2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)
July 6-11, 2014, Beijing, China

A Comparison between T-S Fuzzy Systems and Affine T-S Fuzzy Systems as Nonlinear Control System Models

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***Abstract*— In model based fuzzy control, almost all considered control system models are T-S and affine T-S fuzzy control systems. However, there is a lack of systematic and theoretic understanding what the similarity and difference between these two dominated fuzzy control systems models to provide the guidance to choose the right models to the right application problems. To fill in such a gap, this paper gives a systematic comparison between T-S and affine T-S fuzzy control systems. The main results obtained are, firstly, the similarity between T-S and affine T-S fuzzy control systems is that both can and can only approximate affine nonlinear control models and have the similar representation capability for smooth (continuous differentiable) nonlinear control systems. As a result, T-S fuzzy systems are better choice as the stabilization analysis and control design simpler; Secondly, one of the main dissimilarities is that affine T-S fuzzy systems have better representation for continuous only (i.e., not differentiable) nonlinear control systems and can accurately approximate some continuous only nonlinear systems which cannot be accurately represented by T-S fuzzy systems. Another main dissimilarity is that affine T-S fuzzy systems are more accurate when representing high dimensional nonlinear systems. As a result, affine T-S fuzzy systems often could be the better choice for non-smooth or high dimensional nonlinear control systems.**

Keywords*— *affine nonlinear control systems; T-S fuzzy control systems; affine T-S fuzzy control systems; nonlinear control system models

I. INTRODUCTION

IN model based fuzzy control[1], T-S fuzzy control systems [1]-[14] and affine T-S fuzzy control systems [15]-[21] are dominated nonlinear control system models during the last two decades and a huge amount of research results have been published in this topic, please see the survey paper [1] for more details. Despite of the dominated fuzzy control system models for two decades, there is almost no result to analyze and compare these two classes of fuzzy control models such as what are the similarity, dissimilarity, advantage and disadvantage. As far as we know, the commonly cited reason for the need of affine T-S fuzzy control systems [16]-[21] is the better approximation accuracy based on [22]. However, the analysis and result obtained in [22] is based on T-S and affine T-S fuzzy systems rather than T-S and affine T-S fuzzy *control* systems and is only for SISO systems. In other words, the result given in [22] is related to basic T-S and affine T-S fuzzy systems and is not directly related control system modeling. Given the great importance and fundamental role of T-S fuzzy control systems and affine T-S fuzzy control systems, we feel that there is a need to fill in such a gap in the theory and applications of model based fuzzy control and this is the main motivation to this paper.

The comparison between T-S and affine T-S fuzzy control systems as nonlinear control system models given in this paper focuses on two fundamental aspects as follows:

- The similarity and dissimilarity in the representation capability. That is, what type of nonlinear control systems can or cannot be accurately represented by T-S and affine T-S fuzzy control systems?
- The approximation accuracy. That is, for a nonlinear system which can be accurately approximated by both T-S and affine T-S fuzzy control systems, which class of T-S fuzzy control systems can produce better accuracy and under what conditions?

The remaining part of this paper is organized as below: In section II, some preliminaries are given, including the introduction of T-S and affine T-S fuzzy control systems. Then in Section III, the similarity between T-S and affine T-S fuzzy control systems as nonlinear control system models are analyzed and the corresponding results are given. After that, in Section IV, the dissimilarity between T-S and affine T-S fuzzy control systems is analyzed and the corresponding results are given. Further and more detailed analysis related to the similarity and difference is given in Section V. Finally a brief conclusion is given. Due to the space limitations, all proofs are omitted from the paper. .

II. PRELIMINARIES

Consider a nonlinear control system given by the following:

$$\frac{dx(t)}{dt} = G[x(t), u(t)] \tag{1}$$

or

$$x(t+1) = G[x(t), u(t)] \tag{2}$$

where $x(t) = [x_1(t),...,x_n(t)]^\tau \in X = \prod_{i=1}^{n} X_i \subset R^n$ with $x_0 = [0,...,0]^\tau \in X$, $u(t) = [u_1(t),...,u_m(t)]^\tau \in U = \prod_{i=1}^{m} U_i \subset R^m$ with $u_0 = [0,...,0] \in U$, $G(x,u) = [G_1(x,u),...,G_n(x,u)]^\tau$ and τ represents vector or matrix transpose. Throughout this paper, it is always assumed that

$x_0 = [0,...,0]^\tau$ is the equilibrium of the system. (That is, $G[x_0, u_0] = 0$) and $G(x,u)$ is a continuous function on $X \times U$. Further it is assumed that $X \times U$ is a compact set on $R^n \times R^m$.

In the remaining part of this paper, in order to simplify the statement, $\frac{dx(t)}{dt} = x(t+1)$ is assumed if the considered system is a discrete system. In other words, we will no longer give the explicit representations of discrete system as (2). But any discussion, analysis and result obtained for continuous control systems are true for discrete systems by replacing $\frac{dx(t)}{dt}$ by $x(t+1)$.

In this paper, three most important classes of nonlinear systems described in (1) or (2) is going to be analyzed. The first is the most important class of nonlinear systems in conventional control theory – affine nonlinear control systems; the second one is the most widely used class of nonlinear systems in fuzzy control –T-S fuzzy control systems; the third one is another important class of nonlinear systems in fuzzy control – affine T-S fuzzy control systems. Therefore, in the following subsections, the brief description for each of the above classes of control systems is to be given.

A. Affine Nonlinear Control Systems

The general nonlinear system given in (1) or (2) is called an affine nonlinear system if $G(x,u)$ is an affine function on control. That is,

$$G(x,u) = f(x) + g(x)u \tag{3}$$

where

$$f(x) = \begin{bmatrix} f_1(x) \\ ... \\ f_n(x) \end{bmatrix} \quad g(x) = \begin{bmatrix} g_{11}(x) & ... & g_{1m}(x) \\ ... & ... & ... \\ g_{n1}(x) & ... & g_{mm}(x) \end{bmatrix} \tag{4}$$

In other words, an affine nonlinear system is given by the following:

$$\frac{dx(t)}{dt} = f[x(t)] + g[x(t)]u(t) \tag{5}$$

When $G(x,u)$ is an affine function on control, then it has the following properties.

Property 1. If $G(x,u)$ is an affine nonlinear continuous function on $X \times U$ given in (3) and $G(0,0) = 0$, then 1) $f(0) = 0$; 2) $f(x)$ and $g(x)$ are continuous on X .

B. T-S Fuzzy Control Systems

The most widely used T-S control systems in fuzzy control literature are given as follows: the rule base of such a system is given by

R^l : IF x_1 *is* F_1^l and … and x_n *is* F_n^l ,

$$\text{THEN } \frac{dx(t)}{dt} = A_l x(t) + B_l u(t) \quad l = 1,2,...,N \tag{6}$$

Under the center-average defuzzifier, product inference and singleton fuzzifier, the mathematical formula of the resulting T-S control system is

$$\frac{dx(t)}{dt} = F[x(t), u(t)] \tag{7}$$

with

$$\begin{aligned} F(x,u) &= \left[\sum_{l=1}^{N} \mu_l(x) A_l\right] x + \left[\sum_{l=1}^{N} \mu_l(x) B_l\right] u \\ &= f_A(x)x + g_B(x)u \end{aligned} \tag{8}$$

where

$$\mu_l(x) = \frac{F_1^l(x_1) * ... * F_n^l(x_n)}{\sum_{l=1}^{N} F_1^l(x_1) * ... * F_n^l(x_n)} \quad l = 1,2,...,N \tag{9}$$

are fuzzy basis functions [23]-[26] (also called normalised membership functions) and * is the product operator. Further, it is implied immediately from (9) that

$$\sum_{i=1}^{N} \mu_l(x) = 1 \tag{10}$$

From (7) - (10) and the definition of affine nonlinear control systems, the following property is obvious:

Property 2. A T-S control system is an affine nonlinear control system. In other words, the set of all T-S control systems is a subset of all affine nonlinear control systems.

C. Affine T-S Fuzzy Control Systems

In addition to the above most widely used T-S control systems in fuzzy control literature, another important class of T-S fuzzy control systems, which are called as affine T-S control systems, are also widely used. The affine T-S systems are given as follows: the rule base of such a system is given by

R^l : IF x_1 *is* F_1^l and … and x_n *is* F_n^l ,

$$\text{THEN } \frac{dx(t)}{dt} = c_l + A_l x(t) + B_l u(t) \;\; l = 1,...,N \tag{11}$$

The main difference between a commonly used T-S fuzzy control system and an affine T-S fuzzy control systems is that the Then-part of the above fuzzy rules of an affine T-S system includes an extra constant term c_l , comparing the rules of a commonly used T-S fuzzy control system given in (6). Under the center-average defuzzifier, product inference and singleton fuzzifier, the mathematical formula of the resulting affine T-S control system is

$$\frac{dx(t)}{dt} = F[x(t), u(t)] \tag{12}$$

with

$$
\begin{aligned}
&F(x,u) \\
&= \left[\sum_{l=1}^{N} \mu_l(x) c_l \right] + \left[\sum_{l=1}^{N} \mu_l(x) A_l \right] x + \left[\sum_{l=1}^{N} \mu_l(x) B_l \right] u \\
&= C(x) + f_A(x) x + g_B(x) u
\end{aligned} \tag{13}
$$

where $\mu_l(x)$ ($l = 1,2,...,N$) is given in (9). Similar to Property 2, the following property is obvious from (6) - (13) and the definition of affine nonlinear systems.

Property 3. A T-S fuzzy control system is an affine T-S fuzzy control system. In other words, the set of all T-S fuzzy control systems is a subset of all affine T-S fuzzy control systems.

Property 4. An affine T-S fuzzy control system is an affine nonlinear control system. In other words, the set of all affine T-S fuzzy systems is a subset of all affine nonlinear control systems.

D. *Notations*

Before the end of this section, a few notations which are going to be used in the next section are introduced here.

1. For a vector $x = [x_1,...,x_n]^\tau$, its norm is defined as

$$\| x \|_\infty = \max_{1 \le i \le n} | x_i |;$$

2. For a matrix

$$A = \begin{pmatrix} a_{11} & ... & a_{1m} \\ ... & ... & ... \\ a_{n1} & ... & a_{nm} \end{pmatrix}$$

its norm is defined as $\| A \|_\infty = \max_{1 \le i \le n, 1 \le j \le m} | a_{ij} |$. Note that the norm defined here for matrices is not the standard matrix norm rather the vector norm in $R^{n \times m}$. As all the vector and matrix norms used in the paper are the ∞ norms defined above, we will omit the subscript ∞ in most cases;

3. For a real value continuous function $f(x)$ on a compact set $X \subset R^m$, its norm is defined as

$$\| f \|_\infty = \max_{x \in X} | f(x) |.$$

4. For a vector or matrix value continuous function $F(x)$ on a compact set $X \subset R^m$, i.e., $F(x) = [f_1(x),..., f_n(x)]^\tau$ or

$$F(x) = \begin{pmatrix} f_{11}(x) & ... & f_{1m}(x) \\ ... & ... & ... \\ f_{n1}(x) & ... & f_{nm}(x) \end{pmatrix}$$

its norm is defined as $\| F \|_\infty = \max_{1 \le i \le n} \| f_i(x) \|_\infty$ or $\| F \|_\infty = \max_{1 \le i \le n, 1 \le j \le m} \| f_{ij}(x) \|_\infty$. To distinguish the ∞ vector and matrix norms defined in 1 and 2, the subscript ∞ will be kept for all real vector or matrix value functions.

III. Similarity Between T-S Fuzzy Systems and Affine T-S Fuzzy Systems as Control System Models

In this section, the similarity between T-S fuzzy systems and affine T-S fuzzy systems as control system models are going to be analyzed and presented.

When using T-S fuzzy systems or affine T-S fuzzy systems to represent a nonlinear control system, it is sufficient that the nonlinear control system can be approximate to any degree of accuracy whose definition is given below.

Definition 1. Let a nonlinear control system be given in (1) and S be a set of approximation models defined by

$$S = \left\{ \frac{dx(t)}{dt} = F[x(t),u(t)] \,\middle|\, F(x,u) \in F \right\} \tag{14}$$

in which F is a set of functions defined on $X \times U$. Then it is said that the given nonlinear system can be approximated to any degree of accuracy by S, if for any given $\varepsilon > 0$, there exists a function $F(x,u) \in F$ such that

$$\| G - F \|_\infty = \max_{(x,u) \in X \times U} \| G(x,u) - F(x,u) \| < \varepsilon$$

In the following, we use S_{T-S} represents the set of all T-S fuzzy control systems. That is,

$$S_{T-S} = \left\{ \frac{dx(t)}{dt} = F[x(t),u(t) \,\middle|\, F(x,u) \in F_{T-S} \right\} \tag{15}$$

in which F_{T-S} is the set of all T-S fuzzy systems given in (8). Similarly S_{A-T-S} represent the set of all affine T-S fuzzy control systems as below

$$S_{A-T-S} = \left\{ \frac{dx(t)}{dt} = F[x(t),u(t) \,\middle|\, F(x,u) \in F_{A-T-S} \right\} \tag{16}$$

in which F_{A-T-S} is the set of all affine T-S fuzzy systems given in (13).

As pointed out by Property 3, at the first look, it seems obvious that affine T-S fuzzy control systems are more general than T-S fuzzy control systems as the latter is a special case of the former when the constant terms c_l $(l = 1,2,...,N)$ in (11) are zero. However, this is not true in most cases due to the following theorem.

Theorem 1. Let a nonlinear control system be given by (1) and $G(x,u)$ be a continuously differentiable on compact set $X \times U$. If the given nonlinear control system can be approximated to any degree of accuracy by S_{A-T-S}, then can be approximated to any degree of accuracy by S_{T-S}.

Remark 1. The meaning of Theorem 1 is that, for any continuously differentiable nonlinear system, if it can be accurately represented by affine T-S fuzzy systems, it can be accurately represented by T-S fuzzy systems. As nonlinear control systems are continuously differentiable in most cases of applications, then Theorem 1 shows that T-S fuzzy systems are as general as affine T-S fuzzy systems in most cases.

Now define

$$S^i_{Affine} = \left\{ \frac{dx(t)}{dt} = G[x(t),u(t) \middle| G(x,u) \in C^i(X \times U), \right.$$
$$\left. G(x,u) = f(x) + g(x)u \right\} \quad i = 0,1,... \tag{17}$$

where $C^0(X \times U)$ is the set of all continuous function on $X \times U$ and $C^i(X \times U)$ is the set of all $i\,(\geq 1)$ time continuously differentiable functions on $X \times U$. Further if S be a set of control system models, we use $\overline{S}$ to represent the set of all control systems which can be approximated to any degree of accuracy by S. With these notations, we have the following theorem.

Theorem 2. $S^1_{Affine} \subset \overline{S}_{T-S} \subset \overline{S}_{A-T-S} \subset S^0_{Affine}$

Remark 2. Theorem 2 shows that, for any nonlinear control system, if it can be approximated to any degree of accuracy by T-S fuzzy systems or affine T-S fuzzy systems, then it is an affine nonlinear system. In other words, T-S fuzzy systems or affine T-S fuzzy systems can be used and can only be used as affine nonlinear control system models.

Due to Remark 2, in the remaining part of this paper, we always assume that the nonlinear control system to be approximated or represented is an affine nonlinear system as below:

$$\frac{dx}{dt} = f(x) + g(x)u \tag{18}$$

For affine nonlinear control systems given above, we have the following theorem.

Theorem 3. Let an affine nonlinear control system given in (18) in which $f(x)$ are continuously differentiable and $g(x)$ is continuous on X. Then for any given $\varepsilon > 0$, the following statements hold: 1) There exists a T-S fuzzy system

$$F(x,u) = \left[\sum_{l=1}^{N} \mu_l(x) A_l\right] x + \left[\sum_{l=1}^{N} \mu_l(x) B_l\right] u$$
$$= f_A(x)x + g_B(x)u$$

such that

$$f(x) = [f_A(x) + \varepsilon_A(x)]x \tag{19}$$
$$g(x) = g_B(x) + \varepsilon_B(x) \tag{20}$$
$$\| \varepsilon_A(x) \|_\infty < \varepsilon \qquad \| \varepsilon_B(x) \|_\infty < \varepsilon \tag{21}$$

2) There exists an affine T-S fuzzy system

$$F(x,u)$$
$$= \left[\sum_{l=1}^{N} \mu_l(x) C_l\right] + \left[\sum_{l=1}^{N} \mu_l(x) A_l\right] x + \left[\sum_{l=1}^{N} \mu_l(x) B_l\right] u$$
$$= f_C(x) + f_A(x)x + g_B(x)u$$

such that

$$f(x) = f_C(x) + f_A(x)x + \varepsilon_A(x) \tag{22}$$
$$g(x) = g_B(x) + \varepsilon_B(x) \tag{23}$$
$$\| \varepsilon_A(x) \|_\infty < \varepsilon \qquad \| \varepsilon_B(x) \|_\infty < \varepsilon \tag{24}$$

The above analysis and result show the similarity between T-S and affine T-S fuzzy control systems as nonlinear control system models in the following sense:

- Both classes of fuzzy control systems are a subset of affine nonlinear control systems (Properties 2 and 4) and can only be used to represent and approximate affine nonlinear control systems (Theorem 2).
- For any continuously differentiable nonlinear system, if it can be modeled by affine T-S fuzzy systems to any desired degree of accuracy; it can be modeled by T-S fuzzy systems to any desired degree of accuracy (Theorem 1). The vice versa is true too (Property 3).
- If a nonlinear control system to be modeled is continuously differentiable affine nonlinear systems, then both classes of systems can be used to model and represent such a control system to any degree of accuracy (Theorem 3). Further the robust stabilization of T-S or affine T-S fuzzy control systems can ensure not only the stabilization of T-S or affine T-S fuzzy control systems themselves but also the stabilization of the original control system based on equation (18). .

For any stabilization and other control design problems of continuously differentiable affine control systems, due to the above similarity between T-S and affine T-S fuzzy control systems, it is sufficient and without loss of generality by applying T-S fuzzy control systems, as their stabilization and control design is simpler than affine T-S fuzzy control systems. This conclusion gives the further support why T-S fuzzy control systems are deserved as the most widely used fuzzy control system models due to their simplicity in stability analysis and stabilization design and capability to accurately represent most affine nonlinear control systems faced in applications.

IV. Dissimilarity Between T-S Fuzzy Systems and Affine T-S Fuzzy Systems as Control System Models

In the last section, the similarity between T-S fuzzy systems and affine T-S fuzzy systems as control system models are analyzed and presented. In this section, the focus is to analyze and present dissimilarity between these two classes of fuzzy systems as control system models.

As all nonlinear control systems that can be accurately represented by T-S or affine T-S fuzzy control systems are affine nonlinear systems (based on Theorem 2), without loss of generality and for simplifying the statements, this section only considers affine nonlinear control systems given in (18). In other words, this section will analyze and identify the difference when using T-S or affine T-S fuzzy control systems to model affine nonlinear control systems.

Firstly, as it is discovered in the last section that T-S and affine T-S fuzzy systems are similar when being applied to model continuously differentiable affine nonlinear control systems, we turn our attention to ask the question: Are T-S and affine T-S fuzzy systems are similar when being applied to model continuous only affine nonlinear control systems? The answer to this question is negative as shown in the following theorem and the example.

Theorem 4. Let an affine nonlinear control system be given in (18) in which $f(x)$ and $g(x)$ are continuous on X. Then for any given $\varepsilon > 0$, there exists an affine T-S fuzzy system

$$F(x,u) = \left[\sum_{l=1}^{N}\mu_l(x)C_l\right] + \left[\sum_{l=1}^{N}\mu_l(x)A_l\right]x + \left[\sum_{l=1}^{N}\mu_l(x)B_l\right]u = f_C(x) + f_A(x)x + g_B(x)u$$

such that

$$f(x) = f_C(x) + f_A(x)x + \varepsilon_A(x) \quad (25)$$

$$g(x) = g_B(x) + \varepsilon_B(x) \quad (26)$$

$$\| \varepsilon_A(x) \|_\infty < \varepsilon \qquad \| \varepsilon_B(x) \|_\infty < \varepsilon \quad (27)$$

Further $\overline{S}_{A-T-S} = S^0_{Affine}$.

Theorem 4 shows that, for an affine nonlinear system given by (18) in which $f(x)$ is only continuous but not continuously differentiable, the second part conclusion of Theorem 3 still holds. That is, such an affine nonlinear system can be approximated to any degree of accuracy by affine T-S fuzzy systems. However, this is not true for T-S fuzzy systems. In other words, the first part conclusion in Theorem 3 is no longer valid for such an affine nonlinear system. In the following, an example is given to verify this claim.

Example 1. Consider an affine nonlinear control system given below

$$\frac{dx}{dt} = \sqrt{x} + u$$

on $X \times U = [0,1]\times[-1,1]$ in which $f(x) = \sqrt{x}$ is continuous but not continuously differentiable at $0 \in X = [0,1]$. Now we prove by contradiction that it is impossible to find a T-S fuzzy system

$$F(x,u) = \left[\sum_{l=1}^{N}\mu_l(x)A_l\right]x + \left[\sum_{l=1}^{N}\mu_l(x)B_l\right]u$$

such that (19) - (21) hold. Suppose that such a T-S fuzzy system exists, then

$$\sqrt{x} = [f_A(x) + \varepsilon_A(x)]x$$

Which implies for $x \in X$ and $x > 0$,

$$1 = [f_A(x) + \varepsilon_A(x)]\sqrt{x} .$$

Noticing $f_A(x)$ and $\sqrt{x}$ are continuous and $\varepsilon_A(x)$ is bounded on $X = [0,1]$, this implies

$$1 = \lim_{x\to 0+, x>0}[f_A(x) + \varepsilon_A(x)]\sqrt{x} = 0$$

This contradiction shows that, when $f(x)$ is only continuous but not continuously differentiable on X, the conclusion of Theorem 3 does not hold.

The above theorem and the example show the first main difference between T-S and affine T-S fuzzy control systems as affine nonlinear control system models. For continuous but not continuously differentiable affine nonlinear control systems, affine T-S fuzzy control systems are still capable of approximating such control systems to any degree of accuracy but T-S fuzzy systems are not always capable of achieving the same. This difference concludes that sometimes we will have to use affine T-S fuzzy control systems in order to obtain the needed approximation accuracy and achieve desired stabilization and control design objectives.

So far, all our comparison has been focusing on what affine nonlinear control systems can or can not be represented by T-S or affine T-S fuzzy control systems to any degree of accuracy. Now we turn our attention to the analysis of the approximation complexity when both T-S and affine T-S fuzzy control systems can be used to represent an affine control system. The complexity here is defined as which class of fuzzy systems needs more fuzzy rules and parameters in order to achieve the same approximation accuracy. In other words, we want to understand, in the case where both class of fuzzy systems are usable, which class of systems can do better. For this purpose, we will first present the approximation bounds when applying T-S or affine T-S fuzzy control systems to model affine nonlinear control systems.

Theorem 5. Assume that a) $f(x) = f(x_1,...,x_n)$ and $g(x) = g(x_1,...,x_n)$ in (18) be k (≥ 3) time continuous differentiable function on $X = \times_{i=1}^{n} X_i = \times_{i=1}^{n}[\alpha_i, \beta_i]$ b) For each state variable x_i $(i = 1,2,...,n)$, its membership functions given in the state space $X_i = [\alpha_i, \beta_i]$ are the triangle shape functions $F^l_i(x_i) = \Delta_{i,l}(x_i, e_{i,l-1}, e_{i,l}, e_{i,l+1})$ $\quad (i = 1,2,...,n,$

$l = 1,2,\ldots,N)$, in which $e_{i,0} = e_{i,1} = \alpha_i$, $e_{i,N} = e_{i,N+1} = \beta_i$ and $e_{i,1} < e_{i,2} \ldots < e_{i,N_i}$; c) $h_i = \max_{1 \le l \le N}\left(e_{i,l+1} - e_{i,l}\right)$, $i = 1,\ldots,n$, $h = \max_{1 \le i \le n}\{h_i\}$. Then 1) its corresponding T-S fuzzy system

$$F_{T-S}(x,u) = \left[\sum_{l=1}^{N} \mu_l(x) A_l\right] x + \left[\sum_{l=1}^{N} \mu_l(x) B_l\right] u$$
$$= f_A(x)x + g_B(x)u$$

has the following approximation error bound:

$$\max_{x \in X}\left\| f(x) - f_A(x)x \right\| \le A_{T-S} h^2 \quad (28)$$

$$\max_{x \in X}\left\| g(x) - g_B(x) \right\| \le B_{T-S} h^2 \quad (29)$$

where A_{T-S} and B_{T-S} are some positive real number.
2) its corresponding affine T-S fuzzy system

$$F_{A-T-S}(x,u)$$
$$= \left[\sum_{l=1}^{N} \mu_l(x) C_l\right] + \left[\sum_{l=1}^{N} \mu_l(x) A_l\right] x + \left[\sum_{l=1}^{N} \mu_l(x) B_l\right] u$$
$$= f_C(x) + f_A(x)x + g_B(x)u$$

has the following approximation error bound:

$$\max_{x \in X}\left\| f(x) - \left[f_C(x) + f_A(x)x\right] \right\| \le A_{A-T-S} h^3 \quad (30)$$

$$\max_{x \in X}\left\| g(x) - g_B(x) \right\| \le B_{A-T-S} h^2 \quad (31)$$

where A_{A-T-S} and B_{A-T-S} are some positive real number;

Theorem 5 shows the second main difference between T-S and affine T-S fuzzy control systems as affine control system models. To approximate affine control systems given in (18), affine T-S fuzzy control systems are third-order approximators for $f(x)$ based on (30) whereas T-S fuzzy control systems are only second-order approximators based on (28). Please see [26] for the detailed concepts about the different orders of approximation. Therefore affine T-S fuzzy control systems are more accuracy and required less rules and parameters than T-S fuzzy systems when highly accuracy is required. The more accuracy and less complexity (in the sense of less rules and parameters) achieved by affine T-S fuzzy systems could be very useful when modeling a high dimensional control systems.

Unfortunately, as the components to approximate $g(x)$ in both T-S and affine T-S fuzzy systems are the same as $\sum_{l=1}^{N} \mu_l(x) B_l$ which are only the second-order approximators based on (29) and (31), the improvement in accuracy and complexity by using affine T-S fuzzy systems may not really achievable if $g(x)$ is a more complicated and highly nonlinear function than $f(x)$. In order to overcome such a weakness and achieve the overall improvement in accuracy and complexity, bilinear T-S fuzzy systems [27]-[28] are good choice. In order to discuss and analyze this idea in more details, bilinear T-S fuzzy control systems are introduced first.

For a bilinear T-S fuzzy control system, its rule base is given by

R^l: IF x_1 *is* F_1^l and … and x_n *is* F_n^l,

$$\text{THEN } \frac{dx}{dt} = c_l + A_l x(t) + B_l u(t) + [x(t) \otimes D_i] u(t)$$
$$l = 1,2,\ldots,N \quad (32)$$

where $D_l = [D_{l,1},\ldots,D_{l,n}]^\tau$ and $D_{l,j}$ $(j = 1,2,\ldots,n)$ is an $n \times m$ matrix, $x \otimes D_l = \begin{bmatrix} x^\tau D_{l,1} \\ \ldots \\ x^\tau D_{l,n} \end{bmatrix}$

Under the center-average defuzzifier, product inference and singleton fuzzifier, the mathematical formula of the resulting affine T-S control system is

$$\frac{dx}{dt} = F[x(t), u(t)] \quad (33)$$

with

$$F(x,u) = \left[\sum_{l=1}^{N} \mu_l(x) c_l\right] + \left[\sum_{l=1}^{N} \mu_l(x) A_l\right] x$$
$$+ \left\{ \left[\sum_{l=1}^{N} \mu_l(x) B_l\right] + x^\tau \left[\sum_{l=1}^{N} \mu_l(x) \begin{bmatrix} D_{i,1} \\ \ldots \\ D_{i,n} \end{bmatrix}\right] \right\} u$$
$$= f_C(x) + f_A(x)x + [g_B(x) + x^\tau g_D(x)] u \quad (34)$$

where $\mu_l(x)$ $(l = 1,2,\ldots,N)$ is given in (9).

For bilinear T-S fuzzy systems, we have the following result.

Property 6. An bilinear T-S fuzzy control system is an affine nonlinear control system. In other words, the set of all bilinear T-S fuzzy systems is a subset of all affine nonlinear control systems.

Theorem 6. Under the same conditions as Theorem 5, the approximation error bounds for the corresponding bilinear T-S fuzzy system given in (34) are as follows:

$$\max_{x \in X}\left\| f(x) - \left[f_C(x) + f_A(x)x\right] \right\| \le A_{B-T-S} h^3 \quad (35)$$

$$\max_{x \in X}\left\| g(x) - \left[g_B(x) + x^\tau g_D(x)\right] \right\| \le B_{B-T-S} h^3 \quad (36)$$

where A_{B-T-S} and B_{B-T-S} are some positive real number.

Theorem 6 shows that bilinear T-S fuzzy control systems are third-order approximators for both $f(x)$ and $g(x)$ based on (35) and (36). As a result, they are the overall third-order approximators (rather than just one side third-order approximation as affine T-S fuzzy systems) and have better approximate accuracy than both T-S and affine T-S fuzzy systems. Therefore bilinear T-S fuzzy control systems could be very useful for representing and controlling high dimensional nonlinear systems as fuzzy systems, similar to many other control system models, suffer the curse of dimensionality. However, the analysis and design of bilinear T-S fuzzy systems are much more difficult and there are only a limited of research results [27]-[28] available.

V. Further Discussion

In the last section, it is shown that one main difference between T-S and affine T-S fuzzy control systems as affine control system models is that the former can not approximate some continuous only (i.e., not continuously differentiable) affine control systems but the latter can. However, it should be emphasize that the example and analysis given in the last section does not mean that any continuous but not continuously differentiable affine control systems can not be approximated to any degree of accuracy by T-S fuzzy systems. It can be very easy to find many examples where T-S fuzzy systems can approximate such affine systems to any degree of accuracy. To illustrate and verify this point, a simple example is given below.

Example 2. Consider an affine system given below

$$\frac{dx}{dt} = \left| x \sin\left(\frac{1}{x}\right) \right| x + u$$

on $X \times U = [-1,1] \times [-1,1]$ in which $f(x) = \left| x \sin\left(\frac{1}{x}\right) \right| x$ is continuous but not continuously differentiable at $0 \in X = [-1,1]$. However, as function $\left| x \sin\left(\frac{1}{x}\right) \right|$ is continuous, by the universal approximation property of fuzzy systems [23]-[26], it can be approximated by a fuzzy system in the form of $f_A(x) = \sum_{l=1}^{N} \mu_l(x) A_l$ to any degree of accuracy. In other words, there exists $f_A(x) = \sum_{l=1}^{N} \mu_l(x) A_l$ such that

$$\left| x \sin\left(\frac{1}{x}\right) \right| x = [f_A(x) + \varepsilon_A(x)]x$$

and $\| \varepsilon_A(x) \|_\infty < \varepsilon$ with any given accuracy $\varepsilon > 0$.

Examples 1 and 2 shows that T-S fuzzy systems sometimes can approximate a continuous but not continuously differentiable affine control system to any degree of accuracy but can not in the other times. Therefore a natural question is whether there is some measurement or condition to determine when T-S fuzzy systems can be used to model such an affine control system? The answer to this question is that such a measurement or condition does exist and can be given as below: For an affine control system given in (18), if there exists a continuous matrix function

$$F(x) = \begin{bmatrix} f_{11}(x) & \ldots & f_{1n}(x) \\ \ldots & \ldots & \ldots \\ f_{n1}(x) & \ldots & f_{nn}(x) \end{bmatrix}$$

such that

$$f(x) = F(x)x \tag{37}$$

Then such an affine control system can be approximated to any degree of accuracy by T-S fuzzy system.

Now we use this measurement or condition to check the result obtained in the last section and Examples 1 and 2 above. When $f(x)$ is continuously differentiable as we assume in Theorem 3, then by defining

$$f_{ij}(x) = f_{ij}(x_1, \ldots, x_n) = \begin{cases} \dfrac{f_i(0,\ldots,0,x_j,x_{j+1}\ldots,x_n) - f_i(0,\ldots,0,0,x_{j+1}\ldots,x_n)}{x_j} & if \ x_j \neq 0 \\ \dfrac{\partial f_i(0,\ldots,0,0,x_{j+1}\ldots,x_n)}{\partial x_j} & if \ x_j = 0 \end{cases}$$

$$i, j = 1,2,\ldots,n$$

It can be proved that $f(x) = F(x)x$ and $F(x)$ is continuous. As $F(x)$ is continuous and then can be approximated to any degree of accuracy by fuzzy system $f_A(x) = \sum_{l=1}^{N} \mu_l(x) A_l$ due to universal approximation property, $f(x) = [f_A(x) + \varepsilon_A(x)]x$ and $\| \varepsilon_A(x) \|_\infty < \varepsilon$ for any given accuracy $\varepsilon > 0$. In fact, what stated here is the main idea behind the proof of Theorem 3, which is omitted from the paper due to the space limitation.

For the same analysis as the above case where $f(x)$ is continuously differentiable, the reason that the continuous but not continuously differentiable affine system given in Example 2 can be accurately modelled by T-S fuzzy systems is that condition (37) holds. However, for Example 1, it can be easily verified that condition (37) being not satisfied and this is the reason why the given affine control system in Example 1 cannot be approximated accurately by T-S fuzzy systems.

In the applications when model based fuzzy control approach is applied to model-unknown nonlinear systems, condition (37) may be impossible or very difficult to verify. In such a case, two intuitive judgements as below can be used to decide whether to apply T-S or affine T-S fuzzy systems:

1. Whether the considered real system is smooth around its equilibrium.
2. Whether the model errors around its equilibrium by using a T-S fuzzy system in the system identification based on historical data is small enough.

If one of the above checks is true, then T-S fuzzy systems can be a better choice as such fuzzy control systems are easier to analyze and design. However, if both checks turn out not to be true, then affine T-S fuzzy systems will have to use in order to ensure an accurate fuzzy control system model to be obtained and then the quality of control outcomes.

VI. CONCLUSION

In model based fuzzy control, almost all considered control system models are T-S and affine T-S fuzzy control systems. However, there is a lack of systematic and theoretic understanding what the similarity and difference between these two dominated fuzzy control systems models to provide the guidance to choose the right models to the right application problems. To fill in such a gap, this paper gives a systematic comparison between T-S and affine T-S fuzzy control systems. The main results obtained are, firstly, the similarity between T-S and affine T-S fuzzy control systems is that both can and can only approximate affine nonlinear models and have the similar representation capability for smooth (continuous differentiable) nonlinear control systems. As a result, T-S fuzzy systems are better choice as the stabilization analysis and control design simpler; Secondly, one of the main dissimilarities is that affine T-S fuzzy systems have better representation for continuous only (i.e., not differentiable) nonlinear control systems and can accurately approximate some continuous only nonlinear systems which cannot be accurately represented by T-S fuzzy systems. Another main dissimilarity is that affine T-S fuzzy systems are more accurate when representing high dimensional nonlinear systems. As a result, affine T-S fuzzy systems often could be the better choice for non-smooth or high dimensional nonlinear control systems.

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