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Stability Analysis of Nonlinear Time-delay System with Delayed Impulsive Effects

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***Abstract*— This paper concerns time-delay systems with delayed impulses. "Average impulsive intervals" is used to replace the lower and upper bounds of impulsive intervals, which weakens the limitations to the impulsive sequences. Under some modified conditions on the impulsive functions and the Lyapunov-based function, the exponential stability of systems is established. A specific impulsive delayed system with linear input time-delays is investigated by virtue of the obtained results. Finally, some numerical example is given to demonstrate the applicability of our results.**

I. Introduction

IMPULSIVE SYSTEMS have been widely applied to model practical problems in various fields such as mechanical systems [1], control systems [2], complex networks [3], [4], ect. Moreover, impulsive effects, as a non-continuous control, provides an important method to modify dynamical behaviors of impulsive systems. Therefore, during the past decade, stabilities of impulsive systems have been extensively investigated by virtue of impulsive control(See [2], [5], [6], [8]).

Since delays can not be avoided in the transmission of impulsive information, it is necessary to investigate time-delay impulsive systems and many significant results have been obtained (See[7], [9], [11]). Note that in these results, the delays of impulsive effects are not considered, which naturally exist in realistic problems. Recently, [12]investigated delay-free autonomous systems with delayed impulsive effects. Using exponential estimates for delay-free systems, they obtained system state estimates at impulse times and derived a sufficient condition of asymptotic stability. This method can not be applied to deal with time-delay systems. [13] investigated time-delay neural networks with destabilizing delayed impulses. Using the differential inequality method they obtained the global exponential stability of systems. [14] considered some nonlinear time-delay systems with more general delayed impulses. By virtue of a class of Lyapunov-based functions, they provided some sufficient conditions to ensure the exponential stability of systems. We note that both [13] and [14] imposed some restrictions on delays in relation to impulsive intervals, which lead to the results in [13] and [14] to be somewhat conservative.

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In this paper, we also concerns time-delay systems with delayed impulses. Here we address weakening the limitations to the impulsive sequences [13] and [14]. We use the so-called "average impulsive interval" (see Definition 2) recently developed by[15] to replace the lower and upper bounds of impulsive intervals. For an impulsive sequence $\{t_k\}$ with average impulsive interval T_a, there may exist some intervals $(t_{i-1}, t_i)(i \in \mathbb{N})$ satisfying $|t_k - t_{k-1}|$ is arbitrarily small or large enough(see the example in the end for details). Therefore, the results obtained in this paper seem to be less conservative than those in[13] and [14]. Under some modified conditions on impulsive function g_k and the function V of class ν_0, we first establish the exponential stability of the system. Then we apply our results to study a specific impulsive delayed system with linear input time-delays and establish it's exponential stability. Finally we give some numerical example to demonstrate the applicability of our results.

II. Preliminaries

For $\rho \geq 0$, let $\mathscr{B}(\rho) = \{x \in \mathbb{R}^n \big| |x| \leq \rho\}$. For $r > 0$, let $PRC([-r, 0], \mathbb{R}^n) = \{\phi : [-r, 0] \to \mathbb{R}^n \big| \phi \text{ is piecewise right continuous}\}$, which is endowed with norm $\|\cdot\| : \|\phi\|_r = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$. For $\phi \in PRC([t_0 - r, +\infty), \mathbb{R}^n)$ and $t \geq t_0$, define $\phi_t \in PRC([-r, 0], \mathbb{R}^n)$ by $\phi_t(s) = \phi(t + s)$. Suppose that $D \subset \mathbb{R}^n$ is an open set, and for some $\rho \geq 0$, $\mathscr{B}(\rho) \subset D$. $f : \mathbb{R}^+ \times PRC([-r, 0], D) \to \mathbb{R}^n$ satisfies $f(t, 0) = 0$. $g_k : D \times D \to \mathbb{R}^n, k \in \mathbb{N}$.

We consider the following nonlinear time-delay system with delayed impulses

$$\begin{cases} \dot{x}(t) = f(t, x_t); & t > t_0, t \neq t_k \\ x(t) = g_k(x(t^-), x(t - d_k)^-), & t = t_k, k \in \mathbb{N} \\ x(t_0 + \theta) = \phi(\theta), & \theta \in [-\tau, 0], \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state of the system. $x(t^+)$ and $x(t^-)$ are the right and left derivatives of x at t, respectively. Suppose any solution of (1) is right continuous, i.e., $x(t^+) = x(t)$. $\{d_k \geq 0, k \in \mathbb{N}\}$ are impulsive delays satisfying $\max_k\{d_k\} = d < \infty$. $t_0 \geq 0$ is the initial time. $\{t_k\}$ is an impulsive sequence on $[t_0, +\infty)$, $t_0 < t_1 < \cdots < t_k < \cdots \to +\infty$. $\phi \in PRC([-\tau, 0], \mathbb{R}^n)$ is the initial state, where $\tau = \max\{r, d\}$.

Definition1: For a given impulsive sequence $\{t_k\}$, the trivial solution of (1) is said to be exponentially stability, if there exists positive numbers ρ_0, M and λ such that if $\|\phi\|_\tau < \rho_0$,

$$|x(t, t_0, \phi)| \leq M\|\phi\|_\tau e^{-\lambda(t - t_0)}, \quad t \geq t_0.$$

Moreover, if the trivial solution of (1) is exponentially stable, and the number ρ_0 can be an arbitrarily large finite number, then the trivial solution is called to be globally exponentially stable.

In order to guarantee the existence of solutions of (1), generally, one needs make the following assumptions(see[10])

($\mathbf{A_{01}}$) f is composite PRC, that is for each $t_0 \in \mathbb{R}^+$ and $\alpha > 0$, if $x \in PRC([t_0 - r, t_0 + \alpha], D)$ and x is continuous at $t \neq t_k$, $t \in (t_0, t_0 + \alpha]$, then the composite function $f(t, x_t) \in PRC([t_0, t_0 + \alpha], \mathbb{R}^n)$.

($\mathbf{A_{02}}$) f is quasi-bounded on $\mathbb{R}^+ \times PRC([-r, 0], D)$, that is for each $t_0 \geq 0$, $\alpha > 0$ and compact subset F of D, there exists $M > 0$ such that for each $(t, x) \in [t_0, t_0 + \alpha] \times PRC([-r, 0], F)$, $|f(t, x)| \leq M$.

($\mathbf{A_{03}}$) For each fixed $t \in \mathbb{R}^+$, $f(t, x)$ is continuous with respect to $x \in PRC([-r, 0], D)$.

Definition2: **(Average Impulsive Interval[15])** The average impulsive interval of $\zeta = \{t_1, t_2, \cdots\}$ is T_a, if there exist $N_0 \in \mathbb{Z}^+$ and $T_a > 0$ such that

$$\frac{T-t}{T_a} - N_0 \leq N_\zeta(T, t) \leq \frac{T-t}{T_a} + N_0, \ \forall \ T \geq t \geq 0,$$

where $N_\zeta(T, t)$ is the impulsive times of ζ on (t, T).

Remark1: The concept of average impulsive interval was first introduced in [15] to investigate some systems with nonunited distributive impulses, whose impulsive interval length may be arbitrarily small or large. Specific examples can be found in [15]. See also the numerical example in the end.

III. Exponential Stability

In this section we investigate exponential stability of solutions of (1) with nonunited distributive impulses. We make assumptions as follows

($\mathbf{A_1}$) There exists $L_1 > 0$ such that for each $\varphi \in PRC([-r, 0], \mathscr{B}(\rho))$, $|f(t, \varphi)| \leq L_1 \|\varphi\|_r$.

($\mathbf{A_2}$) There exists L_2 and $L_3 > 0$ such that for all $k \in \mathbb{N}$ and $x, y_1, y_2 \in \mathscr{B}(\rho)$, $|g_k(x, 0)| \leq L_2 |x|$, $|g_k(x, y_1) - g_k(x, y_2)| \leq L_3 |y_1 - y_2|$.

($\mathbf{A_3}$) The average impulsive interval of $\zeta = \{t_k\}$ is $T_a > 0$, that is there exist $N_0 \in \mathbb{N}, T_a > 0$ such that for all $T \geq t > t_0$, $\frac{T-t}{T_a} - N_0 \leq N_\zeta(T, t) \leq \frac{T-t}{T_a} + N_0$. Thus, there exist at most $l = \lceil \frac{d}{T_a} \rceil + N_0$ impulses on $(t_0, t_0 + d]$, where $\lceil \frac{d}{T_a} \rceil$ means the upper integer of $\frac{d}{T_a}$. Let $\nabla = \max\{1, L_2 + L_3\}, \varrho = \nabla^l e^{L_1 d}$.

Suppose that the impulsive instants on $(t_0, t_0 + d]$ are $\{t_i\}, \ i = 1, 2, \cdots, m, \ m \leq l$.

Lemma1: If the system (1) satisfies $(A_{01}) - (A_{03})$ and $(A_1) - (A_3)$, then for any $\phi \in PRC([-\tau, 0], \mathscr{B}(\rho/\varrho))$, $|x(t, t_0, \phi)| \leq \varrho \|\phi\|_\tau, t \in [t_0 - \tau, t_0 + d]$.

Proof: Obviously, $|x(t_0 + \theta)| = |\phi(\theta)| \leq \|\phi\|_\tau < \rho(\theta \in [-\tau, 0])$. Then for $t \in [t_0 - \tau, t_0]$, $|x(t)| < \rho$. We show this is also holds on $[t_0 - \tau, t_1)$. If not, there exists $t^* \in (t_0, t_1)$ such that for $t \in [t_0 - \tau, t^*)$, $|x(t)| < \rho$ and $|x(t^*)| = \rho$.

For $t \in [t_0, t^*], \theta \in [-r, 0]$, without loss of generality, we assume $t + \theta > t_0$. Integrating the first formula of (1) from t_0 to $t + \theta$ and using (A_1) yields

$$\begin{aligned} |x(t+\theta)| &= |x(t_0) + \int_{t_0}^{t+\theta} f(s, x_s) ds| \leq |\phi(\theta)| \\ &+ \int_{t_0}^{t} |f(s, x_s)| ds \leq \|\phi\|_r + L_1 \int_{t_0}^{t} \|x_s\|_r ds. \end{aligned}$$

Then,

$$\|x_t\|_r \leq \|\phi\|_r + L_1 \int_{t_0}^{t} \|x_s\|_r ds, \quad t \in [t_0, t^*].$$

By the Gronwall's inequality,

$$\|x_t\|_r \leq \|\phi\|_r e^{L_1(t-t_0)}, \quad t \in [t_0, t^*].$$

Therefore,

$$|x(t^*)| \leq \frac{\rho}{\varrho} e^{L_1 d} < \rho,$$

a contradictionary.

From the above argument, we also obtain that

$$\begin{aligned} |x(t)| &\leq \|x_t\|_r \leq \|\phi\|_r e^{L_1(t-t_0)} \\ &\leq \|\phi\|_\tau e^{L_1(t-t_0)}, \quad t \in [t_0 - \tau, t_1). \end{aligned} \tag{2}$$

By (A_2), for $k \in \mathbb{N}$ and $x, y \in \mathscr{B}(\rho)$,

$$|g_k(x, y)| \leq L_2 |x| + L_3 |y|.$$

Hence,

$$\begin{aligned} |x(t_1)| &= |g_k(x(t_1^-), x((t_1 - d_1)^-))| \\ &\leq L_2 |x(t_1^-)| + L_3 |x((t_1 - d_1)^-)| \\ &\leq (L_2 + L_3) \|\phi\|_\tau e^{L_1(t_1 - t_0)} \\ &\leq \nabla \|\phi\|_\tau e^{L_1(t_1 - t_0)}. \end{aligned}$$

Thus, we have proved that

$$|x(t)| \leq \nabla \|\phi\|_\tau e^{L_1(t-t_0)}, \ t \in [t_0, t_1].$$

Repeatedly, we obtain

$$|x(t)| \leq \nabla^m \|\phi\|_\tau e^{L_1(t-t_0)} \leq \nabla^l \|\phi\|_\tau e^{L_1(t-t_0)}, \ t \in [t_0, t_m].$$

Since there are not impulses on $(t_m, t_0 + d]$for $t \in [t_0 - \tau, t_0 + d]$, we have

$$|x(t)| \leq \nabla^l \|\phi\|_\tau e^{L_1(t-t_0)} \leq \nabla^l \|\phi\|_\tau e^{L_1 d} = \varrho \|\phi\|_\tau.$$

We will apply some functional related to system (1) to investigate the stability of it's solutions.

Definition3: A function $V : [-\tau, \infty) \times D \to \mathbb{R}^+$ is said to belong to the class ν_0 if

(1) V is continuous in each of the sets $[t_{k-1}, t_k) \times D$and for each $k \in \mathbb{N}, \lim_{(t,y) \to (t_k^-, x)} V(t, y) = V(t_k^-, x)$ exists

(2) $V(t, x)$ is local Lipschitz with respect to $x \in D$, and for all $t \geq t_0$, $V(t, 0) \equiv 0$.

Definition4: Let $V \in \nu_0$, for each $(t, \varphi) \in [t_0, \infty) \times PRC([-\tau, 0], \mathscr{B}(\rho))$, the upper right-hand derivative of V with respect to (1) is defined by

$$\begin{aligned} &D^+ V(t, \varphi(0)) \\ &= \limsup_{h \to 0} \frac{1}{h} [V(t+h, \varphi(0) + h f(t, \varphi)) - V(t, \varphi(0))]. \end{aligned}$$

Theorem1: Suppose (1) satisfies $(A_{01})-(A_{03})$ and $(A_1)-(A_3)$, and there exists $V \in \nu_0, a, b, c, p \geq 1, \kappa > 1$ satisfying

$(\mathbf{S_1})$ For all $(t, x) \in [-\tau, \infty) \times D$, $a|x|^p \leq V(t, x) \leq b|x|^p$.

$(\mathbf{S_2})$ For $t \in [t_0, \infty)$, $t \neq t_k$ and $\varphi \in PRC([-\tau, 0], \mathscr{B}(\rho))$, $\kappa V(t, \varphi(0)) \geq V(t+\theta, \varphi(\theta))$ $(\theta \in [-\tau, 0])$ leads to $D^+V(t, \varphi(0)) \leq -cV(t, \varphi(0))$.

If there exist $\nu, k_1, k_2 > 0$ satisfying

$(\mathbf{S_3})$ For all $t = t_k$ and $x \in \mathscr{B}(\rho/(\nabla + L_3))$, $V(t, g_k(x, x)) \leq \nu V(t^-, x)$.

$(\mathbf{S_4})$ For all $t = t_k$ and $x, y \in \mathscr{B}(\rho)$ with $x + y \in \mathscr{B}(\rho)$, $V(t, x+y) \leq k_1 V(t, x) + k_2 V(t, y)$. There exists $d \geq 0$ such that

$$k_1\nu + k_2\frac{b}{a}L_3^p[dL_1 + l(1 + L_2 + L_3)]^p < 1, \tag{3}$$

then if the time-delays d_k satisfying $d_k \leq d, k \in \mathbb{N}$, (1) is exponentially stable.

Proof: It is easy to see that if (3) is true, there exists $\lambda \in (0, \min\{c, \frac{\ln \kappa}{r}\})$ such that

$$\begin{aligned} k_1\nu + k_2\frac{b}{a}L_3^p[dL_1 e^{\lambda(r+d)/p} \\ +l(1 + L_2 + L_3)e^{2\lambda d/p}]^p < 1. \end{aligned} \tag{4}$$

Take $\delta \in (0, (\sqrt[p]{\frac{b}{a}}(\nabla + L_3)\varrho)^{-1}\rho)$. Suppose that the solution of (1) subject to $(t_0, \phi) \in \mathbb{R}^+ \times PRC\ ([-\tau, 0], \mathscr{B}(\delta))$ is $x(t) = x(t, t_0, \phi)$, with maximal existence interval $[t_0 - \tau, \bar{T})$, where $\bar{T} > t_0$ is a positive number. We show that $\bar{T} = \infty$ and

$$V(t) = V(t, x(t)) \leq b\varrho^p\|\phi\|_\tau^p e^{-\lambda(t-t_0-d)}, \quad t \in [t_0 - \tau, \bar{T}). \tag{5}$$

For simplicity, we also denote the sequence on $(t_0 + d, \bar{T})$ by $\{t_i\}, i = 1, 2, \cdots$. Let $t_k^* = \min\{t_k, \bar{T}\}$. For $t \in [t_k, t_{k+1}^*), k \in \mathbb{N}$, define

$$W(s) = e^{\lambda(s-t_0-d)}V(s), \ \ s \in [t_0 - \tau, t]. \tag{6}$$

We first show that, for $s \in [t_0 - \tau, t_k)$,

$$W(s) \leq b\varrho^p\|\phi\|_\tau^p. \tag{7}$$

The method is mathematical induction. By Lemma 1,

$$|x(t)| \leq \varrho\|\phi\|_\tau, t \in [t_0 - \tau, t_0 + d]. \tag{8}$$

Combing $(6), (8)$ and (S_1), we see (7) holds.

We claim that for $t \in [t_0 - \tau, t_1)$, (7) also holds. If not, there exists $t^* \in [t_0 + d, t_1)$ and $0 < \epsilon < b[(\nabla + 2L_3)^p - 1]$ such that

$$W(t^*) = (b + \epsilon)\varrho^p\|\phi\|_\tau^p, \quad D^+W(t^*) \geq 0 \tag{9}$$

and for $s \in [t_0 - \tau, t^*)$,

$$W(s) < W(t^*). \tag{10}$$

By (9), (10) and (S_1), as $s \in (t_0, t^*)$, we have

$$|x(s)| \leq \sqrt[p]{\frac{b+\epsilon}{a}}\varrho\|\phi\|_\tau < \rho.$$

For $s \in [t^* - r, t^*)$, (10) implies

$$V(t^*) > e^{-\lambda(t^*-s)}V(s) \geq e^{-\lambda r}V(s) \geq \frac{1}{\kappa}V(s).$$

Thus from (S_2), we obtain $D^+V(t^*) \leq -cV(t^*)$, which leads to

$$D^+W(t^*) \leq -(c - \lambda)e^{\lambda(t^*-t_0-d)}V(t^*) < 0,$$

a contradiction with (9). Hence, for $[t_0 - \tau, t_1)$, (7) holds on $[t_0 - \tau, t_0 + d]$.

We now assume that for $[t_0 - \tau, t_m)$, (7) is true, Then

$$W(t_m^-) \leq b\varrho^p\|\phi\|_\tau^p. \tag{11}$$

In the following we show that $W(t_m) \leq b\varrho^p\|\phi\|_\tau^p$.

Since $N_\zeta(t_m, t_m - d_m) \leq \frac{d_m}{T_a} + N_0 \leq \frac{d}{T_a} + N_0$, there exist at most $l = \lceil \frac{d}{T_a} \rceil + N_0$ impulses on $(t_m - d_m, t_m)$, which are assumed that $t_{m_1}, t_{m_2}, \cdots, t_{m_{l_0}}, l_0 \leq l$.

By (11) and (S_1), for $s \in [t_0 - \tau, t_m)$,

$$|x(s)| \leq \sqrt[p]{\frac{b}{a}}\varrho\|\phi\|_\tau e^{-\lambda(s-t_0-d)/p} < \rho. \tag{12}$$

From (12) and $(A_1), (A_2)$,

$$\begin{aligned} | & \ x(t_m^-) - x(t_m - d_m)^-| \\ = & \ \Big|\int_{t_m-d_m}^{t_m^-}\dot{x}(s)ds + \sum_{i=1}^{l_0}\Delta x(t_{m_i})\Big| \\ \leq & \ \Big|\int_{t_m-d_m}^{t_m^-}|f(s, x_s)|ds\Big| \\ + & \ \sum_{i=1}^{l_0}|g_{m_i}(x(t_{m_i}^-), x((t_{m_i} - d_{m_i})^-)) - x(t_{m_i}^-)| \\ \leq & \ L_1\Big|\int_{t_m-d_m}^{t_m^-}\|x_s\|_r ds\Big| \\ + & \ \sum_{i=1}^{l_0}[(1 + L_2)|x(t_{m_i}^-)| + L_3|x(t_{m_i} - d_{m_i})^-|] \\ \leq & \ [L_1 de^{\lambda(r+d)/p} \\ + & \ l(1 + L_2 + L_3)e^{2\lambda d/p}]\Big(\frac{b}{a}\Big)^{1/p}\varrho\|\phi\|_\tau e^{-\lambda(t_m-t_0-d)/p}. \end{aligned} \tag{13}$$

Let $\Delta g_m = g_m(x(t_m^-), x(t_m - d_m)^-) - g_m(x(t_m^-), x(t_m^-))$. Then by $(12), (13)$, (A_2) and the choice of δ, we obtain

$$\begin{aligned} |\Delta g_m|^p \leq L_3^p[L_1 de^{\lambda(r+d)/p} \\ +l(1 + L_2 + L_3)e^{2\lambda d/p}]^p\Big(\frac{b}{a}\Big)\varrho^p\|\phi\|_\tau^p e^{-\lambda(t_m-t_0-d)}. \end{aligned}$$

It is also easy to check

$$\begin{aligned} |g_m(x(t_m^-), x(t_m^-))| < \rho, \\ |g_m(x(t_m^-), x((t_m - d_m)^-))| < \rho, \\ |\Delta g_m| < \rho. \end{aligned} \tag{14}$$

Combing (11), (13), (15), (S_1), (S_3) and (S_4) gives

$$\begin{aligned} V(t_m) &= V(t_m, x(t_m)) \\ &= V(t_m, g_m(x(t_m^-), x((t_m - d_m)^-))) \\ &= V(t_m, g_m(x(t_m^-), x(t_m^-)) + \Delta g_m) \\ &\leq k_1 V(t_m, g_m(x(t_m^-), x(t_m^-))) + k_2 V(t_m, \Delta g_m) \\ &\leq k_1 \nu V(t_m^-) + k_2 b |\Delta g_m|^p \\ &\leq \{k_1 \nu + \frac{b}{a} k_2 L_3^p [d L_1 e^{\lambda(r+d)/p} \\ &+ l(1 + L_2 + L_3) e^{2\lambda d/p}]^p\} b \varrho^p \|\phi\|_\tau^p e^{-\lambda(t_m - t_0 - d)} \\ &\leq b \varrho^p \|\phi\|_\tau^p e^{-\lambda(t_m - t_0 - d)}. \end{aligned}$$

Therefore, $W(t_m) \leq b\varrho^p \|\phi\|_\tau^p$.

In summary, we have proved, for $s \in [t_0 - \tau, t_m]$,

$$W(s) \leq b\varrho^p \|\phi\|_\tau^p. \quad (15)$$

We can establish by contradiction that (15) holds on $[t_m, t_{m+1})$ as same as that on $[t_0 - \tau, t_1)$. By the mathematical induction method, for any $k \in \mathbb{N}$, (15) holds on $[t_0 - \tau, t_k)$. Similar to the above argument, we can establish $W(t_k) \leq b\varrho^p \|\phi\|_\tau^p$. Again using the contradiction method, we can see(15) is true on $[t_k, t_{k+1}^*)$. By the theorem of continuity in[10], we obtain $\bar{T} = +\infty$, and (1) is exponentially stable.

Theorem 1 can be used to investigate the Robust stability of some continuous time-delay systems with delayed impulses, where the condition (S_2) implies the continuous system in (1) is stable. If the constant c in (S_2) is negative, then the continuous system is not stable. In this case, the Lyapunov function associated with this system is not necessary to be decreased along it's orbits. Thus, the stability of the whole system greatly depends on the effect of the input impulses. The next theorem concerns this case.

Theorem2: Suppose (1) satisfies $(A_{01}) - (A_{03})$ and $(A_1) - (A_3)$. There exist $V \in \nu_0, a, b, p \geq 1, \kappa > 1$ and $c \leq 0$ such that $(S_1) - (S_2)$ holds. Let $\triangle = N_0 T_a$. If there exist $\nu, k_1, k_2 > 0$ such that $(S_3) - (S_4)$ is true. Moreover, we can find $d \geq 0$ satisfying

$$\begin{aligned} & e^{-c\triangle} < \min\{\kappa, [k_1 \nu \\ & + \frac{b}{a} k_2 L_3^p (d L_1 + l(1 + L_2 + L_3))^p]^{-1}\}, \end{aligned} \quad (16)$$

then the system (1) is exponentially stable as long as the input impulsive delays $d_k \leq d, k \in \mathbb{N}$.

Proof: It is easy to see that, if (16) is true, there exist $\lambda, \sigma > 0$ such that

$$\begin{aligned} & e^{(-c+\lambda)\triangle} < \sigma \\ & < \min\{\kappa e^{-\lambda r}, \{k_1 \nu + \frac{b}{a} k_2 L_3^p (d L_1 e^{\lambda(r+d)/p} \\ & + l(1 + L_2 + L_3) e^{2\lambda d/p})^p\}^{-1}\}. \end{aligned} \quad (17)$$

Take $\delta \in (0, (\sqrt[p]{\frac{b\sigma}{a}} (\nabla + L_3)\varrho)^{-1}\rho)$. Suppose $x(t) = x(t, t_0, \phi)$ is the solution of (1) subject to $(t_0, \phi) \in \mathbb{R}^+ \times PRC\ ([-\tau, 0], \mathscr{B}(\delta))$, with maximal existence interval $[t_0 - \tau, \bar{T})$, where $\bar{T} > t_0$ is a positive number. We show $\bar{T} = \infty$ and

$$V(t) \leq b\sigma\varrho^p \|\phi\|_\tau^p e^{-\lambda(t - t_0 - d)}, \quad t \in [t_0 - \tau, \bar{T}). \quad (18)$$

We still denote the impulsive sequence on $(t_0 + d, \bar{T})$ by $\{t_i\}, i = 1, 2, \cdots$. Let $t_k^* = \min\{t_k, \bar{T}\}$. For $t \in [t_k, t_{k+1}^*)$, define

$$W(s) = e^{\lambda(s - t_0 - d)} V(s), \ s \in [t_0 - \tau, t]. \quad (19)$$

Similar to the proof of Theorem 1, we will apply the method of mathematical induction to establish that for $k \geq 1, s \in [t_0 - \tau, t_k)$,

$$W(s) \leq b\sigma\varrho^p \|\phi\|_\tau^p. \quad (20)$$

By (S_1) and Lemma 1

$$W(s) \leq b\varrho^p \|\phi\|_\tau^p, \quad s \in [t_0 - \tau, t_0 + d].$$

Then (20) holds on $[t_0 - \tau, t_0 + d]$.

We claim that for $t \in [t_0 - \tau, t_1)$, (20) also is true. Otherwise, there exists $s \in [t_0 + d, t_1)$ such that

$$W(s) > b\sigma\varrho^p \|\phi\|_\tau^p.$$

Let $t^* = \inf\{t \in (t_0 + d, t_1); W(s) > b\sigma\varrho^p \|\phi\|_\tau^p\}$. Then $t^* \in (t_0 + d, t_1)$ and $W(t^*) = b\sigma\varrho^p \|\phi\|_\tau^p\}$. Let $\bar{t} = \sup\{t \in (t_0 - \tau, t^*); W(s) \leq b\varrho^p \|\phi\|_\tau^p\}$. Then $\bar{t} \in [t_0 + d, t^*)$ and $W(\bar{t}) = b\varrho^p \|\phi\|_\tau^p$.

For $s \in [\bar{t}, t^*)$,

$$W(s) \geq b\varrho^p \|\phi\|_\tau^p = \frac{1}{\sigma} \cdot b\sigma\varrho^p \|\phi\|_\tau^p \geq \frac{1}{\sigma} W(s+\theta), \ \theta \in [-r, 0].$$

by which and (17), we have

$$\begin{aligned} V(s) &\geq \frac{1}{\sigma} e^{\lambda\theta} V(s + \theta) \geq \frac{1}{\sigma} e^{-\lambda r} V(s + \theta) \\ &\geq \frac{1}{\kappa} V(s + \theta), \quad \theta \in [-r, 0]. \end{aligned}$$

For $s \in [\bar{t}, t^*)$, using $W(s) \leq b\sigma\varrho^p \|\phi\|_\tau^p$ and (S_1), we obtain

$$|x(s)| \leq \sqrt[p]{\frac{b\sigma}{a}} \varrho \|\phi\|_\tau < \rho.$$

Then by (S_2),

$$D^+ V(s) \leq -cV(s), \quad s \in [\bar{t}, t^*).$$

Integrating the above formula from $\bar{t}$ to t^* yields

$$\begin{aligned} V(t^*) &\leq e^{-c(t^* - \bar{t})} V(\bar{t}) = e^{-c(t^* - \bar{t})} e^{-\lambda(\bar{t} - t_0 - d)} b\varrho^p \|\phi\|_\tau^p \\ &= e^{(-c+\lambda)(t^* - \bar{t})} e^{-\lambda(t^* - t_0 - d)} b\varrho^p \|\phi\|_\tau^p. \end{aligned} \quad (21)$$

Since there are not impulses on $(\bar{t}, t^*)$, $N_\zeta(t^*, \bar{t}) = 0$. By the definition of average impulsive interval, $t^* - \bar{t} \leq N_0 T_a = \triangle$. By (21),

$$\begin{aligned} V(t^*) &\leq e^{(-c+\lambda)\triangle} e^{-\lambda(t^* - t_0 - d)} b\varrho^p \|\phi\|_\tau^p \\ &< \sigma e^{-\lambda(t^* - t_0 - d)} b\varrho^p \|\phi\|_\tau^p. \end{aligned} \quad (22)$$

Then $W(t^*) < b\sigma\varrho^p \|\phi\|_\tau^p$, a contradiction.

Now, assume for $m \in \mathbb{N}, 1 \leq m \leq k - 1$,

$$W(s) \leq b\sigma\varrho^p \|\phi\|_\tau^p, \ s \in [t_0 - \tau, t_m). \quad (23)$$

we will show

$$W(s) \le b\sigma\varrho^p \|\phi\|_\tau^p,\ s \in [t_m, t_{m+1}). \tag{24}$$

By(23) and (S_1),

$$|x(s)| \le \sqrt[p]{\frac{b\sigma}{a}}\varrho\|\phi\|_\tau e^{-\lambda(s-t_0-d)/p} < \rho, \quad s \in [t_0-\tau, t_m).$$

Since $N_\zeta(t_m, t_m-d_m) \le \frac{d_m}{T_a}+N_0 \le \frac{d}{T_a}+N_0$, there are at most $l = [\frac{d}{T_a}]+N_0$ impulses on (t_m-d_m, t_m).

As the proof of Theorem 1, we can establish that

$$\begin{aligned}&|x(t_m^-) - x(t_m-d_m)^-|\\ \le\ & [L_1 d e^{\lambda(r+d)/p} + l(1+L_2+L_3)e^{2\lambda d/p}]\sqrt[p]{\frac{b\sigma}{a}}\\ &\varrho\|\phi\|_\tau e^{-\lambda(t_m-t_0-d)/p}.\end{aligned} \tag{25}$$

Then

$$\begin{aligned}& V(t_m)\\ \le\ & \{k_1\nu + \frac{b}{a}k_2 L_3^p [L_1 d e^{\lambda(r+d)/p}\\ +\ & l(1+L_2+L_3)e^{2\lambda d/p}]^p\} b\sigma\varrho^p\|\phi\|_\tau^p e^{-\lambda(t_m-t_0-d)}\\ \le\ & b\sigma\varrho^p\|\phi\|_\tau^p e^{-\lambda(t_m-t_0-d)}.\end{aligned}$$

That is

$$W(t_m) \le b\sigma\varrho^p\|\phi\|_\tau^p.$$

We can establish by contradiction that (15) holds on $[t_m, t_{m+1})$ as same as that on $[t_0-\tau, t_1)$. By the mathematical induction method, for any $k \in \mathbb{N}$, (15) holds on $[t_0-\tau, t_k)$. Similar to the above argument, we can establish $W(t_k) \le b\varrho^p\|\phi\|_\tau^p$. Again using the contradiction method, we can see(15) is true on $[t_k, t_{k+1}^*)$. By the theorem of continuity in [10], we obtain $\bar{T} = +\infty$, and (1) is exponentially stable.

IV. An application of the main results

We now apply the above results to investigate the following time-delay system with delayed impulses.

$$\begin{cases} \dot{x}(t) = Ax(t) + \Phi(t, x(t-r)); & t > t_0, t \ne t_i\\ x(t^+) = \mu x(t^-) + B_i(x(t-d_i)^-), & t = t_i, i \in \mathbb{N}\\ x(t) = \phi(t-t_0), & t_0-\tau \le t \le t_0,\end{cases} \tag{26}$$

where A is an $n\times n$ constant matrix, $r \ge 0$ is a time-delay, $\Delta x(t_i) = x(t_i) - x(t_i^-)$. Impulsive sequence $\{t_i\}$ satisfies $t_0 < t_1 < t_2 < \cdots < t_n < \cdots$, $\lim_{n\to\infty} t_n = \infty$. d_i is the impulsive delay. $d = \max_i\{d_i\} \ge 0$. $\phi(t) \in C^1([t_0-\tau, 0])$, where $\tau = \max\{r, d\}$. $B_i, i = 1, 2, \cdots$ are $n\times n$ matrices. For a matrix B, define it's norm by $\|B\| = \sqrt{\lambda_{\max}(B^TB)}$, where $\lambda_{\max}(B^TB)$ is the most maximum eigenvalue of B^TB.

Recently, [12] investigated (26) without time-delays(i.e., $r = 0$). They established that the solution of (26) is (uniformly) equi-attractive in large. Here we consider the exponential stability of (26). Suppose

There exist M_0, M_1, L_0 such that $\|B_i + \mu I\| \le M_0, \|B_i\| \le M_1, i = 1, 2, \cdots .|\Phi(t,x)| \le L_0|x|, (t,x) \in \mathbb{R}^+\times\mathbb{R}^n$.

This guarantees for each $(t_0, \phi) \in \mathbb{R}^+ \times C([-\tau, 0], \mathbb{R}^n)$, the system (26) has a local solution $x(t) = x(t, t_0, \phi)$ ([10]). Here we assume this solution is right-hand continuous on it's existence interval.

For the impulsive sequence $\{t_i\}$, we suppose it's average impulsive interval is T_a, i.e., there exist $N_0 \in \mathbb{N}, T_a > 0$ such that $\frac{T-t}{T_a} - N_0 \le \mathscr{N}(T,t) \le \frac{T-t}{T_a} + N_0$.

Let $f(t, x_t) = Ax + \Phi(t, x(t-r)), g_i(x,y) = \mu x + B_i y$. Then for $L_1 = \|A\| + L_0, L_2 = \mu$ and $L_3 = M_1$, $(A_1), (A_2)$ are satisfied.

Choose $V(\psi) = |\psi|$ for $\psi \in PRC([-\tau, 0], \mathbb{R}^n)$. Obviously, $V \in \nu_0$ and V satisfies (S_1) for $a = b = p = 1$. It is not difficult to calculate that

$$\begin{aligned} D^+V(x(t)) &= \frac{(x(t), Ax(t)) + (x(t), \Phi(t, x(t-r)))}{|x(t)|}\\ &\le \lambda_{\max}(A)|x(t)| + L_0|x(t-r)|.\end{aligned}$$

Then whenever $\kappa V(\psi(t)) \ge V(\psi(t+\theta))$, $\theta \in [-\tau, 0]$, (S_2) holds for $c = -(\lambda_{\max}(A) + \kappa L_0)$.

Since $V(g_i(x,x)) = |(B_i + \mu I)x| \le \|B_i + \mu I\||x| \le M_0 V(x)$, for $\nu = M_0$, (S_3) holds. For $k_1 = k_2 = 1$, (S_4) is true.

If $\bar\lambda + \kappa L_0 < 0$ and there exists $d \ge 0$ such that

$$\begin{aligned}&M_0 + M_1[d(\|A\| + L_0)\\ &+ (\lceil\frac{d}{T_a}\rceil + N_0)(1+\mu+M_1)] < 1,\end{aligned} \tag{27}$$

then (3) holds;

If $\bar\lambda + \kappa L_0 \ge 0$ and there exists $d \ge 0$ such that

$$\begin{aligned} e^{(\bar\lambda+\kappa L_0)N_0T_a} &< [M_0 + M_1(d(\|A\| + L_0)\\ &+ (\lceil\frac{d}{T_a}\rceil + N_0)(1+\mu+M_1)]^{-1},\end{aligned} \tag{28}$$

then (16) is true.

By Theorem 1 and Theorem 2, for arbitrary input delays $d_k \le d$, the solution of (26) is exponentially stable.

V. Numerical examples

In this section, as a specific example of the above subsection, we consider the following time-delay system with delayed impulses:

$$\begin{cases} \dot{x}_1(t) = ax_1(t) + x_1(t-r)\sin(x_2(t-r)) & t \ne t_k\\ \dot{x}_2(t) = ax_2(t) + x_2(t-r)\cos(x_1(t-r)) & t \ne t_k\\ x_1(t^+) = \mu x_1(t^-) + bx_1((t-d_k)^-) & t = t_k\\ x_2(t^+) = \mu x_2(t^-) + bx_2((t-d_k)^-) & t = t_k\end{cases} \tag{29}$$

where a, b, μ are constants, $r, d_k \ge 0, k \in \mathbb{N}$. Here the impulsive sequence is taken by $\zeta = \{\epsilon, 2\epsilon, \cdots, (N_0-1)\epsilon, N_0T_a, N_0T_a+\epsilon, N_0T_a+2\epsilon, \cdots, N_0T_a+(N_0-1)\epsilon, 2N_0T_a, \cdots\}$, which was first constructed in [15]. Obviously, $\inf_{k\in\mathbb{N}^+}\{t_k - t_{k-1}\} = \epsilon, \sup_{k\in\mathbb{N}^+}\{t_k - t_{k-1}\} = N_0(T_a - \epsilon) + \epsilon$. When ϵ is small enough, the smallest length of impulsive intervals can be arbitrarily small. While if N_0 is large enough, the supremum of impulsive intervals can be very large. Assume the average impulsive interval is T_a. Here we take $T_a = 0.5, N_0 = 5, r = 1$.

Let $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $B_i = B = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix}$, $\Phi(t,x) = \begin{bmatrix} x_1 \sin x_2 \\ x_2 \cos x_1 \end{bmatrix}$. Then $\|A\| = |a|, M_0 = |b+\mu|, M_1 = b, L_0 = 1, \lambda_{\max}(A) + \kappa L_0 = a + \kappa$.

In (29), for $\kappa = \frac{3}{2}$, if we choose $a = -2, b = \mu = \frac{1}{14}, d = 1$, then $\lambda_{\max}(A) + \kappa L_0 = -\frac{1}{2} < 0$ and (27) holds; If we choose $a = 1, b = \frac{1}{3\times 10^3}, \mu = \frac{1}{6\times 10^2}, d = 1$. Then$\lambda_{\max}(A) + \kappa L_0 = \frac{5}{2} > 0$ and (28) holds. In both cases, for any d_k satisfying $0 \le d_k \le 1$, the system is exponentially stable by virtue of the results in the above subsection.

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