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# Non-quadratic Stabilization Of Second Order Continuous Takagi-Sugeno Descriptor Systems via Line-Integral Lyapunov Function

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***Abstract*—In this paper, a line-integral fuzzy Lyapunov function is proposed for stability and stabilization of continuous-time Takagi-Sugeno descriptor models. The scheme enhances the relaxation due to the Lyapunov function by combining it with an application of the Finsler's lemma which allows an independent controller design up to second order systems. The proposed approach includes and outperforms former results on the same subject as shown both theoretically and via illustrative examples.**

## I. Introduction

As a result of their ability to exactly represent nonlinear models in a compact set of the state space, Takagi-Sugeno (TS) models [1] have attracted many researchers from the control community, which have taken advantage of their convex structure for stability analysis and controller design. A TS model can be obtained via the sector nonlinearity approach [2]; the representation thus obtained is a convex sum of linear models blended together with membership functions (MFs) which contain the nonlinearities of the system [3]. In order to obtain linear matrix inequality (LMI) conditions for stability analysis, controller or observer design, the direct Lyapunov method is used altogether with the convex structure of the TS models [4], [5]. Expressing results as LMI constraint problems is an essential feature of this framework, since they can be solved by efficient algorithms already available in software implementing convex optimization techniques [6].

Sufficient conditions are derived when a common quadratic Lyapunov function is used. There exist several works to deal with the inherent conservativeness of the LMI-TS approach; they can be classified into three categories. First, in order to obtain LMI conditions, MFs should be removed from nested convex sums; some strategies have been proposed in [7], [8], and [9]: they are referred to as sum relaxations. Second, different types of Lyapunov functions may produce more relaxed conditions; among others, the following have been proposed: piecewise [10], [11], fuzzy (also known as non-quadratic) [12], [13], and line-integral [14]. Third, other convex models besides the TS ones have been used: polynomial [15] and descriptor [16]. This work focuses in the latter case as well as in the use of a line-integral Lyapunov function.

The descriptor structure appeared in [17] with the main interest of describing nonlinear families of systems in a more natural way than the standard state-space one. In [16], stability and stabilization of fuzzy descriptor systems have been presented under a quadratic scheme; this work takes advantage of the descriptor structure to reduce the number of LMI constraints, thus reducing the computational burden.

The use of non-quadratic Lyapunov functions in continuous time faces the designer with the time-derivatives of the MFs, a problem that has been considered in several works [18], [19], [20], [21] for standard models and in [22], [23] for descriptor models. In order to avoid the time-derivatives of the MFs, the line-integral approach has been proposed for controller design, resulting in bilinear matrix inequalities (BMIs), which are no longer as efficiently solved as LMIs [14]. LMI constraints were found for the two-rule second-order case in [24].

This work proposes an approach for stability analysis and controller design based on the Finsler's lemma [25] and a line-integral Lyapunov function [14], [24]: it breaks the link between controller gains and the Lyapunov function; Moreover, it offers parameter-dependent LMI conditions instead of BMI constraints. The result for stabilization is guaranteed for the second-order case.

The paper is organized as follows. Section II presents the TS descriptor model, provides basic notation and useful lemmas. In section III the main result is developed: it combines a Finsler-based approach with line-integral Lyapunov functions for TS controller design; the two-rule second-order case is analyzed in detail. Section IV gives some examples to illustrate the effectiveness of the proposed approach. Finally, section V briefs the paper results.

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## II. Definitions and Notations

Consider the following continuous-time TS model in the descriptor form:

$$E_v \dot{x}(t) = A_h x(t) + B_h u(t) , \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input, the sums $A_h = \sum_{i=1}^{r} h_i(z(t)) A_i$, $B_h = \sum_{i=1}^{r} h_i(z(t)) B_i$, and $E_v = \sum_{k=1}^{r_e} v_k(z(t)) E_k$ depend on $A_i$, $B_i$, $i \in \{1,\dots,r\}$, as matrices of appropriate dimensions, and $E_k$, $k \in \{1,\dots,r_e\}$, as matrices of adequate size, with $E_v$ as a regular matrix. Two sets of MFs $0 \le h_i(z(t)) \le 1$, $i \in \{1,\dots,r\}$, and $0 \le v_k(z(t)) \le 1$, $k \in \{1,\dots,r_e\}$, holding the convex sum property $\sum_{i=1}^{r} h_i(z(t)) = 1$ and $\sum_{k=1}^{r_e} v_k(z(t)) = 1$ in a compact set of the state variables are defined: they depend on a premise vector $z(t) \in \mathbb{R}^p$ which depends on the state $x(t)$. Note that model (1) might arise from a wide variety of nonlinear models in the descriptor form via the sector nonlinearity approach [2].

An asterisk $(*)$ for inline expressions denotes the transpose of the terms on its left-hand side; for matrix expressions denotes the transpose of its symmetric block-entry. The standard notation for the Lie-derivative of $V(x)$ on the vector field $g(x)$ is adopted, i.e., $L_g V(x)$. When convenient, arguments will be omitted.

LMI-based controller design in the TS context requires MFs to be removed from nested convex sums via different schemes which differ in generality and complexity. A good compromise between effectiveness and computational burden is given by the following result:

**Relaxation lemma** [7]**:** Let $\Upsilon_{ij}^{k}$ be matrices of appropriate dimensions. Then $\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r_e} h_i h_j v_k \Upsilon_{ij}^{k} < 0$ holds if

$$\begin{gathered} \Upsilon_{ii}^{k} < 0, \ \forall i, \ \forall k \\ \frac{2}{r-1}\Upsilon_{ii}^{k} + \Upsilon_{ij}^{k} + \Upsilon_{ji}^{k} < 0, \ \forall i, j, \ \forall k, \ i \ne j \end{gathered} \qquad (2)$$

for $i \in \{1,\dots,r\}$ and $k \in \{1,\dots,r_e\}$.

The next lemma has been useful in many recent results on LMI-based controller and observer design for TS models [25], [26]: it increases the design flexibility by allowing the Lyapunov function and the controller/observer gains to be designed independently as well as with progressively more relaxed results via controllers/observers with nested convex sums:

**Finsler's Lemma** [27]: Let $x \in \mathbb{R}^n$, $Q = Q^T \in \mathbb{R}^{n\times n}$, and $R \in \mathbb{R}^{m\times n}$ such that $rank(R) < n$; the following expressions are equivalent:

a) $x^T Q x < 0$, $\forall x \in \{x \in \mathbb{R}^n : x \ne 0, Rx = 0\}$.

b) $\exists M \in \mathbb{R}^{n\times m} : Q + MR + R^T M^T < 0$.

The following line-integral Lyapunov function candidate will be considered [14]:

$$V(x) = 2\int_{\Gamma(0,x)} f(\psi)\, d\psi , \qquad (3)$$

where $\Gamma(0,x)$ is any path from the origin to the current state $x$, $\psi \in \mathbb{R}^n$ is a dummy vector for the integral, $d\psi \in \mathbb{R}^n$ is an infinitesimal displacement vector.

While $V(x)$ is a continuously differentiable function, its dependency on path $\Gamma(0,x)$ obliges to consider the following lemma in order to impose path-independency on $V(x)$, thus satisfying positive-definiteness as well as radially unboundedness [14]:

**Lemma 1:** Let $f(x) = [f_1(x),\dots,f_n(x)]^T$. A necessary and sufficient condition for $V(x)$ to be a path-independent function is

$$\frac{\partial f_i(x)}{\partial x_j} = \frac{\partial f_j(x)}{\partial x_i} \qquad (4)$$

for $i, j = 1,\dots,n$.

*Proof:* It is the condition for a line-integral to be path-independent [28].

In [14], a solution satisfying (4) and leading to LMIs (for stability analysis) or BMIs (for stabilization) in ordinary state space, i.e., $\dot{x}(t) = A_z x(t) + B_z u(t)$, has been proposed as follows:

$$f(x) = \left( \sum_{i=1}^{r} h_i(x)(\bar{P} + D_i) \right) x , \qquad (5)$$

with

$$\bar{P} = \begin{bmatrix} 0 & p_{12} & \cdots & p_{1n} \\ p_{12} & 0 & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{1n} & p_{2n} & \cdots & 0 \end{bmatrix}, \quad D_i = \begin{bmatrix} d_{11}^{\alpha_{i1}} & 0 & \cdots & 0 \\ 0 & d_{22}^{\alpha_{i2}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^{\alpha_{in}} \end{bmatrix},$$

$h_i(x) = \prod_{j=1}^{n} w_j^{\alpha_{ij}}(x_j)$ where $\mu_j^{\alpha_{ij}}(x_j)$ are the weight-functions, and $\bar{P} + D_i = (\bar{P} + D_i)^T > 0$.

## III. Main Results

Let $\bar{x}(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}$ so (1) can be written as:

$$\bar{E}\dot{\bar{x}}(t) = \bar{A}_{hv}\bar{x}(t) + \bar{B}_h u(t) \qquad (6)$$

with $\bar{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, $\bar{A}_{hv} = \begin{bmatrix} 0 & I \\ A_h & -E_v \end{bmatrix}$, and $\bar{B}_h = \begin{bmatrix} 0 \\ B_h \end{bmatrix}$.

Consider the following control law as in [25]:

$$u(t) = \begin{bmatrix} F_{hv} & 0 \end{bmatrix} Y_{hhv}^{-1}\bar{x}(t) = \bar{F}_{hv} Y_{hhv}^{-1}\bar{x}(t) \qquad (7)$$

where $Y_{hhv} = \begin{bmatrix} Y_{hv}^{1} & Y_{hv}^{2} \\ Y_{hh}^{3} & Y_{hh}^{4} \end{bmatrix}$, with $F_{hv} = \sum_{i=1}^{r}\sum_{k=1}^{r_e} h_i v_k F_{ik}$ and

$Y_{hhv} = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r_e} h_i h_j v_k Y_{ijk}$ formed by matrices $F_{ik}$, $Y_{ik}^1$, $Y_{ik}^2$, $Y_{ij}^3$, and $Y_{ij}^4$, $i, j \in \{1,\ldots,r\}$, $k \in \{1,\ldots,r_e\}$.

Substituting (7) in (6) and properly grouping terms, the closed-loop TS model constraint equality is obtained:

$$\begin{bmatrix} \bar{A}_{hv} + \bar{B}_h \bar{F}_{hv} Y_{hhv}^{-1} & -I \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{E}\dot{\bar{x}} \end{bmatrix} = 0 . \qquad (8)$$

*A. Stability*

Considering $u = 0$, (6) yields

$$\bar{E}\dot{\bar{x}}(t) = \bar{A}_{hv}\bar{x}(t) \qquad (9)$$

**Lemma 1:** The function $f(\bar{x})$ satisfies path-independent conditions in (4) if it has the next structure:

$$f(\bar{x}) = \bar{E}P_h\bar{x} \; ; \; \bar{E}P_h = P_h^T \bar{E} , \qquad (10)$$

where $P_h = \begin{bmatrix} P_h^1 & 0 \\ P_h^3 & P_h^4 \end{bmatrix}$, $P_h^1 = \sum_{i=1}^{r} h_i(x) P_i^1$, $P_i^1 = (P_i^1)^T > 0$, $P_i^1 = \bar{P} + D_i$ as in (5), $P_h^3 = \sum_{i=1}^{r} h_i(x) P_i^3$, $P_h^4 = \sum_{i=1}^{r} h_i(x) P_i^4$.

*Proof:* From (10), the following expression

$$f(\bar{x}) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} P_h^1 & 0 \\ P_h^3 & P_h^4 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} P_h^1 x \\ 0 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(\bar{x}) \end{bmatrix}, \qquad (11)$$

leads to $f_1(x) = P_h^1 x$ (this has the same structure than (5)) and $f_2(\bar{x}) = 0$ (this plays no role). In order to satisfy path-independency for function (10), it is necessary that $f_1(x)$ holds conditions in (4). Due to $f_1(x) \Leftrightarrow (5)$ it satisfies path-independency condition (4).

**Theorem 1:** The TS descriptor model (1) with $u = 0$ is asymptotically stable if there exist matrices $P_j^1 = (P_j^1)^T > 0$, $P_j^3$, and $P_j^4$, $j \in \{1,\ldots,r\}$, such that:

$$\Upsilon_{ij}^k = \begin{bmatrix} A_i^T P_j^3 + (*) & (*) \\ P_j^1 - E_k^T P_j^3 + (P_j^4)^T A_i & -(E_k^T P_j^4 + (*)) \end{bmatrix}. \qquad (12)$$

*Proof:* The time-derivative of the Lyapunov function candidate in (3) is:

$$\dot{V}(\bar{x}) = L_g V(\bar{x}) = f^T(\bar{x}) g(\bar{x}) + g^T(\bar{x}) f(\bar{x}) , \qquad (13)$$

where $g(\bar{x}) = \dot{\bar{x}}$. Using (10) and (9), (13) yields

$$\begin{aligned} \dot{V}(\bar{x}) &= \bar{x}^T P_h^T \bar{A}_{hv} \bar{x} + \bar{x}^T \bar{A}_{hv}^T P_h \bar{x} \\ &= \bar{x}^T \left(P_h^T \bar{A}_{hv} + \bar{A}_{hv}^T P_h\right) \bar{x} < 0. \end{aligned} \qquad (14)$$

Then

$$P_h^T \bar{A}_{hv} + \bar{A}_{hv}^T P_h < 0 . \qquad (15)$$

Recalling the definitions of $P_h$ and $\bar{A}_{hv}$, (15) is rewritten as:

$$\begin{bmatrix} P_h^1 & 0 \\ P_h^3 & P_h^4 \end{bmatrix}^T \begin{bmatrix} 0 & I \\ A_h & -E_v \end{bmatrix} + \begin{bmatrix} 0 & I \\ A_h & -E_v \end{bmatrix}^T \begin{bmatrix} P_h^1 & 0 \\ P_h^3 & P_h^4 \end{bmatrix} < 0, \qquad (16)$$

which is equivalent to:

$$\begin{bmatrix} A_h^T P_h^3 + (*) & (*) \\ P_h^1 - E_v^T P_h^3 + (P_h^4)^T A_h & -(E_v^T P_h^4 + (*)) \end{bmatrix} < 0 . \qquad (17)$$

By relaxation Lemma in (2), the previous conditions are implied by (12), thus concluding the proof. □

*B. Stabilization*

Consider the following path-independent function

$$f(\bar{x}) = \bar{E}Z_{hhv}\bar{x} \; ; \; \bar{E}Z_{hhv} = Z_{hhv}^T \bar{E} , \qquad (18)$$

with block-matrix $Z_{hhv} = \begin{bmatrix} Z_h^1 & 0 \\ Z_{hhv}^3 & Z_{hhv}^4 \end{bmatrix}$, $Z_h^1 = (P_h^1)^{-1} = \sum_{i=1}^{r} h_i(x) P_i^1$, $P_i^1 = (P_i^1)^T > 0$, $P_i^1 = \bar{P} + D_i$, $Z_{hhv}^3 = -(P_{hhv}^4)^{-1} P_{hhv}^3 (P_h^1)^{-1}$, and $Z_{hhv}^4 = (P_{hhv}^4)^{-1}$ where $P_{hhv}^3 = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r_e} P_{ijk}^3$ and $P_{hhv}^4 = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r_e} P_{ijk}^4$, with $P_{hhv}^4$ as a regular matrix.

The inverse matrix of $Z_{hhv}$ is

$$Z_{hhv}^{-1} = \begin{bmatrix} P_h^1 & 0 \\ P_{hhv}^3 & P_{hhv}^4 \end{bmatrix}. \qquad (19)$$

From [24], it is known that for a 2nd-order case, the Lyapunov function is path independent if and only if

$$(P_h^1)^{-1} = \begin{bmatrix} d_{11}^{\alpha_{i1}} & q \\ q & d_{22}^{\alpha_{i2}} \end{bmatrix}, \qquad (20)$$

where $d_{11}^{\alpha_{i1}} = w_1\bar{d}_1 + (1-w_1)\underline{d}_1$ and $d_{22}^{\alpha_{i2}} = w_2\bar{d}_2 + (1-w_2)\underline{d}_2$, with $\bar{d}_1$, $\underline{d}_1$, $\bar{d}_2$, $\underline{d}_2$, and $q$ being constants. Thus, the following inverse can be directly obtained:

$$P_h^1 = \begin{bmatrix} d_{11}^{\alpha_{i1}} & q \\ q & d_{22}^{\alpha_{i2}} \end{bmatrix}^{-1} = \frac{1}{d_{11}^{\alpha_{i1}} d_{22}^{\alpha_{i2}} - q^2} \begin{bmatrix} d_{22}^{\alpha_{i2}} & -q \\ -q & d_{11}^{\alpha_{i1}} \end{bmatrix}. \qquad (21)$$

For convenience, expression (21) will be written as follows:

$$P_h^1 = \frac{X_h^1}{\left|(P_h^1)^{-1}\right|}, \qquad (22)$$

where $X_h^1 = \begin{bmatrix} d_{22}^{\alpha_{i2}} & -q \\ -q & d_{11}^{\alpha_{i1}} \end{bmatrix}$ and $\left|(P_h^1)^{-1}\right| = d_{11}^{\alpha_{i1}} d_{22}^{\alpha_{i2}} - q^2$.

**Theorem 2:** The TS descriptor model (1) under the control law (7) is asymptotically stable if for a given $\varepsilon > 0$, there exist matrices $X_j^1 = (X_j^1)^T > 0$, $X_{ijk}^3$, $X_{ijk}^4$, $K_{jk}$, $Q_{jk}^1$, $Q_{jk}^2$, $Q_{ij}^3$, $Q_{ij}^4$, $i, j \in \{1,\ldots,r\}$, $k \in \{1,\ldots,r_e\}$, such that (2) holds for:

$$\Upsilon_{ij}^k = \begin{bmatrix} G_{ij}^{11} & (*) & (*) & (*) \\ G_{ijk}^{21} & G_{ijk}^{22} & (*) & (*) \\ G_{ijk}^{31} & G_{ijk}^{32} & G_j^{33} & (*) \\ G_{ijk}^{41} & G_{ijk}^{42} & G_{ijk}^{43} & G_{ijk}^{44} \end{bmatrix}, \qquad (23)$$

with $G_{ij}^{11} = Q_{ij}^3 + (*)$, $G_{ijk}^{21} = A_i Q_{jk}^1 - E_k Q_{ij}^3 + B_i K_{jk} + (Q_{ij}^4)^T$,

$G_{ijk}^{31}=Q_{jk}^{1}-X_{j}^{1}+\varepsilon Q_{ij}^{3}$ , $G_{ijk}^{42}=Q_{ij}^{4}-X_{ijk}^{4}+\varepsilon\left(A_iQ_{jk}^{2}-E_kQ_{ij}^{4}\right)$ , $G_{j}^{33}=-\varepsilon X_{j}^{1}+(*)$ , $G_{ijk}^{32}=Q_{jk}^{2}+\varepsilon Q_{ij}^{4}$ , $G_{ijk}^{44}=-\varepsilon X_{ijk}^{4}+(*)$ , $G_{ijk}^{41}=Q_{ij}^{3}-X_{ijk}^{3}+\varepsilon\left(A_iQ_{jk}^{1}-E_kQ_{ij}^{3}+B_iK_{jk}\right)$ , $G_{ijk}^{43}=-\varepsilon X_{ijk}^{3}$ , $G_{ijk}^{22}=A_iQ_{jk}^{2}-E_kQ_{ij}^{4}+(*)$.

*Proof:* The time derivative of the Lyapunov function candidate (3) with (11) and $P_h^1>0$ can be rewritten as:

$$\begin{bmatrix}\bar{x}\\ \bar{E}\dot{\bar{x}}\end{bmatrix}^T\begin{bmatrix}0 & Z_{hhv}^T\\ Z_{hhv} & 0\end{bmatrix}\begin{bmatrix}\bar{x}\\ \bar{E}\dot{\bar{x}}\end{bmatrix}<0\,. \tag{24}$$

Using Finsler's lemma, the next inequality guarantees $\dot{V}(\bar{x})<0$ along the trajectories of the systems (1) restricted by (8):

$$\begin{bmatrix}0 & Z_{hhv}^T\\ Z_{hhv} & 0\end{bmatrix}+\begin{bmatrix}U\\ W\end{bmatrix}\begin{bmatrix}\bar{A}_{hv}+\bar{B}_h\bar{F}_{hv}Y_{hhv}^{-1} & -I\end{bmatrix}+(*)<0\,. \tag{25}$$

Multiplying the previous expression by $\begin{bmatrix}Y_{hhv}^T & 0\\ 0 & Z_{hhv}^{-1}\end{bmatrix}$ on the left-hand side and by its transpose $\begin{bmatrix}Y_{hhv} & 0\\ 0 & Z_{hhv}^{-T}\end{bmatrix}$ on the right-hand side, gives

$$\begin{bmatrix}0 & Y_{hhv}^T\\ Y_{hhv} & 0\end{bmatrix}+\begin{bmatrix}Y_{hhv}^TU\\ Z_{hhv}^{-1}W\end{bmatrix}\begin{bmatrix}\bar{A}_{hv}Y_{hhv}+\bar{B}_h\bar{F}_{hv} & -Z_{hhv}^{-T}\end{bmatrix}+(*)<0 \tag{26}$$

and selecting $U=Y_{hhv}^{-T}$, $W=\varepsilon Z_{hhv}$, $\varepsilon>0$, and recalling (19), renders:

$$\begin{bmatrix}\bar{A}_{hv}Y_{hhv}+\bar{B}_h\bar{F}_{hv}+(*) & (*)\\ Y_{hhv}+\varepsilon\left(\bar{A}_{hv}Y_{hhv}+\bar{B}_h\bar{F}_{hv}\right)-Z_{hhv}^{-1} & -\varepsilon\left(Z_{hhv}^{-1}+Z_{hhv}^{-T}\right)\end{bmatrix}<0\,. \tag{27}$$

Recalling (21) with $P_{hhv}^3=\dfrac{X_{hhv}^3}{\left|\left(P_h^1\right)^{-1}\right|}$ , $P_{hhv}^4=\dfrac{X_{hhv}^4}{\left|\left(P_h^1\right)^{-1}\right|}$ , $Y_{hv}^1=\dfrac{Q_{hv}^1}{\left|\left(P_h^1\right)^{-1}\right|}$ , $Y_{hv}^2=\dfrac{Q_{hv}^2}{\left|\left(P_h^1\right)^{-1}\right|}$ , $Y_{hh}^3=\dfrac{Q_{hh}^3}{\left|\left(P_h^1\right)^{-1}\right|}$ , $Y_{hh}^4=\dfrac{Q_{hh}^4}{\left|\left(P_h^1\right)^{-1}\right|}$ , $F_{hv}=\dfrac{K_{hv}}{\left|\left(P_h^1\right)^{-1}\right|}$ , and after some operations, (27) can be rewritten as:

$$\frac{1}{\left|\left(P_h^1\right)^{-1}\right|}\begin{bmatrix}G_{hh}^{11} & (*) & (*) & (*)\\ G_{hhv}^{21} & G_{hhv}^{22} & (*) & (*)\\ G_{hhv}^{31} & G_{hhv}^{32} & G_{h}^{33} & (*)\\ G_{hhv}^{41} & G_{hhv}^{42} & G_{hhv}^{43} & G_{hhv}^{44}\end{bmatrix}<0\,, \tag{28}$$

with $G_{hh}^{11}=Q_{hh}^3+(*)$ , $G_h^{33}=-\varepsilon X_h^1+(*)$ , $G_{hhv}^{43}=-\varepsilon X_{hhv}^3$ , $G_{hhv}^{42}=Q_{hh}^4-X_{hhv}^4+\varepsilon\left(A_hQ_{hv}^2-E_vQ_{hh}^4\right)$ , $G_{hhv}^{32}=Q_{hv}^2+\varepsilon Q_{hh}^4$ , $G_{hhv}^{21}=A_hQ_{hv}^1-E_vQ_{hh}^3+B_hK_{hv}+\left(Q_{hh}^4\right)^T$ , $G_{hhv}^{44}=-\varepsilon X_{hhv}^4+(*)$ , $G_{hhv}^{22}=A_hQ_{hv}^2-E_vQ_{hh}^4+(*)$ , $G_{hhv}^{31}=Q_{hv}^1-X_h^1+\varepsilon Q_{hh}^3$ , $G_{hhv}^{41}=Q_{hh}^3-X_{hhv}^3+\varepsilon\left(A_hQ_{hv}^1-E_vQ_{hh}^3+B_hK_{hv}\right)$.

Due to the fact that $\dfrac{1}{\left|\left(P_h^1\right)^{-1}\right|}>0$ , (28) can be written as:

$$\begin{bmatrix}G_{hh}^{11} & (*) & (*) & (*)\\ G_{hhv}^{21} & G_{hhv}^{22} & (*) & (*)\\ G_{hhv}^{31} & G_{hhv}^{32} & G_{h}^{33} & (*)\\ G_{hhv}^{41} & G_{hhv}^{42} & G_{hhv}^{43} & G_{hhv}^{44}\end{bmatrix}<0\,. \tag{29}$$

Applying (2) to the previous expression gives the desired result, thus concluding the proof. □

**Remark 1:** If $P_h^1=P^1$ , then LMI conditions in (27) are the same in Theorem 1 of [25], i.e., under this assumption [25] is a particular case of conditions in (27) until second-order systems.

**Remark 2:** Conditions in (23) are parameter-dependent LMIs; they are LMIs up to the choice of $\varepsilon$ . Nevertheless, it has been proved in [29] and [30] that a logarithmically spaced family of values, for instance $\varepsilon\in\left\{10^{-6},10^{-5},\ldots,10^{6}\right\}$ , is adequate to avoid an exhaustive search of feasible solutions.

## IV. Examples

The proposed results are illustrated via the following two numerical examples.

### A. Example 1

Consider the following TS descriptor model:

$$\sum_{k=1}^{r_e}v_k\left(z(t)\right)E_k\dot{x}(t)=\sum_{i=1}^{r}h_i\left(z(t)\right)A_ix(t)\,, \tag{30}$$

with model matrices $A_1=\begin{bmatrix}-4.3 & 4.8\\ -1.7 & a\end{bmatrix}$, $A_2=\begin{bmatrix}b & -4.6\\ -1.9 & -3.9\end{bmatrix}$, $E_1=\begin{bmatrix}0.8 & -0.5\\ 0.21 & 1.3\end{bmatrix}$ , $E_2=\begin{bmatrix}0.8 & 0.7\\ 0.5 & 0.68\end{bmatrix}$ ; number of rules $r=r_e=2$ ; MFs $h_1=\dfrac{x_1^2}{4}$, $h_2=1-h_1$ , $v_1=\dfrac{x_2^2}{4}$ , $v_2=1-v_1$ ; and parameters $a\in[-100,5]$ and $b\in[-120,0]$.

Figure 1 shows the fact that solutions of [16] are all included in those of (12).

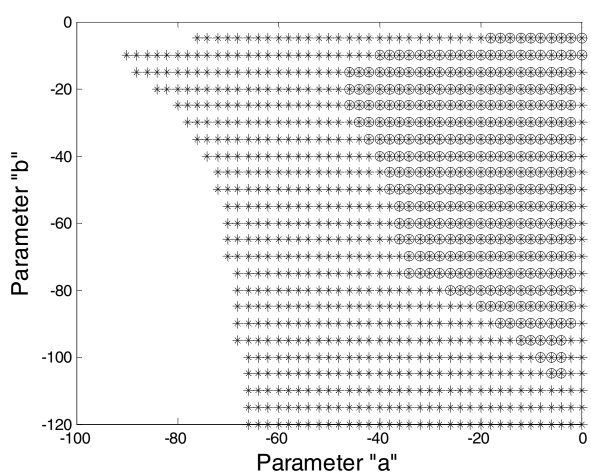


Figure1. Stability: "*" from (12) and " o " from [16].

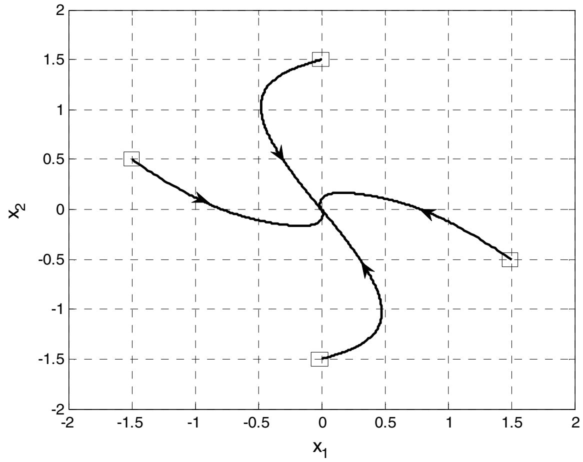


Figure 2. State-trajectories for $a=-5$ and $b=-25$.

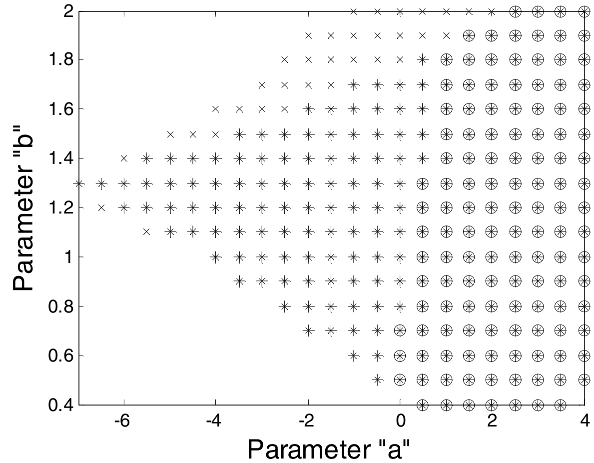


Figure 3. Stabilization: "x" from (23), "+" from [25] and " o " from Th. 2 in [23] with $\phi_{1,2}=-1$ and $\theta_{1,2}=-1$.

Selecting $a=-5$ and $b=-25$, a Lyapunov function of the form (3) can be found via Theorem 1. The Lyapunov matrices are

$$P_1=\begin{bmatrix}9.0426 & 5.8869\\ 5.8869 & 9.0211\end{bmatrix},\ P_2=\begin{bmatrix}4.2679 & 5.8869\\ 5.8869 & 9.0211\end{bmatrix}.$$

Figure 2 illustrates the state-trajectory from four different initial conditions: as expected, they all converge towards the origin.

### B. Example 2

For the sake of comparison, consider the following TS fuzzy model (Example 1 in [25]):

$$\sum_{k=1}^{r_e} v_k\left(z(t)\right)E_k\dot{x}(t)=\sum_{i=1}^{r} h_i\left(z(t)\right)\left(A_i x(t)+B_i u(t)\right), \qquad (31)$$

where $A_1=\begin{bmatrix}-4.3 & 4.8\\ -1.7 & 1\end{bmatrix}$, $A_2=\begin{bmatrix}a & -4.6\\ 3.9 & -1.9\end{bmatrix}$, $B_1=\begin{bmatrix}5.6\\ 0.9\end{bmatrix}$, $B_2=\begin{bmatrix}8.1\\ b\end{bmatrix}$, $E_1=\begin{bmatrix}0.8+a & 0\\ 0.21+b & 0.03\end{bmatrix}$, $E_2=\begin{bmatrix}0.8 & 0.7\\ 0.5 & 0.68\end{bmatrix}$, $r=r_e=2$, $h_1=\frac{x_1^2}{4}$, $h_2=1-h_1$, $v_1=\frac{x_2^2}{4}$, $v_2=1-v_1$, with $a\in[-7,4]$ and $b\in[0.4,2]$.

Figure 3 shows that all the solutions from [25] and Theorem 2 in [23] are included in those of (23).

Conditions in Theorem 2 are able to found a controller for cases where the previous approaches in [25] and [23] do not: for instance, when $a=-2$ and $b=1.8$ with $\varepsilon=1$, Theorem 2 finds a stabilizing controller of the form (7) with the following gains and Lyapunov matrices:

$$P_1=\begin{bmatrix}5.5431 & -1.2761\\ -1.2761 & 0.3003\end{bmatrix},\ P_2=\begin{bmatrix}22.7103 & -1.2761\\ -1.2761 & 0.3003\end{bmatrix},$$

$$F_{11}=\begin{bmatrix}-21.0092\\ -117.3489\end{bmatrix}^T,\ F_{12}=\begin{bmatrix}-22.3257\\ -114.7307\end{bmatrix}^T,\ F_{21}=\begin{bmatrix}0.1825\\ 0.9114\end{bmatrix}^T,$$

and $F_{22}=\begin{bmatrix}0.2059\\ 0.8113\end{bmatrix}^T$.

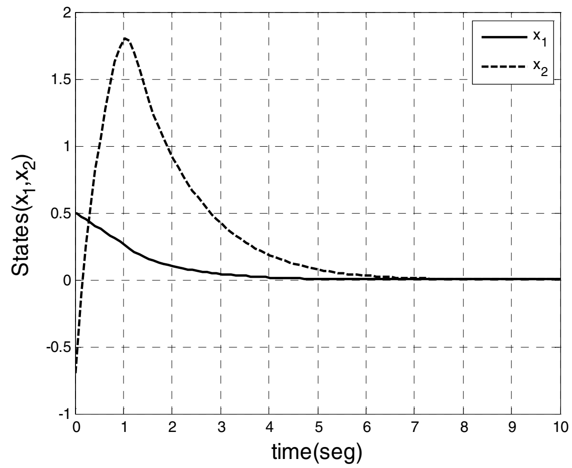


Figure 4. Time evolution of the state.

The simulation in Figure 4 has been performed from the initial condition $x(0)=[0.5 \quad -0.7]^T$.

## V. Conclusions

New stability and stabilization schemes via line-integral fuzzy Lyapunov functions and Finsler's lemma for continuous-time Takagi-Sugeno descriptor models have been presented. The new approach cut the link between the Lyapunov function and the controller gains; it also offered less conservative LMI conditions instead of BMIs. Some examples were given to show that the previous results on the same subject were outperformed.

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