0] \rangle$ is a d-directed set with separable bottom element $[0, 0]$, and hence a VID.

Theorem 2: According to [13], [28], given two distinct intervals $X, Y \in \mathbb{I}(\mathbb{R})$, let be the set of euclidean distances between the elements of X and Y, respectively: $D_{XY} = \{d(x, y); x \in X \text{ and } y \in Y\}$. The function $d_{km} : \mathbb{I}(\mathbb{R}) \times \mathbb{I}(\mathbb{R}) \to \langle \mathbb{I}(\mathbb{R})^+, \leq_{km}, \ll, [0, 0] \rangle$ defined by:

$$d_{km}(X, Y) = \begin{cases} [0, 0] & \text{, if } X = Y \\ [\min D_{XY}, \max D_{XY}] & \text{, if } X \neq Y \end{cases}, \tag{1}$$

which is an interval i-metric.

An important issue about interval functions is related to its computation. In order to answer this question, Santana in [13] provided the following theorem:

Theorem 3: Given X, $Y \in \mathbb{I}(\mathbb{R})$, we have: $d_{km}(X, Y) =$

$$\begin{cases} [0, 0] & \text{, if } X = Y \\ [d(\overline{x}, \underline{y}), d(\underline{x}, \overline{y})] & \text{, if } \overline{x} < \underline{y} \\ [d(\underline{x}, \overline{y}), d(\overline{x}, \underline{y}) & \text{, if } \overline{y} < \underline{x} \\ [0, d(\underline{x}, \overline{y})] & \text{, if } X <_{km} Y \text{ and } X \cap Y \neq \emptyset \\ [0, d(\overline{x}, \underline{y}) & \text{, if } Y <_{km} X \text{ and } X \cap Y \neq \emptyset \\ [0, \max(d(\overline{x}, \underline{y}); d(\underline{x}, \overline{y}))] & \text{, if } X = Y \text{and } (X \subset Y \text{or} Y \subset X) \end{cases} \tag{2}$$

The main idea is to use an interval-based i-metric instead of a metric to measure the distance between two interval data. It lies on the fact that the interval data carries an imprecision and this imprecision is completely lost when a standard metric maps them onto a single and exact real number. Since D_{XY} contains the values of the distance between all pairs $(x, y) \in X \times Y$, it is an interval which contains the exact distance, $d(x, y)$, between the exact values, x and y, codified respectively by the intervals X and Y.

III. Clustering Algorithms

As mentioned previously, there are several clustering algorithms proposed in the literature and they can be broadly divided into two categories: hard (crisp) and fuzzy (soft) clustering. For fuzzy clustering, we will use a well-known fuzzy clustering algorithm, which is Fuzzy C-Means (FCM)[4]. Therefore, this algorithm will be briefly described in the next subsection. We will also describe one of the main aspects when adapting fuzzy clustering algorithms for interval-based data, which is the use of an interval-based fuzzy membership matrix.

A. Fuzzy c-means (FCM)

One of the most popular methods for finding fuzzy partitions is FCM [4]. The FCM algorithm tries to find groups of data, minimizing the objective function in Eq.(3).

$$J_m(U, V) = \sum_{j=1}^{c} \sum_{j=1}^{n} \mu_{ij}^m \ d(x_i, v_j)^2, \tag{3}$$

where: n is the data quantity; c is number of clusters considered in the algorithm, predefined; m is a fuzzification parameter, with $m > 1$; μ_{ij} is membership degree that x_i is in v_j, where x_i is the i-th data and v_j is the center (prototype) of j-th cluster, and $d(x_i, v_j)$, or only d_{ij}, is the distance between x_i and v_j.

Given a data set $X = \{x_1, \ldots, x_n\}$ we cluster them in groups, in which each group is represented by its prototype, i.e an element of $V = \{v_1, \ldots, v_c\}$. FCM is based on fuzzy logic [33], each instance is not only associated to a cluster, but has a membership degree for each of the centroids existing in the process. The set of all fuzzy c-partition is

$$\mathbf{M}_{fcn} = \Big\{ \mathbf{U} \in \mathbb{R}^{cn} | 0 \leq \mu_{ij} \leq 1, \ (1 \leq j \leq c)(1 \leq i \leq n) \ ; \sum_{i=1}^{c} \mu_{ij} = 1, \ \forall i; \ \sum_{j=1}^{n} \mu_{ij} > 0, \forall j \Big\}. \tag{4}$$

Let U $\in \mathbf{M}_{fcn}$, partition matrix, the minimization of $J_m(\mathbf{U}, V)$ is achieved with respect to μ_{ij} in Eq.(5) and with respect to center v_j in Eq.(6).

$$\mu_{ij} = \begin{cases} \dfrac{1}{\sum_{k=1}^{c} \left(\dfrac{d_{ij}}{d_{ik}} \right)^{\frac{2}{m-1}}} & \text{,if } I_i = \emptyset, \quad (5a) \\ \dfrac{1}{|I_i|} & \text{,if } I_i \neq \emptyset \text{ and } j \in I_i, \quad (5b) \\ 0 & \text{,if } I_i \neq \emptyset \text{ and } j \notin I_i, \quad (5c) \end{cases}$$

for all i, j where $I_i = \{j | d_{ij} = 0\}$ and $|I_i|$ is cardinality of I_i.

$$v_j = \frac{\sum_{i=1}^{n} (\mu_{ij})^m x_i}{\sum_{i=1}^{n} (\mu_{ij})^m}, \ \forall j \quad (1 \leq j \leq c). \tag{6}$$

The minimization steps are iterated until the change in any criteria drops below a certain threshold. There are many ways to initialize FCM and, in this paper, we chose c objects randomly from the dataset itself to serve as the initial cluster centers, which seems to work well in almost all cases. This algorithm ends when there are only negligible changes in the objective function, where ϵ is a pre-determined constant.

The following subsection show how algorithm FCM it is adapted for work with interval distances and interval-valued fuzzy set.

B. Interval-based fuzzy membership

A symbolic interval-based variable $X \in \Gamma$, where Γ is a set of s-dimension vector of closed intervals. Given a dataset $\Gamma = \{x_1, \ldots, x_k, \ldots, x_n\}$ with distinct points in s dimensions $x_k = (x_k^1, \ldots, x_k^s) \in \mathbb{I}(\mathbb{R})^s$ such that $x_k^i = [\underline{x}_k^i, \overline{x}_k^i]$.

The datasets X and the set of prototypes V are intervals. In other words, X and $V \in \mathbb{I}(\mathbb{R})^s$. Therefore we can define an interval-based distance, $d : \mathbb{I}(\mathbb{R})^s \times \mathbb{I}(\mathbb{R})^s \longrightarrow \mathbb{I}(\mathbb{R})$. The d_{km} is an interval-based metric (i-metric), such as in Eq.(2), and $X, Y \in \mathbb{I}(\mathbb{R})^s$, the interval-based distance, d, is then defined as:

$$d(X, Y) = \sqrt{\sum_{k=1}^{s} d_{km}(X^k, Y^k)^2}. \tag{7}$$

Once we have defined an interval-based distance metric, we need to adapt some fuzzy clustering algorithms to be used with interval-based data.

For fuzzy clustering using an interval-based distance, the similarity measure μ_{ij} also needs to be an interval. Therefore, the $\mathbf{U}$ matrix needs to be adapted. Now consider the interval-based matrix $\mathbb{U} = [u_{ij}] \subset \mathbb{I}([0,1])^{n \times c}$, where $u_{ij} = [\underline{u}_{ij}, \overline{u}_{ij}]$, like the interval-based distance.

In [8], the authors proposed the best *interval representation* for the classical function as the approach to calculate fuzzy membership Eq.(5), defined by

$$u_{ij} = \begin{cases} \big[\min \mu_{ij}, \max \mu_{ij}\big] & \text{,if } I_i = \emptyset \text{ and } E_i = \emptyset \\ \big[0, \max \mu_{ij}\big] & \text{,if } I_i \neq \emptyset \text{ and } j \notin I_i \\ & \text{and } E_i = \emptyset \\ \big[\min \mu_{ij}, 1\big] & \text{,if } I_i \neq \emptyset \text{ and } j \in I_i \\ & \text{and} E_i = \emptyset \text{ and} |I_i| = 1 \\ \big[0, 1\big] & \text{,if } I_i \neq \emptyset \text{ and } j \in I_i \\ & \text{and} E_i = \emptyset \text{ and} |I_i| > 1 \\ \big[0, 0\big] & \text{,if } E_i \neq \emptyset \text{ and } j \notin E_i \\ & \text{and } I_i = \emptyset \\ \left[\frac{1}{|E_i|}, \frac{1}{|E_i|}\right] & \text{,if } E_i \neq \emptyset \text{ and } j \in E_i \\ & \text{and } I_i = \emptyset \\ \left[0, \frac{1}{1 + |E_i|}\right] & \text{,if } I_i \neq \emptyset \text{ and } E_i \neq \emptyset \\ & \text{and } i \in I_i \\ \left[\frac{1}{|I_i| + |E_i|}, \frac{1}{|E_i|}\right] & \text{,if } I_i \neq \emptyset \text{ and } E_i \neq \emptyset \\ & \text{and } j \in E_i. \end{cases} \tag{8}$$

where $I_i = \{j | 0 \in d_{ij} \text{ and } d_{ij} \neq [0,0]\}$ and $E_i = \{j | d_{ij} = [0,0]\}$, besides $|I_i|$ and $|E_i|$ the cardinality of I_i and E_i, respectively. Moreover, $\min \mu_{ij}$ and $\max \mu_{ij}$ are defined as follows:

$$\min \mu_{ij} = \frac{1}{\sum_{k=1}^{j-1} \left(\frac{\overline{d_{ij}}}{\underline{d_{ik}}} \right)^{\frac{2}{m-1}} + 1 + \sum_{k=j+1}^{c} \left(\frac{\overline{d_{ij}}}{\underline{d_{ik}}} \right)^{\frac{2}{m-1}}}, \tag{9}$$

$$\max \mu_{ij} = \frac{1}{\sum_{k=1}^{j-1} \left(\frac{\underline{d_{ij}}}{\overline{d_{ik}}} \right)^{\frac{2}{m-1}} + 1 + \sum_{k=j+1}^{c} \left(\frac{\underline{d_{ij}}}{\overline{d_{ik}}} \right)^{\frac{2}{m-1}}}. \tag{10}$$

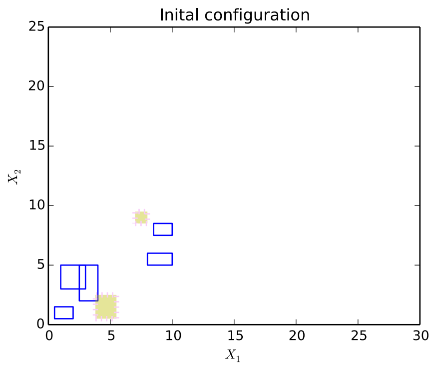
where d is the interval-based distance, i.e., is generated by i-metric Eq.(7).

Nevertheless, $\mathbb{U}$ obtained for Eq.(8) does not satisfy the constraint that $\sum_{j=1}^{c} [\underline{u}_{ij}, \overline{u}_{ij}] = [1,1]$, in the classical FCM algorithm. In other words, the sum of all intervals will only be degenerated if and only if these intervals are degenerated. Therefore, the set of all c-partition interval-valued is

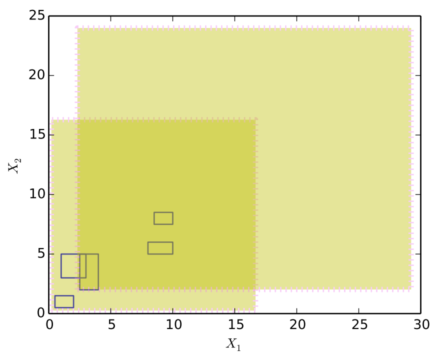
$$\mathbf{M}_{ivfcn} = \{\mathbb{U} \in \mathbb{I}(\mathbb{R})^{cn} | [0,0] \leq u_{ij} \leq [1,1] \quad (1 \leq j \leq c)(1 \leq i \leq n); \ \textstyle\sum_{i=1}^{n} u_{ij} > [0,0], \forall i\}. \tag{11}$$

In addition, the main drawback is the calculation of interval prototypes, because the standard operators of multiplication and division, defined by the Moore interval arithmetic, increased imprecision even further. For example, Fig. 1(a) shown an initial set of five interval variables and two prototypes (dashed rectangles) and Fig. 1(b) shown the new centers obtained from interval-valued memberships fuzzy, which led to inconsistent results.

The following section presents the proposed approaches and it shows how algorithms FCM can be adapted for solving these problem with matrix $\mathbb{U}$.



(a)



(b)

Fig. 1. The initial configuration (a) and the clusters center obtained (b) with $\mathbb{U}$.

IV. The Proposed Approaches

The IbFcM (Interval-based Fuzzy c-Means), proposed in [8], has the same structure of the classical FCM, presented in subsection III-A. In this paper, we propose an adaptation of IbFcM, in which, after calculating interval membership ($\mathbb{U}$), we define a different approach to define (U), which contains degenerate intervals. This definition is because the original $\mathbb{U}$ is not appropriate to calculate the centers of the groups.

The literature of interval-valued fuzzy sets provides some tools that help to transform the interval-valued fuzzy sets in fuzzy set. For example, the Atanassov's operator [3] and K-operator [27]. Therefore, we propose the use of an operator, H-operator, in the context of fuzzy clustering problems, that has two important properties, which are: it does not make modifications if all intervals of matrix $\mathbb{U}$ are degenerated and any admissible total order [5] can be used. In this case, we can ensure that IbFcM can be used with punctual data and the interpretation of the order is preserved.

Now consider $\widehat{\mathbb{U}} \in \mathbf{M}_{ivfcn}$ is an interval (degenerate) representation of matrix $\mathbf{U} \in \mathbf{M}_{fcn}$. In other words, $\widehat{\mathbb{U}}_{ij} = [\mathbf{U}_{ij}, \mathbf{U}_{ij}]$. In addition, let $\mathbb{U} \in \mathbf{M}_{ivfcn} = [u_{ij}]$ where $u_{ij} = [a_{ij}, b_{ij}]$.

Definition 9: A family of functions (H), where H : $\mathbf{M}_{ivfcn} \longrightarrow \mathbf{M}_{fcn}$, is a **H**-operator if it verifies the following conditions:

1) $\mathrm{H}(\widehat{\mathbb{U}}) = \mathbf{U}$,
2) if $\mathbb{U}_{ij} \leq_{km} \mathbb{U}_{ip}$ then $\mathrm{H}(\mathbb{U})_{ij} \leq \mathrm{H}(\mathbb{U})_{ip}$

Proposition 1: H_b (based in boundaries of intervals) is H-operator defined by

$$\mu_{ij} = \begin{cases} \dfrac{a_{ij}}{2\sum\limits_{k=1}^{c} a_{ik}} + \dfrac{b_{ij}}{2\sum\limits_{k=1}^{c} b_{ik}} & , \sum\limits_{k=1}^{c} a_{ik} \neq 0 \\ \dfrac{b_{ij}}{\sum_{k=1}^{c} b_{ik}} & , \text{otherwise} \end{cases} \tag{12}$$

Proof. Based on Definition (1), Let $\mathbb{U} \in \mathbf{M}_{ivfcn}$. It follows that: (1) $\widehat{\mathbb{U}}_{ij} = [\mathbf{U}_{ij}, \mathbf{U}_{ij}]$, then $\mathrm{H}(\widehat{\mathbb{U}})_{ij} = \frac{a_{ij}}{2\sum_{k=1}^{c} a_{ik}} + \frac{b_{ij}}{2\sum_{k=1}^{c} b_{ik}} = \frac{a_{ij}}{2} + \frac{a_{ij}}{2} = a_{ij} = b_{ij} = \mathbf{U}_{ij}$ (2) if $\mathbb{U}_{ij} \leq_{km} \mathbb{U}_{ip}$ then $\frac{a_{ij}}{2\sum_{k=1}^{c} a_{ik}} + \frac{b_{ij}}{2\sum_{k=1}^{c} b_{ik}} \leq \frac{a_{ip}}{2\sum_{k=1}^{c} a_{ik}} + \frac{b_{ip}}{2\sum_{k=1}^{c} b_{ik}}$ because $a_{ij} \leq a_{ip} \wedge b_{ij} \leq b_{ip}$.

Example 2: Motived by H_b, others H$-$operators can be defined as follows:

i) H_{mp} (based on midpoint of intervals) is defined

$$\mu_{ij} = \frac{\frac{(a_{ij}+b_{ij})}{2}}{\sum_{k=1}^{c} \frac{(a_{ik}+b_{ik})}{2}} \tag{13}$$

ii)H_{xu} (based on the Xu and Yager[32] order) is defined

$$\mu_{ij} = \begin{cases} \dfrac{(a_{ij}+b_{ij})}{2\sum_{k=1}^{c}(a_{ik}+b_{ik})} + \dfrac{(b_{ij}-a_{ij})}{2\sum_{k=1}^{c}(b_{ik}-a_{ik})} & \sum_{k=1}^{c}(b_{ik}-a_{ik}) \neq 0 \\ \dfrac{(a_{ij}+b_{ij})}{\sum_{k=1}^{c}(a_{ik}+b_{ik})} & , \text{otherwise} \end{cases} \tag{14}$$

Therefore, the objective function presented in Eq.(15) and the calculation of the centers of the groups in Eq.(16) are adaptations of **U** using the H-operator instead of the original $\mathbb{U}$, obtained by Eq.(8).

$$J_m = \sum_{k=1}^{n} \sum_{i=1}^{c} \mu_{ij}^m d_{IM}(x_i; v_j)^2, \tag{15}$$

where: n is the interval-based data size; c is predefined number of clusters considered in the algorithm; m is a fuzzification parameter, for $m > 1$; u_{ij} is membership (degenerate interval) degree that x_i in v_j; x_i is the i-th interval data; v_j is the center (interval) of j-th clustering and $d(x_i, v_j)$, is an interval-based distance (i-metric) between x_i and v_j.

The input of the generic H-operator algorithm is n interval-based data, the number of prototypes c and m value. The main steps of this algorithm are:

1) Start V with c objects randomly from the dataset to serve as the initial cluster centers;
2) Calculate the matrix $\mathbb{U} \in \mathbf{M}_{ivfnc}$ interval membership function fuzzy as shown in Eq.(8);
3) Calculate the matrix matrix **U** contains intervals degenerated using $\mathrm{H}(\mathbb{U})$, where H is H-operator .
4) Calculate J_m using Eq.(15);
5) Calculate the new prototype of the cluster j using **U**: The value v_j is the interval $[\underline{v_j}, \overline{v_j}]$, which can be obtained;

$$\underline{v_j} = \frac{\sum_{i=1}^{n} \mu_{ij}^m \underline{x_i}}{\sum_{i=1}^{n} \mu_{ij}^m} \quad \overline{v_j} = \frac{\sum_{i=1}^{n} \mu_{ij}^m \overline{x_i}}{\sum_{i=1}^{n} \mu_{ij}^m}; \tag{16}$$

6) Return to step 2 until a convergence criteria is reached. This happens when the minimization process stagnates, $(d_{km}(J_m^{new}, J_m^{old})^2 \leq \epsilon)$, where ϵ is a pre-determined constant.

In this paper, we propose three different approaches. In this case, each H−operator characterize a different approach, because they update the centers of the groups in a different way. They were obtained by different $\mathbf{U}$, given $\mathbb{U}$. Therefore, we propose three different approaches of IbFcM, that were used to update the centers of the clusters, which are:

- **IbFcM-**H_b uses H_b Eq. (12) in the step 3 as an H-operator;
- **IbFcM-**H_{mp} uses H_{mp} Eq. (13) in the step 3 as an H-operator;
- **IbFcM-**H_{xu} uses H_{xu} Eq. (14) in the step 3 as an H-operator.

V. EXPERIMENTAL ANALYSIS

In order to evaluate the feasibility of the proposed approaches, an empirical analysis is conducted. In order to assess the results obtained by the clustering algorithms, an external index, called Corrected Rand (CR), [17], is used. This index compares the goal partition with the partition obtained from the clustering algorithm. The CR index delivers values that lies in the interval $[-1, 1]$, where values close to one indicates the perfect agreement between partitions, whereas values near zero (or negatives) correspond to cluster agreement found by random. This index does not needs any adaptation to work with interval-based data.

In this methodology, we perform 100 replications and we use the average correct Rand Index CR of all 100 replication. In addition, we compared the results obtained by these methods with some existing interval-based methods in the literature.

The algorithms were implemented in Python (www.python.org) using the MPMATH library (http://docs.sympy.org/dev/modules/mpmath) to represent the interval-based data. Moreover, we use the fuzzification parameter, m, equals 2 and a stopping condition of $[1e^{-4}, 1e^{-4}]$.

A. Dataset

As already mentioned, we use seven datasets, four synthetic interval-valued datasets and three real interval-valued datasets, which are: city temperatures, fish and car.

1) Synthetic symbolic interval datasets: In this paper, we recreate the dataset proposed in [6]. In order to test the clustering algorithms, we used four artificial datasets of the seven datasets proposed in [6], which are: 2Dim, 3Dim, 5Dim and Sun. In 3Dim and 5Dim datasets, unlike [6], the number of points used in this paper is 300. Table I presents a brief description of the artificial datasets.

The 2Dim dataset consists of two groups of 200 data points each, linearly separated in to two dimensions. In addition, the Sun dataset consists of five groups oriented in different ways. This set contains 195 data points in two dimensions.

The 3D dataset 3Dim consists of four groups of equal size that are arranged on the vertices's of a tetrahedron in a three-dimensional space. This dataset contains 300 data points.

TABLE I
DESCRIPTION OF THE FOUR INTERVAL-BASED ARTIFICIAL DATASETS

Database name	Number of data points	Number Cluster	Number of Variables
2Dim	200	2	2
3Dim	300	4	3
5Dim	300	4	5
Sun	195	5	2

Finally, the 5Dim dataset contains groups that are of various shapes and sizes in a space with five dimensions.

2) Real Interval-Valued Datasets: In this section we present the real interval-valued datasets that will be used in this empirical analysis, which are:

City temperature symbolic interval dataset: The city temperature symbolic Interval dataset [16] represents the temperature in different cities around the world. These cities were put into groups using expert knowledge. It results in four clusters of cities. The cities belonging to cluster 1 are located between 0° and 40° latitudes and the cities which are classified under cluster 2 are located between 40° and 60° latitudes. Mauritius and Tehran are classified as members of singleton classes 3 and 4, respectively.

The table II presents the cluster distribution based on the human observer which originally created this dataset.

TABLE II
CLUSTER DISTRIBUTION OF THE TEMPERATURE DATASET

Cluster no	Sample name
I	Bahraim Bombay Cairo Calcutta Colombo Dubai Hong Kong Kula Lampur Madras Manila Mexico Nairobi New Delhi Sydney
II	Amsterdam Athens Copenhagen Frankfurt Geneva Lisbon London Madrid Moscow Munich New York Paris Rome San Francisco Seoul Stockholm Tokyo Toronto Vienna Zurich
III	Mauritius
IV	Tehran

For each city (instance), we have 12 interval-based attributes and each one represents the minimum and maximum and temperature of the city in a month.

Car Symbolic Interval Dataset: The car symbolic interval dataset, described in Table III, considers a set of 33 car models (objects) described by 8 interval variables and one categorical variable, which is divided into 4 classes: Utilitarian, Berlina, Sporting and Luxury. These classes have 10, 8, 8 and 7 instances, respectively, as described in table III. The interval-valued variables are Price, Engine Capacity, Top Speed, Acceleration, Step, Length, Width and Height.

TABLE III
DESCRIPTION OF THE CAR SYMBOLIC INTERVAL DATASET

Var. individuals	Price	...	Height	Category
Alfa 145	[27 806, 33 596]	...	[143, 143]	Utility
Alfa 156	[41 593, 62 291]	...	[142,142]	Sedan
⋮	⋮	⋮	⋮	⋮
Rover 25	[21 492, 33 042]	...	[142,142]	Utility
Passat	[39 676, 63 455]	...	[146,146]	Luxury

Fish Symbolic Interval Dataset: Fish symbolic interval data set represents 12 fish species of freshwater, where each specie is described by 13 symbolic interval variables, which are: Length, Weight, Muscle, Intestine, Stomach, Gills, Liver, Kidneys, Liver/Muscle, Kidneys/Muscle, Gills/Muscle, Intestine/Muscle and Stomach/Muscle. These species are grouped four classes, Carnivorous, Detritivorous, Omnivorous and Herbivorous.

VI. Results and Discussion

For the empirical analysis with synthetic and real datasets, 100 replications are considered the mean and standard deviation of the index CR (presented in Subsection V) are presented.

Table IV synthesizes the results for the three proposed approaches, IbFcM-H$_b$, IbFcM-H$_{xu}$ and IbFcM-H$_{mp}$, for the synthetic datasets. As it can be observed from Table IV, the CR values show that the proposed algorithms are capable of producing a good partitioning of the interval-based data, since all CR values are close to 1 (the obtained partitions are close to the optimal partition). For the 2Dim and 3Dim datasets, we obtained CR equal to 1. These datasets are considered as simple and the optimal partition can be easily obtained (classes are easily separable). Of the other two datasets, although we did not obtain CR values equal to 1, they are very close to 1.

Of the proposed approaches, the IbFcM-H$_b$ algorithm showed the best CR results for synthetic datasets.

TABLE IV

The average CR values for all artificial interval-based datasets.

Datasets	IbFcM-H$_b$	IbFcM-H$_{xu}$	IbFcM-H$_{mp}$
2Dim	1(0)	1(0)	1(0)
3Dim	1(0)	1(0)	1(0)
5Dim	0.908(0.911)	0.849(0.997)	0.844(0.175)
Sun	0.912(0.098)	0.9(0.09)	0.849(0.088)

Table V synthesizes the CR values obtained by the three proposed algorithms, IbFcM-H$_b$, IbFcM-H$_{xu}$ and IbFcM-H$_{mp}$, applied to the interval-based real datasets. Generally speaking, the proposed algorithms demonstrated good CR results, in almost all cases. The only exception is the use of IbFcM-H$_{mp}$ for fish dataset (CR value lower than 0.5).

Unlike the synthetic dataset, the IbFcM-H$_{xu}$ approach achieved the best CR result for two datasets (Fish and Temperature datasets), while IbFcM-H$_{mp}$ achieved the best CR value in the remaining dataset.

TABLE V

The average CR values for all real interval-based datasets.

Datasets	IbFcM-H$_b$	IbFcM-H$_{xu}$	IbFcM-H$_{mp}$
Fish	*0.766*(0.301)	*0.767*(0.263)	0.348(0.072)
Car	0.68(0.605)	0.706(0.581)	*0.792*(0.733)
Temperature	0.657(0.477)	*0.877*(0.554)	0.78(0.397)

In order to evaluate the performance of the proposed algorithms, we also make a comparative analysis of the proposed approaches with existing algorithms in the literature. For this analysis, we used the best CR value obtained from 100 replications. Unlike the previous analysis, we use the best CR value because we only found the best CR value in the literature of the analyzed algorithms. In order to perform a fair comparison, we also represent the best CR value of the proposed algorithms.

Table VI synthesizes the results of the proposed approaches to the Fish dataset, comparing with existing algorithms for this interval-based dataset. We can observe that the CR values of two proposed methods are higher than all algorithms found in the literature. The existing algorithm that achieved CR value closest to the proposed algorithms is KKM-IV[23], with CR equals to 0.629. Even though, we have an improvement in CR performance of almost 15% for two proposed approaches, when compared with the KKM-IV algorithm.

TABLE VI

Comparative Results for the Fish dataset.

Algorithm	CR
IbFcM-H$_b$	**0.766**
IbFcM-H$_{xu}$	**0.767**
IbFcM-H$_{mp}$	0.348
MFCMdd-RWL-[11]	0.488
CARD-R[11]	0.160
FKCN[9]	0.209
IFKCN-FD[9]	0.043
IFCM-C [25]	0.161
IPCM [25]	0.318
IPCM-E [25]	0.369
IKCM-F[12]	0.021
IVKCM-K-LP[12]	0.275
IVKMCM-F[12]	0.034
IVKMCM-F-LP[12]	0.275
KM-IV[23]	0.610
KKM-IV[23]	0.629
FCM-IV[24]	0.466
KFCM-IV[24]	0.714

Table VII illustrates the best CR values for the Cars dataset. From Table VII, we can observe that all three proposed algorithms obtained higher CR values than all analyzed existing algorithms. It is important to highlight that even the worst proposed approach, IbFcM-H$_b$, obtained CR higher than all analyzed algorithms, around 5% higher than the best existing algorithm, MFCMdd-RWL-P.

TABLE VII

Comparative Results for the Fish dataset.

Algorithm	CR
IbFcM-H$_b$	0.68
IbFcM-H$_{xu}$	0.706
IbFcM-H$_{mp}$	**0.792**
MFCMdd-RWL-P [11]	0.614
CARD-R [11]	0.525
FKCN[9]	0.061
IFKCN-FD[9]	0.11
IKCM-F [12]	0.225
IVKCM-K [12]	0.225
IVKCM-K-LP [12]	0.499
IVKMCM-F [12]	0.225
IVKMCM-F-LP [12]	0.499
IFCM[10]	0.56
IFCMADC [10]	0.52

Finally, Table VIII illustrates the CR values for the Temperature dataset. The results in this table show that, once again, the quality of the obtained partitions of two proposed methods is higher than all other analyzed algorithms. For the best proposed approach, IbFcM-H$_{xu}$, the improvement in the CR values is around 16%, when compared with the best existing algorithm, IRBF.

TABLE VIII

RESULTS FOUND IN THE LITERATURE FOR THE DATASET *Temperature*.

Algorithm	CR
IbFcM-H_b	0.657
IbFcM-H_{xu}	**0.877**
IbFcM-H_{mp}	0.78
FKCN[9]	0.260
IFKCN-FD[9]	0.285
KM [2]	0.574
IRBF[2]	0.724
IPF[2]	0.499
Interval ckMeans[15]	0.517
KM-IV[23]	0.610
KKM-IV[23]	0.629
FCM-IV[24]	0.466
KFCM-IV[24]	0.714
IFCM [10]	0.46
IFCMADC [10]	0.50

VII. FINAL REMARKS

In this paper, a family of H-operator was proposed to be used in order to obtain three different approaches to define the centers of the groups in interval-based fuzzy clustering algorithms. In this case, we used a well-known fuzzy clustering algorithm, fuzzy c-means, adapted in three versions, which are: IbFcM-H_b, IbFcM-H_{xu} and IbFcM-H_{mp}.

In order to evaluate the feasibility of the proposed approaches, an empirical analysis was conducted. In this analysis, the proposed algorithms were analyzed using seven interval-based datasets. In addition, they were compared with some existing interval-based clustering algorithms. The comparative analysis was based on an external index, called correct rand (CR), using only three interval-based datasets.

For all interval-based real datasets, the obtained CR results showed that the proposed approaches outperformed all existing algorithms, in all the cases. These results are very promising since it can be seen as an efficient way to deal with interval-based data. The use of more datasets and a more extensive comparative analysis are the subject of an on-going research.

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