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Case Study of Zhang Matrix Inverse for Different ZFs Leading to Different Nets

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***Abstract*—This paper primarily demonstrates the effectiveness of the Z-type methodology for solving the problem of time-variant matrix inverse (termed Zhang matrix inverse, ZMI). As a case study of ZMI with examples, the online solution of ZMI is investigated in this paper. Specifically, different Zhang functions (ZFs), which lead to different effective Z-type models (i.e., Zhang neural nets), are proposed and implemented as the error basis functions for ZMI. Meanwhile, a specific relationship between the Z-type model and others' model/method [i.e., the Getz and Marsden (G-M) dynamic system] is presented. Eventually, the MATLAB Simulink modeling and simulative verifications with examples using such different Z-type models are further researched. Both theoretical analysis and modeling results demonstrate the efficacy of the proposed Z-type models which originate from different ZFs for ZMI.**

I. Introduction

In recent years, the problem of solving linear matrix equations [1]–[3], e.g., Sylvester equation, Lyapunov equation and Stein's equation, has been encountered in various science and engineering fields. As a sub-topic of the linear matrix equations solving, matrix inversion is often treated as one of the fundamental issues, for instance, as preliminary steps for robotic kinematics and optimization [2].

For many decades, many algorithms have been proposed for fast matrix inversion [4]. In general, owing to the serial-processing feature performed on digital computers, it may not be efficient enough in large-scale online or real-time applications for most numerical algorithms. Note that, for such numerical algorithms, the minimal arithmetic operations are proportional to the cube of the matrix dimension. In view of this situation, a variety of parallel-processing computational methods (e.g., random neural networks [5]–[7]) have been further developed and implemented on specific architectures.

Facing directly the online solution of various time-variant problems, a special class of recurrent neural net, termed Zhang neural net (i.e., the so-called Z-type model), has been formally proposed by Zhang *et al* [2] for a decade. Specifically, for online solution of time-variant matrix inverse (termed Zhang matrix inverse, ZMI), according to Zhang *et al*'s neural-dynamics design method, such a Z-type model is designed and developed based on a matrix-valued Zhang function (ZF) as the error basis function. Moreover, by employing the time-derivative information of the time-variant coefficient matrix involved in the ZMI problem, the resultant Z-type model can methodologically avoid the lagging errors. Such a Z-type model can thus guarantee superior convergence performance to make every element of the ZF globally exponentially converge to zero.

In this paper, following from the previous work [2][8], we propose, develop and investigate different Z-type models for ZMI by defining different ZFs as the error basis functions. Note that a dynamic system has been proposed by Getz and Marsden [9], which is described in an explicit dynamics; and that such a dynamic system (termed the G-M dynamic system) can converge exponentially to the theoretical ZMI, for sufficiently large design parameter under the condition of hardware permitting and for initial conditions adequately close to the initial theoretical inverse. Then, the direct link between the Z-type model and the G-M dynamic system is discovered. Apart from the theoretical analysis on the convergence properties of the proposed Z-type models, MATLAB Simulink modeling and simulative examples are performed accordingly. Modeling and simulative results further testify the efficacy of the Z-type models derived from different ZFs for online solution of ZMI.

II. Problem Formulation and Solvers

In this section, we introduce different ZFs, propose the resultant Z-type models, and show the relationship between the Z-type model and the G-M dynamic system for ZMI solving. Meanwhile, the relevant theoretical analysis is given.

As a case study of ZMI, the problem investigated in this paper can be described in the following standard form:

$$A(t)X(t) - I = 0, \tag{1}$$

where, with the smoothly time-variant coefficient matrix $A(t) \in R^{n\times n}$ and the identity matrix $I \in R^{n\times n}$, $X(t) \in R^{n\times n}$ is the time-variant unknown matrix to be obtained. The target of this paper is to find $X(t)$ such that (1) holds true for any time instant $t \geqslant 0$, i.e., to invert matrix $A(t)$ in real time $t \geqslant 0$. Note that $A(t)$ together with its time derivative $\dot{A}(t) \in R^{n\times n}$ is assumed to be known or measurable. For convenience of further discussion, $A(t)$ is assumed to be of full rank at any time instant $t \geqslant 0$ in this paper.

A. Z-Type Models Derived from Different ZFs

The ZF is designed as an error basis function for deriving a Z-type model. Evidently, by defining different ZFs and employing the Z-type model design method, different Z-type models can be obtained to solve for ZMI. For presentation convenience, we denote ZF by $E(X(t), t)$ and denote $\dot{E}(X(t), t)$ as the time derivative of $E(X(t), t)$ in this paper.

Following Zhang *et al*'s neural-dynamics design method [2][8], firstly, we define a ZF as the error basis function to monitor and control the process of ZMI solving. Secondly, we choose its time derivative $\dot{E}(X(t), t)$ by applying the following Z-type design formula (i.e., Zhang formula) to make $E(X(t), t)$ converge to zero:

$$\dot{E}(X(t), t) = \frac{\mathrm{d}E(X(t), t)}{\mathrm{d}t} = -\Gamma\mathcal{F}(E(X(t), t)), \quad (2)$$

where $\Gamma \in R^{n\times n}$ is a positive-definite matrix used to scale the convergence rate of the solution, and $\mathcal{F}(\cdot) : R^{n\times n} \to R^{n\times n}$ denotes a matrix-valued activation function array of the neural net. Evidently, the processing array $\mathcal{F}(\cdot)$ here is made of $n\times n$ activation functions denoted by $f(\cdot)$. Generally speaking, any odd monotonically-increasing activation function can be applied to the Z-type models. For simplicity, $\Gamma = \gamma I$ with the scalar design parameter $\gamma > 0$ and a linear activation function array $\mathcal{F}(E) = E$ is chosen in this paper. Consequently, the above Z-type design formula (2) becomes

$$\frac{\mathrm{d}E(X(t), t)}{\mathrm{d}t} = -\gamma E(X(t), t), \quad (3)$$

where design parameter γ, which corresponds to the reciprocal of a capacitance parameter, should be set as large as possible under the condition of hardware permitting [10]. It is worth pointing out that different values of γ can affect the convergence performance of the resultant Z-type models.

Specifically, for online solution of ZMI [depicted in (1)] in this paper, we can define different ZFs as below:

$$E(t) = A^{-1}(t) - X(t), \quad (4)$$
$$E(t) = A(t) - X^{-1}(t), \quad (5)$$
$$E(t) = A(t)X(t) - I, \quad (6)$$
$$E(t) = X(t)A(t) - I, \quad (7)$$
$$E(t) = \big(A(t)X(t)\big)^{-1} - I, \quad (8)$$
$$E(t) = \big(X(t)A(t)\big)^{-1} - I. \quad (9)$$

Before constructing different Z-type models from different ZFs, we present the following lemma for further discussion.

Lemma 1: We have the following facts:

$$\frac{\mathrm{d}(X^{-1}(t))}{\mathrm{d}t} = -X^{-1}(t)\dot{X}(t)X^{-1}(t), \quad (10)$$

$$\frac{\mathrm{d}(A^{-1}(t))}{\mathrm{d}t} = -A^{-1}(t)\dot{A}(t)A^{-1}(t), \quad (11)$$

$$\frac{\mathrm{d}\big(A(t)X(t)\big)^{-1}}{\mathrm{d}t} = -\big(A(t)X(t)\big)^{-1}\frac{\mathrm{d}\big(A(t)X(t)\big)}{\mathrm{d}t}\big(A(t)X(t)\big)^{-1}. \quad (12)$$

Proof: It can be obtained readily by following the proof of Theorem 1 in [11]. □

According to the Z-type model design formula (3), with six different ZFs [i.e., (4) through (9)] used, six different Z-type models are thus derived and presented as follows.

1) Considering Z-type design formula (3), ZF (4) and equation (11), we then have

$$A(t)\dot{X}(t)A(t) = -\gamma\big(A(t)X(t) - I\big)A(t) - \dot{A}(t), \quad (13)$$

which can also be written in the following explicit form:

$$\dot{X}(t) = \dot{X}(t) + \big(A(t)\dot{X}(t) - \gamma\big(A(t)X(t) - I\big)\big)A(t) + \dot{A}(t).$$

Therefore, based on ZF (4), we obtain Z-type model (13) for the ZMI solving.

2) Based on Z-type design formula (3), ZF (5) and equation (10), we can have

$$\dot{X}(t) = -X(t)\dot{A}(t)X(t) - \gamma X(t)\big(A(t)X(t) - I\big). \quad (14)$$

Therefore, we obtain the Z-type model based on ZF (5), which is exactly the G-M dynamic system for ZMI [9]. In other words, a direct link between the Z-type model and others' model/method (i.e., the G-M dynamic system) for ZMI is found. Specifically, the G-M dynamic system can be derived directly from the Z-type design method with ZF (5) and a linear activation function array exploited. Thus, the G-M dynamic system can be regarded as a special case of Z-type models. Evidently, this paper shows an explanation to the G-M dynamic system for ZMI, which is different from the original derivation proposed by Getz and Marsden [9].

3) With Z-type design formula (3) and ZF (6) exploited, the following Z-type model is established:

$$A(t)\dot{X}(t) = -\dot{A}(t)X(t) - \gamma\big(A(t)X(t) - I\big), \quad (15)$$

and similarly we can have the following explicit form:

$$\dot{X}(t) = \big(I - A(t)\big)\dot{X}(t) - \dot{A}(t)X(t) - \gamma\big(A(t)X(t) - I\big).$$

4) Based on Z-type design formula (3) and ZF (7), the Z-type model can be expressed as below:

$$\dot{X}(t)A(t) = -X(t)\dot{A}(t) - \gamma\big(X(t)A(t) - I\big), \quad (16)$$

of which the explicit form can be shown below:

$$\dot{X}(t) = \dot{X}(t)\big(I - A(t)\big) - X(t)\dot{A}(t) - \gamma\big(X(t)A(t) - I\big).$$

5) Considering Z-type design formula (3), ZF (8) and equation (12), we have

$$A(t)\dot{X}(t) = -\dot{A}(t)X(t) - \gamma\big(A(t)X(t) - I\big)A(t)X(t), \quad (17)$$

and then we can further have the following explicit form:

$$\begin{aligned}\dot{X}(t) =& \big(I - A(t)\big)\dot{X}(t) - \dot{A}(t)X(t)\\ &- \gamma\big(A(t)X(t) - I\big)A(t)X(t).\end{aligned}$$

6) Similarly, based on ZF (9), we have

$$\dot{X}(t)A(t) = -X(t)\dot{A}(t) - \gamma X(t)A(t)\big(X(t)A(t) - I\big), \quad (18)$$

of which the explicit form can be shown below:

$$\begin{aligned}\dot{X}(t) =& \dot{X}(t)\big(I - A(t)\big) - X(t)\dot{A}(t)\\ &- \gamma X(t)A(t)\big(X(t)A(t) - I\big).\end{aligned}$$

Thus, we obtain six different types of Z-type models [i.e., Z-type models (13) through (18)] for the ZMI solving, which are based on six different types of ZFs [i.e., ZFs (4) through (9)]. For readers' convenience and also for comparison, the six different Z-type models corresponding to six different ZFs are listed in Table I. In addition, for them, we have a theorem and two propositions with an illustrative theoretical analysis/proof shown in the ensuing subsection.

TABLE I. DIFFERENT Z-TYPE MODELS DEPICTED IN EXPLICIT DYNAMICS FOR MODELING PURPOSES CORRESPONDING TO DIFFERENT ZFS

ZF	Z-type model (in explicit form for modeling purposes)
$A^{-1}(t) - X(t)$	$\dot{X}(t) = \dot{X}(t) + \Big(A(t)\dot{X}(t) + \gamma(A(t)X(t) - I)\Big)A(t) + \dot{A}(t)$
$A(t) - X^{-1}(t)$	$\dot{X}(t) = -X(t)\dot{A}(t)X(t) - \gamma X(t)(A(t)X(t) - I)$
$A(t)X(t) - I$	$\dot{X}(t) = \Big(I - A(t)\Big)\dot{X}(t) - \dot{A}(t)X(t) - \gamma\Big(A(t)X(t) - I\Big)$
$X(t)A(t) - I$	$\dot{X}(t) = \dot{X}(t)\Big(I - A(t)\Big) - X(t)\dot{A}(t) - \gamma\Big(X(t)A(t) - I\Big)$
$(A(t)X(t))^{-1} - I$	$\dot{X}(t) = \Big(I - A(t)\Big)\dot{X}(t) - \dot{A}(t)X(t) - \gamma\Big(A(t)X(t) - I\Big)A(t)X(t)$
$(X(t)A(t))^{-1} - I$	$\dot{X}(t) = \dot{X}(t)\Big(I - A(t)\Big) - X(t)\dot{A}(t) - \gamma X(t)A(t)\Big(X(t)A(t) - I\Big)$

B. Theoretical Analysis and Results

From the theoretical results of [9] and [8], we summarize and present the following general observations on the convergence properties of the presented Z-type models.

Theorem 1: Let us consider a full-rank time-variant matrix $A(t) \in R^{n\times n}$ in the ZMI problem (1). Starting from an initial state $X(0) \in R^{n\times n}$, the state matrix $X(t)$ of Z-type model (13) derived from ZF (4) globally and exponentially converges to the theoretical ZMI $A^{-1}(t)$.

Proof: From the compact form of Z-type design formula $\dot{E}(t) = -\gamma E(t)$, a set of $n \times n$ decoupled differential equations can be written equivalently as follows.

$$\dot{e}_{ij}(t) = -\gamma e_{ij}(t), \tag{19}$$

for any $i, j \in \{1, 2, 3, \cdots, n\}$. Thus, we define a Lyapunov function candidate $v_{ij}(t) = e_{ij}^2(t)/2 \geqslant 0$ with its time derivative

$$\frac{\mathrm{d}v_{ij}(t)}{\mathrm{d}t} = e_{ij}(t)\dot{e}_{ij}(t) = -\gamma e_{ij}^2(t) \leqslant 0,$$

which guarantees the negative-definiteness of $\dot{v}_{ij}$ (i.e., $\dot{v}_{ij} < 0$ for $e_{ij} \neq 0$ while $\dot{v}_{ij} = 0$ for $e_{ij} = 0$ only). By Lyapunov theory, equilibrium point $e_{ij} = 0$ of (19) is globally asymptotically stable; i.e., $e_{ij}(t)$ globally converges to zero, for any $i, j \in \{1, 2, 3, \cdots, n\}$. In other words, the matrix-valued error-function $E(t) = [e_{ij}(t)] \in R^{n\times n}$ is globally convergent to zero. In addition, we have $E(t) = A^{-1}(t) - X(t)$; or equivalently, $X(t) = A^{-1}(t) - E(t)$. Since $E(t) \to 0$ as $t \to +\infty$, we have $X(t) \to A^{-1}(t)$ as $t \to +\infty$. That is, state matrix $X(t)$ of Z-type model (13) derived from ZF (4) globally converges to the theoretical ZMI $A^{-1}(t)$ of matrix $A(t)$. The proof on global convergence is thus completed.

Furthermore, in view of $\dot{e}_{ij} = -\gamma e_{ij}$, solving the linear first-order differential equation yields readily $e_{ij}(t) = \exp(-\gamma t)e_{ij}(0)$. In other words, the matrix-valued error-function $E(t) \in R^{n\times n}$ can be expressed explicitly as

$$\begin{aligned} E(t) &= \begin{bmatrix} e_{11}(0) & e_{12}(0) & \cdots & e_{1n}(0) \\ e_{21}(0) & e_{22}(0) & \cdots & e_{2n}(0) \\ \vdots & \vdots & \ddots & \vdots \\ e_{n1}(0) & e_{n2}(0) & \cdots & e_{nn}(0) \end{bmatrix} \exp(-\gamma t) \\ &= E(0)\exp(-\gamma t). \end{aligned} \tag{20}$$

With $\alpha = E(0)$, equation (20) can be written as

$$A^{-1}(t) - X(t) = \alpha \exp(-\gamma t),$$

which indicates that $X(t)$ exponentially converges to $A^{-1}(t)$ with convergence rate $\gamma > 0$. That is, starting from an initial state $X(0) \in R^{n\times n}$, the state matrix $X(t)$ of Z-type model (13) originated from ZF (4) exponentially converges to the theoretical ZMI $A^{-1}(t)$.

In summary, the state matrix $X(t)$ of Z-type model (13), starting from an initial state $X(0)$, globally and exponentially converges to the theoretical ZMI $A^{-1}(t)$. The proof is thus completed. □

Proposition 1: Consider a full-rank time-variant matrix $A(t) \in R^{n\times n}$ in the ZMI problem (1). Starting from an initial state $X(0) \in R^{n\times n}$ that is close enough to the initial theoretical inverse $A^{-1}(0)$, the state matrix $X(t)$ of Z-type model (14) derived from ZF (5), the state matrix $X(t)$ of Z-type model (17) derived from ZF (8) and the state matrix $X(t)$ of Z-type model (18) derived from ZF (9) exponentially converge to the theoretical ZMI $A^{-1}(t)$ of matrix $A(t)$.

Proposition 2: Consider a full-rank time-variant matrix $A(t) \in R^{n\times n}$ in the ZMI problem (1). Starting from an initial state $X(0) \in R^{n\times n}$, the state matrix $X(t)$ of Z-type model (15) derived from ZF (6) and the state matrix $X(t)$ of Z-type model (16) derived from ZF (7) globally and exponentially converge to the theoretical ZMI $A^{-1}(t)$ of matrix $A(t)$.

III. MATLAB SIMULINK MODELING

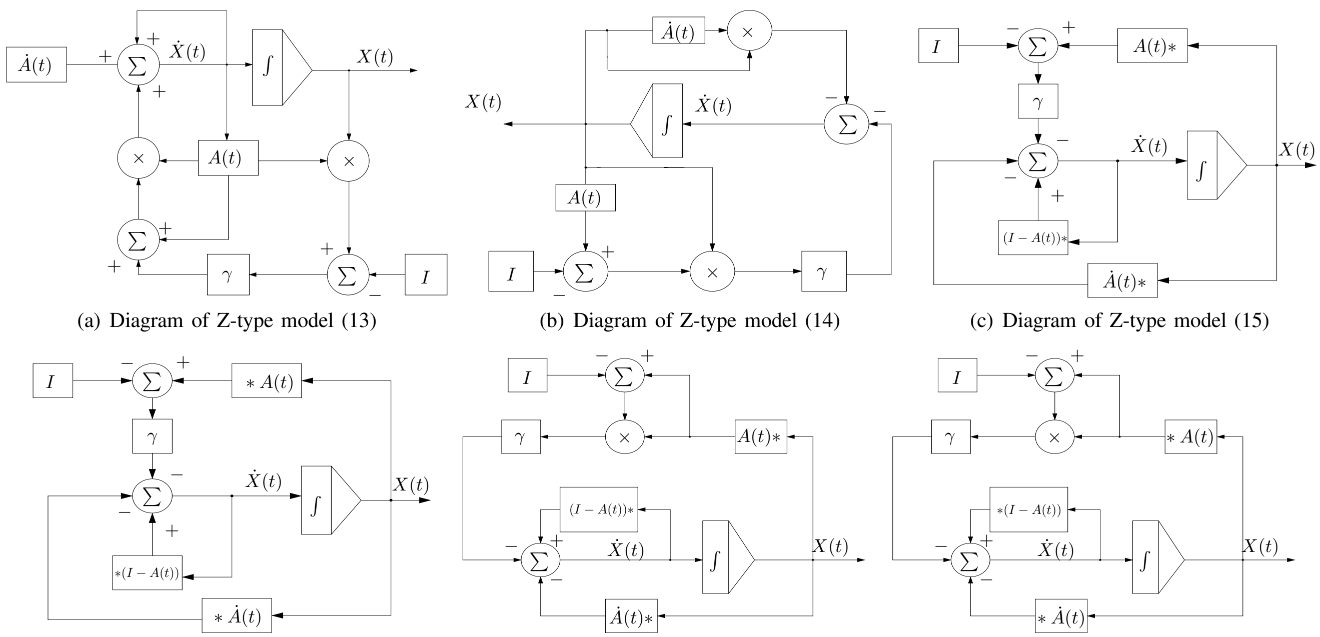
For possible circuit implementation and also for the final purpose of FPGA and ASIC realization, the MATLAB Simulink modeling of the proposed Z-type models [i.e., Z-type models (13) through (18)] is researched in this section. More specifically, the corresponding block diagrams of such Z-type models are shown in Fig. 1.

Specifically, it is worth noting that, in order to make clear the block diagrams of Z-type models (15) and (16) [shown in Fig. 1(c) and (d), respectively] as well as Z-type models (17) and (18) [shown in Fig. 1(e) and (f), respectively], we indicate the left multiplication and the right multiplication via the position of the symbol "$*$", e.g., "$A(t)*$" and "$*A(t)$" stand for "$A(t) * X(t)$" and "$X(t) * A(t)$", respectively. In addition, we take Z-type model (15) as an example for further investigation on hardware implementation. Evidently, from (15) that the ijth neuron dynamics of Z-type model (15) can be expressed as the following dynamic equation:

$$\dot{x}_{ij} = \sum_{k=1}^{n} \mu_{ik}\dot{x}_{kj} - \sum_{k=1}^{n} \dot{a}_{ik}x_{kj} - \gamma\left(\sum_{k=1}^{n}(a_{ik}x_{kj} - \delta_{ij})\right),$$

where

- x_{ij} denotes the ijth neuron state of Z-type model (15) corresponding to the ijth entry of state matrix $X(t)$, with $i, j = 1, 2, ..., n$;



(a) Diagram of Z-type model (13) (b) Diagram of Z-type model (14) (c) Diagram of Z-type model (15)

(d) Diagram of Z-type model (16) (e) Diagram of Z-type model (17) (f) Diagram of Z-type model (18)

Fig. 1. Block diagrams of Z-type models (13)-(18) for online solution of ZMI depicted in (1)

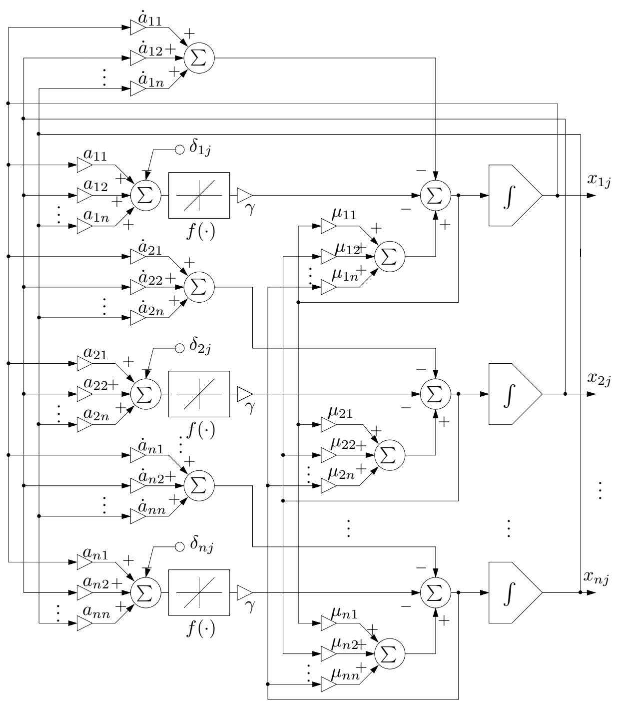


Fig. 2. Circuit schematic of the jth column of neurons of Z-type model (15) for online solution of ZMI depicted in (1)

- time-variant weights a_{ij} and $\dot{a}_{ij}$ are defined respectively as the ijth entries of matrix $A(t)$ and its time-derivative measurement $\dot{A}(t)$;
- δ_{ij} is the Kronecker delta defined here as the ijth entry of the identity matrix I, and $\mu_{ij} = \delta_{ij} - a_{ij}$.

The jth-column circuit schematic of Z-type model (15) is thus shown in Fig. 2.

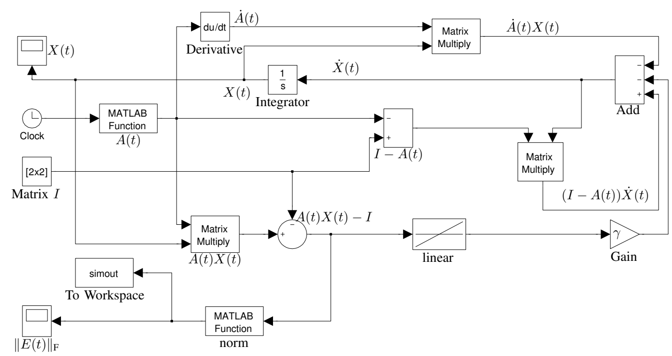


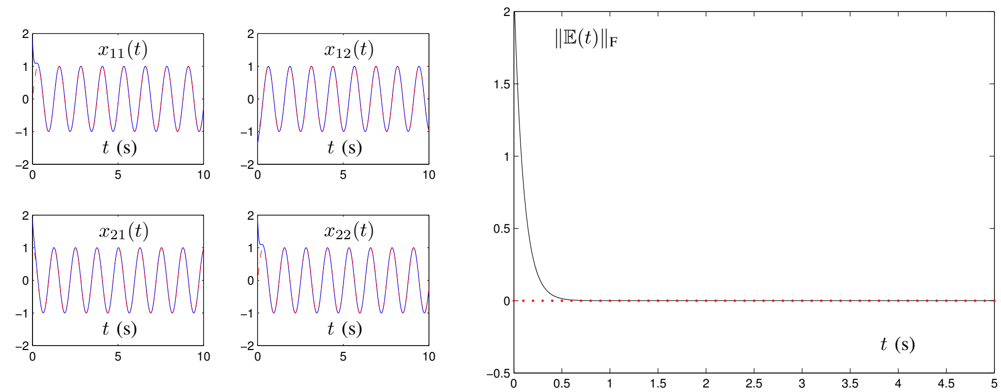
Fig. 3. Overall Simulink model of Z-type model (15)

Hence, based on the above analysis, the overall Simulink model of Z-type model (15) for ZMI solving is shown in Fig. 3, where $A(t)$ is generated by employing the *"MATLAB Function"* block with the *"Clock"* block used as its input (and the clock rate set as 5 Hz in the ensuing section). Note that other Simulink models of Z-type models [i.e.,(13), (14), and (16) through (18)] can also be obtained through the above processing, but omitted here due to space limitation.

IV. Simulations and Illustrative Examples

In the two previous sections, Z-type models (13) through (18) based on different ZFs are presented, studied and modeled for ZMI solving, together with corresponding theoretical analysis and results. In this section, based on the above-mentioned overall Simulink modeling technique, the following illustrative examples are shown to demonstrate the efficiency of the presented Z-type models for ZMI solving.

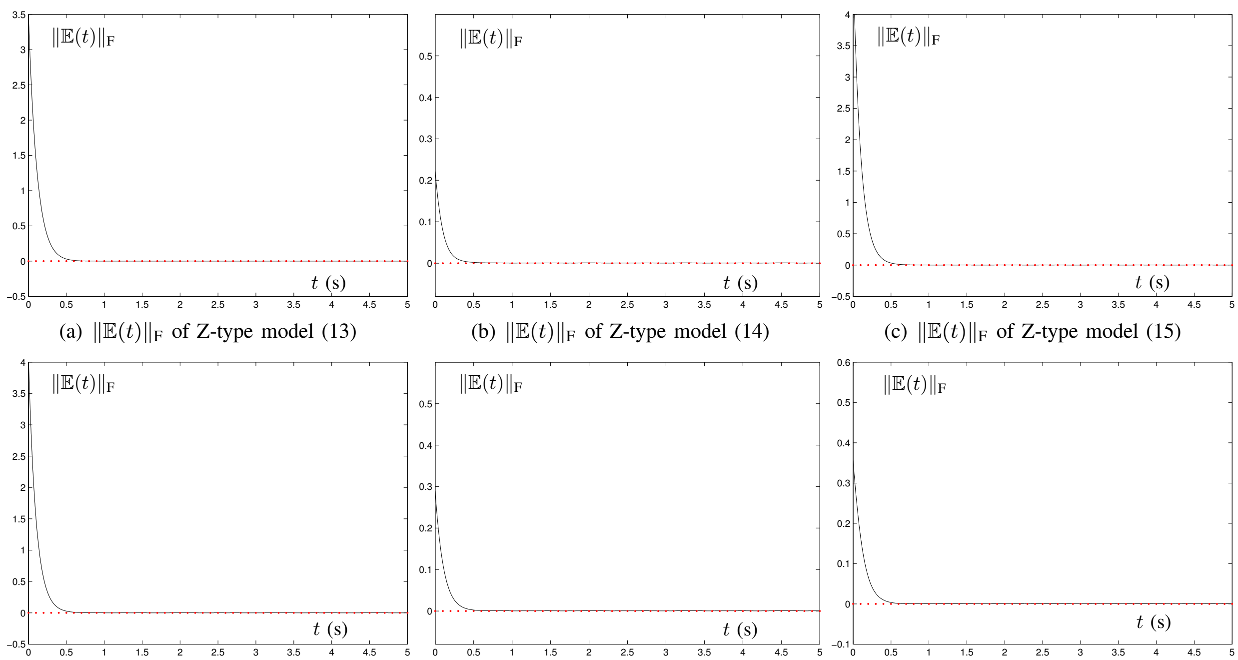
Example 1: Let us consider the problem of ZMI



(a) Entry trajectories of state matrix $X(t)$ of (13)

(b) Residual error $\|\mathbb{E}(t)\|_{\rm F}$ of (13)

Fig. 4. Convergence performance of Z-type model (13) with $\gamma = 10$ for inverting time-variant matrix (21)



(a) $\|\mathbb{E}(t)\|_{\rm F}$ of Z-type model (13)

(b) $\|\mathbb{E}(t)\|_{\rm F}$ of Z-type model (14)

(c) $\|\mathbb{E}(t)\|_{\rm F}$ of Z-type model (15)

(d) $\|\mathbb{E}(t)\|_{\rm F}$ of Z-type model (16)

(e) $\|\mathbb{E}(t)\|_{\rm F}$ of Z-type model (17)

(f) $\|\mathbb{E}(t)\|_{\rm F}$ of Z-type model (18)

Fig. 5. Residual errors $\|\mathbb{E}(t)\|_{\rm F}$ of Z-type models (13) through (18) for inverting time-variant Toeplitz matrix (22)

$A(t)X(t) = I$ with the following matrix $A(t)$:

$$A(t) = \begin{bmatrix} \sin(5t) & \cos(5t) \\ -\cos(5t) & \sin(5t) \end{bmatrix} \in R^{2\times 2}. \tag{21}$$

By algebraic operations, the theoretical ZMI is given as

$$X^*(t) = A^{-1}(t) = \begin{bmatrix} \sin(5t) & -\cos(5t) \\ \cos(5t) & \sin(5t) \end{bmatrix} \in R^{2\times 2}.$$

Thus, we can use such a theoretical solution to compare with the solutions of corresponding Z-type models and then check the correctness of the models' solutions.

The proposed Z-type models [i.e., (13) through (18)] are exploited to solve such a problem, and the corresponding simulative results based on Z-type model (13) are illustrated in Fig. 4. Specifically, as shown in Fig. 4(a), with design parameter $\gamma = 10$, state matrix $X(t)$ of Z-type model (13) denoted by solid curves converges rapidly to the theoretical solution $X^*(t)$ denoted by dash-dotted curves. In addition, to further investigate the convergence performance of Z-type model (13), we monitor the residual error $\|\mathbb{E}(t)\|_{\rm F} = \|A(t)X(t) - I\|_{\rm F}$ (with $\|\cdot\|_{\rm F}$ denoting the Frobenius norm of a matrix) during the inverting process. As seen from Fig. 4(b), by applying (13) to inverting the time-variant matrix, the residual error converges to zero fast and accurately in about 1 second. For other Z-type models [i.e., (14) through (18)], we have the same or similar observations which are omitted because of space limitation.

Example 2: To further substantiate the efficacy of Z-type models (13) through (18) for more complicated situations, we consider (1) with the following Toeplitz matrix $A(t)$:

$$A(t) = \begin{bmatrix} a_1(t) & a_2(t) & \cdots & a_n(t) \\ a_2(t) & a_1(t) & \cdots & a_{n-1}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_n(t) & a_{n-1}(t) & \cdots & a_1(t) \end{bmatrix} \in R^{n\times n}. \tag{22}$$

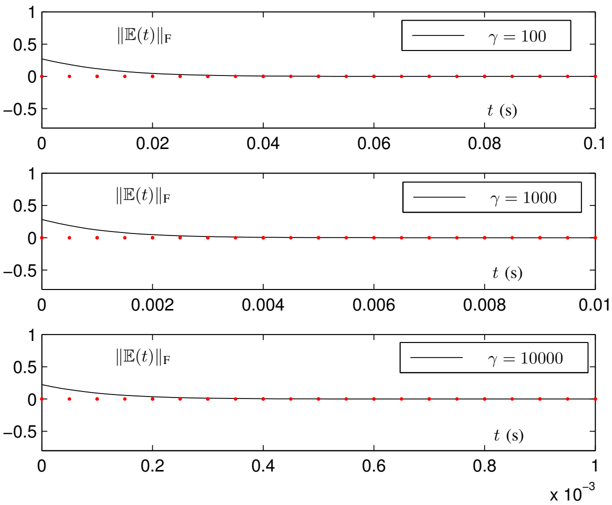


Fig. 6. Residual errors $\|\mathbb{E}(t)\|_{\mathrm{F}}$ of Z-type model (17) with different values of γ (i.e., $\gamma = 100$, 1000, and 10000) for inverting Toeplitz matrix (22)

Let $a_1(t) = n + \sin(5t)$, and $a_k(t) = \cos(5t)/(k-1)$ with $k = 2, 3, \cdots, n$. Evidently, Toeplitz matrix $A(t)$ is strictly diagonally dominant for any time instant $t \geqslant 0$ and is thus invertible. Figure 5 shows the simulative results by using Z-type models (13) through (18) with $\gamma = 10$ for the ZMI of the above Toeplitz matrix $A(t)$ under the condition of $n = 4$. As seen from the figure, residual errors $\|\mathbb{E}(t)\|_{\mathrm{F}}$ of the proposed Z-type models all converge to zero, which means that their corresponding state matrices converge to $A^{-1}(t)$. These results manifest again the efficacy of the proposed Z-type models (13) through (18) on ZMI solving.

Meanwhile, the residual errors $\|\mathbb{E}(t)\|_{\mathrm{F}}$ of Z-type model (17) with different values of γ are illustrated in Fig. 6. As shown in the figure, the convergence time of Z-type model (17) can be expedited from around 0.04 second to 0.004 second and even to 0.0004 second, when design parameter γ is increased from 100 to 1000 and to 10000, respectively. This observation tells that Z-type model (17) has an exponential-convergence property, which can be expedited effectively by increasing the value of γ. Note that, for other Z-type models [i.e., (13) through (16) and (18)], we have the same or similar observations which are omitted here due to results' similarity and space limitation.

Besides, in addition to the above results for $A(t)$ with $n = 4$, more simulations are performed for $A(t)$ with a larger n value (i.e., $n = 6$ and $n = 8$) by using the proposed Z-type models (13) through (18). The related results (omitted due to space limitation) still show the efficacy of such Z-type models on ZMI solving (with residual errors being convergent to zero). Furthermore, based on these results, we can have the conclusion that, for the Z-type models, the convergence time would not increase with the cube of the matrix order.

In summary, the above illustrative simulative results have demonstrated the efficacy of the proposed Z-type models (13) through (18) which are derived from different ZFs for online solution of ZMI depicted in (1). In addition, they have substantiated the main points of the theoretical analysis and results which are presented in Subsection II-B.

V. Conclusions

In this paper, originating from different Zhang functions (ZFs) as error basis functions, different Z-type models [i.e., (13) through (18)] have been derived, analyzed and simulated to solve for time-variant matrix inverse (i.e., Zhang matrix inverse, ZMI). In addition, the clear and direct link between the Z-type model and the G-M dynamic system has been found and presented. Moreover, theoretical analysis and results have been given to substantiate the exponential-convergence properties of the proposed Z-type models for ZMI solving. Besides, for possible hardware implementation based on electronic circuits, the MATLAB Simulink modeling of the proposed Z-type models has been shown and studied in this paper. Through computer simulations and illustrative examples, the efficacy of the proposed Z-type models (13) through (18) has been further demonstrated for ZMI solving.

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